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Dynamic stiffness matrix development and free vibration analysis of a moving beam

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Abstract

The dynamic stiffness matrix of a moving Bernoulli–Euler beam is developed and used to investigate its free flexural vibration characteristics. In order to develop the dynamic stiffness matrix, it is necessary to derive and solve the governing differential equation of motion of the moving beam in closed analytical form. The solution is then used to obtain the general expressions for both responses and loads. Boundary conditions are applied to determine the constants in the general solution, leading to the formation of the frequency dependent dynamic stiffness matrix of the moving beam, relating the amplitudes of the harmonically varying loads to those of the corresponding responses. The application of the resulting dynamic stiffness matrix using the Wittrick–Williams algorithm is demonstrated by some illustrative examples. Numerical results for both simply supported and fixed–fixed end conditions of the beam are discussed, and wherever possible, some are compared with those available in the literature.

1. Introduction

There are many engineering structures that can be modelled as axially moving beams. Examples include power transmission belt and chain drives, high-speed magnetic tapes, aerial cable tramways, band saws, pipe-conveying fluids and many other technological devices. The free vibration analysis of such structures is of considerable importance in achieving a safe, reliable and where possible, an optimum design. Preliminary research on the subject can be traced back nearly half a century ago [1] and then about a decade later, Mote [2] and Barakat [3] approached the problem using, respectively, the Galerkin method and an eigen-function series superposition procedure. Subsequently, Simpson [4], Tabarrok et al. [5], Buffinton and Kane [6] and Wickert and Mote [7] extended the work by developing the governing differential equations of motion more accurately, and sought classical solutions for the problem. On the other hand Hwang and Perkins [8,9] examined both the stability and free vibration characteristics of moving beams by considering geometric nonlinearity resulting from the large deformation. Their investigation showed that the critical speed could be sensitive to system imperfections, such as initial curvature. Later Sreeram and Sivaneri [10] obtained a finite element based solution for both free and forced vibration response of a moving beam using the Lagrangian multiplier

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Nomenclature

$\mathbf{C}$	constant vector
c	axial velocity of the moving beam
EI	bending or flexural rigidly of the beam
i	$\sqrt{-1}$
$\mathbf{K}$	dynamic stiffness matrix
L	length of the beam
M	bending moment at a cross-section of the beam
m	mass per unit length of the beam, $m=\rho A$
P	axially applied tensile load
S	shear force at a cross-section of the beam
t	time
W	amplitude of bending or flexural displacement
w	bending or flexural displacement of the beam
X, Y	rectangular Cartesian coordinate system
λ	non-dimensional natural frequency, see Eq. (8)
μ	non-dimensional parameter, see Eq. (8)
ν	non-dimensional parameter, see Eq. (8)
ω	circular or angular frequency
θ	bending rotation
ξ	non-dimensional length

method. For further reading on this and related research, see Jha and Parker [11], Oz [12,13], Kong and Parker [14,15], Lee [16], Pellicano [17] and Andrianov and Awrejcewicz [18]. The theory used in this paper is classical and linear, namely, that of the dynamic stiffness method, which is a powerful alternative to other methods. However, it is recognised that the application of nonlinear theory can enhance the understanding of the problem, particularly at higher speeds, but such an analysis is beyond the scope of this paper and interested readers are referred to Refs. [19–22].

2. Theory

In a rectangular two-dimensional Cartesian coordinate system (XY), Fig. 1 shows a beam of length L, flexural rigidly EI, mass per unit length m, moving with a constant velocity c, over two supports. (Note that other support conditions may occur, but simple supports are shown only for convenience.) An axially applied constant tensile load P, as shown, is acting on the beam. The governing differential equation of motion of the moving beam and the expressions for shear force (S) and bending moment (M) at a cross-section x from the origin are given by [14,16]

$$EI\frac{\partial^4 w}{\partial x^4} - P\frac{\partial^2 w}{\partial x^2} + \rho A\left(\frac{\partial^2 w}{\partial t^2} + 2c\frac{\partial^2 w}{\partial x\,\partial t} + c^2\frac{\partial^2 w}{\partial x^2}\right) = 0, \tag{1}$$

$$S(x,t) = EI\frac{\partial^3 w}{\partial x^3} + \rho Ac\left(\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x}\right) - P\frac{\partial w}{\partial x}, \tag{2}$$

$$M(x,t) = -EI\frac{\partial^2 w}{\partial x^2}, \tag{3}$$

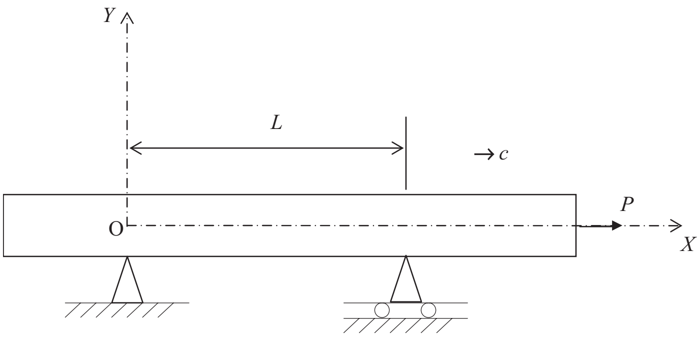

Fig. 1. Notation and coordinate system of a moving beam.

where w is the flexural displacement in the y direction of a point on the axis of the beam at a distance x form the origin and t is time.

Harmonic oscillation is assumed so that w may be written in the form

$$w(x,t) = W(x)\mathrm{e}^{\mathrm{i}\omega t}, \tag{4}$$

where W is the amplitude of the bending or flexural displacement, ω is the circular (or angular) frequency and $\mathrm{i} = \sqrt{-1}$.

Using Eq. (4), Eq. (1) can be converted into an ordinary differential equation as follows:

$$EI\,\frac{\mathrm{d}^4 W}{\mathrm{d}x^4} - P\,\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} + \rho A\left(c^2\,\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} + 2\mathrm{i}\omega c\,\frac{\mathrm{d}W}{\mathrm{d}x} - \omega^2 W\right) = 0. \tag{5}$$

Introducing the non-dimensional length ξ so that

$$\xi = \frac{x}{L}. \tag{6}$$

Eq. (5) can now be written in the following non-dimensional form:

$$[D^4 - (p^2 - v^2)D^2 + 2\mathrm{i}\mu^2 D - \lambda^2]W = 0, \tag{7}$$

where

$$p^2 = \frac{PL^2}{EI}, \quad \mu^2 = \frac{\rho A\omega c L^3}{EI}, \quad v^2 = \frac{\rho A c^2 L^2}{EI}, \quad \lambda^2 = \frac{\rho A \omega^2 L^4}{EI} \tag{8}$$

and

$$D = \frac{\mathrm{d}}{\mathrm{d}\xi} \tag{9}$$

denotes differentiation with respect to ξ.

The fourth-order differential Eq. (7) can now be solved assuming a solution of the type

$$W(\xi) = C\mathrm{e}^{\mathrm{i}k\xi}. \tag{10}$$

Substituting Eq. (10) into Eq. (7) yields the auxiliary equation

$$k^4 + (p^2 - v^2)k - 2\mu^2 k - \lambda^2 = 0. \tag{11}$$

The above equation is a quartic in k and can be solved using standard procedures [23,24]. (Note that the roots are in general, complex.)

Let the four roots of k in Eq. (11) be k_j ($j = 1, 2, 3$ and 4) so that the solution $W(\xi)$ of the differential Eq. (7) is given by

$$W(\xi) = C_1\mathrm{e}^{\mathrm{i}k_1\xi} + C_2\mathrm{e}^{\mathrm{i}k_2\xi} + C_3\mathrm{e}^{\mathrm{i}k_3\xi} + C_4\mathrm{e}^{\mathrm{i}k_4\xi} = \sum_{j=1}^{4} C_j\mathrm{e}^{\mathrm{i}k_j\xi}. \tag{12}$$

The expression for the bending rotation θ (ξ) is given by

$$\theta(\xi) = \frac{1}{L}\,\frac{\mathrm{d}W}{\mathrm{d}\xi} = \frac{\mathrm{i}}{L}\sum_{j=4}^{4} C_j k_j \mathrm{e}^{\mathrm{i}k_j\xi}. \tag{13}$$

With the help of Eq. (12), the amplitude of the shear force bending moment can be obtained by substituting it into Eqs. (2) and (3) and using Eq. (4) to give

$$S(\xi) = -\frac{EI}{L^3}\sum_{j=1}^{4} \mathrm{i}\{k_j^3 + k_j(p^2 - v^2) - \mu^2\}C_j\mathrm{e}^{\mathrm{i}k_j\xi} \tag{14}$$

and

$$M(\xi) = \frac{EI}{L^2} \sum_{j=1}^{4} k_j^2 C_j e^{ik_j \xi}. \tag{15}$$

The expressions for bending displacement $W(\xi)$, bending rotation $\theta(\xi)$, shear force $S(\xi)$ and bending moment $M(\xi)$ given by Eqs. (12)–(15) can now be used to derive the dynamic stiffness matrix of the moving beam by imposing the boundary conditions. Referring to Fig. 2 for the sign convention, the boundary conditions for responses and loads are given as follows.

Responses:

$$\begin{aligned} &\text{At } \xi = 0 (x = 0) : W = W_1 \text{ and } \theta = \theta_1, \\ &\text{At } \xi = 1 (x = L) : W = W_2 \text{ and } \theta = \theta_2. \end{aligned} \tag{16}$$

Loads:

$$\begin{aligned} &\text{At } \xi = 0 (x = 0) : S = S_1 \text{ and } M = M_1, \\ &\text{At } \xi = 1 (x = L) : S = -S_2 \text{ and } M = -M_2. \end{aligned} \tag{17}$$

Substituting Eq. (16) into Eqs. (12) and (13) gives

$$W_1 = C_1 + C_2 + C_3 + C_4, \tag{18}$$

$$\theta_1 = \frac{i}{L}(C_1 k_1 + C_2 k_2 + C_3 k_3 + C_4 k_4), \tag{19}$$

$$W_2 = C_1 e^{ik_1} + C_2 e^{ik_2} + C_3 e^{ik_3} + C_4 e^{ik_4}, \tag{20}$$

$$\theta_2 = \frac{i}{L}(C_1 k_1 e^{ik_1} + C_2 k_2 e^{ik_2} + C_3 k_3 e^{ik_3} + C_4 k_4 e^{ik_4}). \tag{21}$$

Eqs. (18)–(21) can be written in the following matrix:

$$\begin{bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{ik_1}{L} & \frac{ik_2}{L} & \frac{ik_3}{L} & \frac{ik_4}{L} \\ e^{ik_1} & e^{ik_2} & e^{ik_3} & e^{ik_4} \\ \frac{i}{L}k_1 e^{ik_1} & \frac{i}{L}k_2 e^{ik_2} & \frac{i}{L}k_3 e^{ik_3} & \frac{i}{L}k_4 e^{ik_4} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{22}$$

or,

$$\boldsymbol{\delta} = \mathbf{QC}, \tag{23}$$

where $\boldsymbol{\delta}$ is the vector of displacements W_1, θ_1, W_2, θ_2, and $\mathbf{C}$ is the constant vector with elements C_1, C_2, C_3 and C_4 whilst $\mathbf{Q}$ is the 4×4 square matrix as given above.

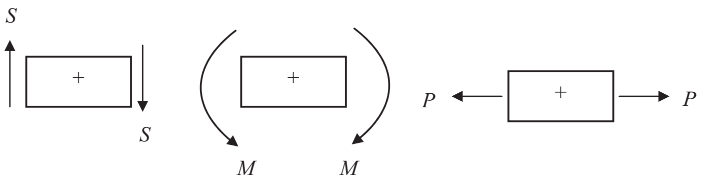

Fig. 2. Sign convention for positive shear force (*S*), positive bending moment (*M*) and positive axial load (*P*).

Substituting Eq. (17) into Eqs. (14) and (15) gives

$$S_1 = -\frac{EI}{L^3}\sum_{j=1}^{4} \mathrm{i}\{k_j^3 + k_j(p^2 - v^2) - \mu^2\}C_j, \tag{24}$$

$$M_1 = \frac{EI}{L^2}\sum_{j=1}^{4} k_j^2 C_j, \tag{25}$$

$$S_2 = \frac{EI}{L^3}\sum_{j=1}^{4} \mathrm{i}\{k_j^3 + k_j(p^2 - v^2) - \mu^2\}C_j \mathrm{e}^{\mathrm{i}k_j}, \tag{26}$$

$$M_2 = -\frac{EI}{L^2}\sum_{j=1}^{4} k_j^2 C_j \mathrm{e}^{\mathrm{i}k_j}. \tag{27}$$

Eqs. (24)–(27) can be written in the following matrix form:

$$\begin{bmatrix} S_1 \\ M_1 \\ S_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{28}$$

or,

$$\mathbf{F} = \mathbf{RC}, \tag{29}$$

where $\mathbf{F}$ is the vector of forces S_1, M_1, S_2, M_2, and $\mathbf{C}$ is the constant vector already defined before (see Eqs. (22) and (23)) whilst $\mathbf{R}$ is the 4×4 square matrix as above whose elements are

$$R_{1j} = -\frac{EI}{L^3}\mathrm{i}\{k_j^3 + k_j(p^2 - v^2) - \mu^2\}, \tag{30}$$

$$R_{2j} = \frac{EI}{L^2}k_j^2, \tag{31}$$

$$R_{3j} = \frac{EI}{L^3}\sum_{\mathrm{j}=1}^{4} \mathrm{i}\{k_j^3 + k_j(p^2 - v^2) - \mu^2\}\mathrm{e}^{\mathrm{i}k_j}, \tag{32}$$

$$R_{4j} = -\frac{EI}{L^2}k_j^2 \mathrm{e}^{\mathrm{i}k_j}. \tag{33}$$

By eliminating the constant vector $\mathbf{C}$ from Eqs. (23) and (29), the 4×4 frequency dependent dynamic stiffness matrix follows as

$$\mathbf{K} = \mathbf{RQ}^{-1}. \tag{34}$$

The above dynamic stiffness matrix can now be used to compute the natural frequencies and mode shapes of either an individual moving beam or an assembly of them. An effective way is to apply the well-known algorithm of Wittrick and Williams [25] generally used in solving transcendental eigenvalue problems, and which has featured in literally hundred of papers (see for example, Refs. [26–29]). It uses the Sturm sequence property of the dynamic stiffness matrix and ensures that no natural frequencies of the structure are missed. Computer implementation of the algorithm is very simple, but for a detailed understanding, interested readers are referred to the original work of Wittrick and Williams [25] which was significantly enhanced later by Williams [26,27] to include the important case when the dynamic stiffness matrix is Hermitian rather than real, as in the present investigation.

3. Scope and limitations of the theory

The dynamic stiffness matrix developed in this paper is limited in that it is based on linear small deflection theory that may be adequate in the lower ranges of the moving speed, but may not be so at higher speeds. This is particularly significant in the vicinity of the critical speed when the beam becomes unstable. The present theory shows that as the moving speed increases, the natural frequency of the beam decreases and eventually, tends to zero, leading to the instability called divergence phenomenon, which is strictly speaking a nonlinear phenomenon. For a better understanding of the instability, it will be prudent to use a well-judged nonlinear analysis, which represents the physical system more accurately. Thurman and Mote [19] made a direct comparison of the free vibration results of a moving strip using both linear and nonlinear theories. They generally concluded that the effect of nonlinear terms become more pronounced at higher moving speeds and particularly nearer the critical speed, when a simple minded linear analysis may not be adequate. Thus, caution should be exercised when using the present theory, particularly within close proximity of the critical speed, i.e. in the vicinity of the divergence instability, when nonlinear (velocity dependent) terms become significant [19]. Nevertheless, the interpolated values of the critical speed computed from the present theory are helpful in establishing trends and providing a qualitative appraisal of the instability. A recent paper by Chen and Yang [20] who used both linear and nonlinear theories to solve the problem shed some more light on the subject. The authors drew similar conclusions to those of Thurman and Mote [19] that the nonlinear terms become more pronounced at higher moving speeds and for higher-order modes, but significantly, they produced a formula relating the frequency of nonlinear free vibration to that of the linear one, see their

Table 1
Natural frequencies of a moving beam with simply supported and fixed–fixed boundary conditions for a range of axial load (p^2) and moving speed (v^2) parameters

Axial load parameter (p^2)	Moving speed parameter (v^2)	Natural frequencies ($\lambda_i = \omega_i\sqrt{(\rho AL^4)/EI}$)					
		Simply supported			Fixed–fixed		
		λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
0.00	0.00	9.87 (9.87)	39.48 (—)	88.83	22.37 (22.37)	61.67 (61.67)	120.90
	1.00	9.37 (9.26)	39.00 (39.07)	88.39	22.15 (21.95)	61.46 (61.37)	120.51
	2.00	7.73 (7.31)	37.84 (37.85)	86.70	21.45 (20.68)	61.22 (60.45)	120.11
0.25	0.00	9.99	39.60	88.95	22.44	61.76	121.00
	1.00	9.52	39.08	88.66	22.22	61.66	120.68
	2.00	8.05	37.99	88.08	21.52	61.53	120.33
0.50	0.00	10.11	39.72	89.07	22.51	62.31	121.11
	1.00	9.65	39.24	88.78	22.29	62.26	120.72
	2.00	8.05	38.25	88.08	21.60	62.17	120.42
0.75	0.00	10.23	39.85	89.20	22.60	62.89	121.20
	1.00	9.78	39.36	89.09	22.36	62.79	120.86
	2.00	8.20	38.26	88.89	21.67	62.71	120.21
1.00	0.00	10.35	39.97	90.32	22.64	63.26	121.31
	1.00	9.90	39.48	90.16	22.43	63.21	121.01
	2.00	8.35	38.66	90.06	21.74	63.12	120.64
1.50	0.00	10.59	40.22	92.77	22.78	63.58	121.51
	1.00	10.15	39.72	92.72	22.59	63.42	121.35
	2.00	8.65	38.84	91.06	21.96	63.19	120.80
2.00	0.00	10.82	40.44	92.82	22.91	63.85	121.71
	1.00	10.39	39.96	93.12	22.62	63.47	121.56
	2.00	8.94	39.24	91.88	22.03	63.23	121.46

Results from Ref. [2] are shown in the parenthesis.

Eq. (27), so that a direct comparison can be made. For further studies on moving beams using linear and nonlinear analyses, the survey paper of Chen [21] is recommended. Moreover, the present theory assumes that the beam is elastic and the material is isotropic and homogeneous. It also ignores damping in the system. Thus, the analysis is restrictive and may impose limitations when solving some practical problems. For instance, the dynamic behaviour of viscoelastic moving systems (e.g. certain belt drives) can be very different from the elastic ones [22].

4. Results and discussion

The dynamic stiffness theory developed above is applied through the use of the Wittrick–Williams algorithm as a solution technique to compute the natural frequencies of a moving beam with both simply supported and fixed–fixed boundary conditions. Table 1 shows results in non-dimensional form for the two cases, respectively. The first three natural frequencies (λ_i, $i = 1$, 2 and 3) are shown for a range of axial load (p^2) and moving speed (v^2) parameters. The first two natural frequencies reported in Ref. [2], which does not account for the axial load (i.e. $p^2 = 0$) are shown in the parenthesis. The results from the present theory agreed very well with those of Ref. [2] as can be seen. The effect of the moving speed is to reduce the natural frequencies and the effect is more pronounced for simply supported boundary conditions than the fixed–fixed ones, as expected. By contrast there is no cause for surprise that the effect of the applied tensile load is to increase the natural frequencies (see Table 1) and the effect is more pronounced for the simply supported case than the fixed–fixed one.

For both simply supported and fixed–fixed boundary conditions, Figs. 3(a) and (b) show, respectively, the first two natural frequencies as functions of the moving speed, for a zero and non-zero value of the axially applied tensile load. The trends shown in these figures are in accord with earlier investigations [3,12,13]. It must be stressed that the divergence instability that corresponds to the critical moving speed at which the natural frequency tends to zero (i.e. when the curves touch the horizontal axis, see Fig. 3) is based on linear small deflection theory whose validity in the vicinity of the instability may become questionable. Nevertheless, the extrapolated value of the critical speed computed from the linear theory provides useful information as an

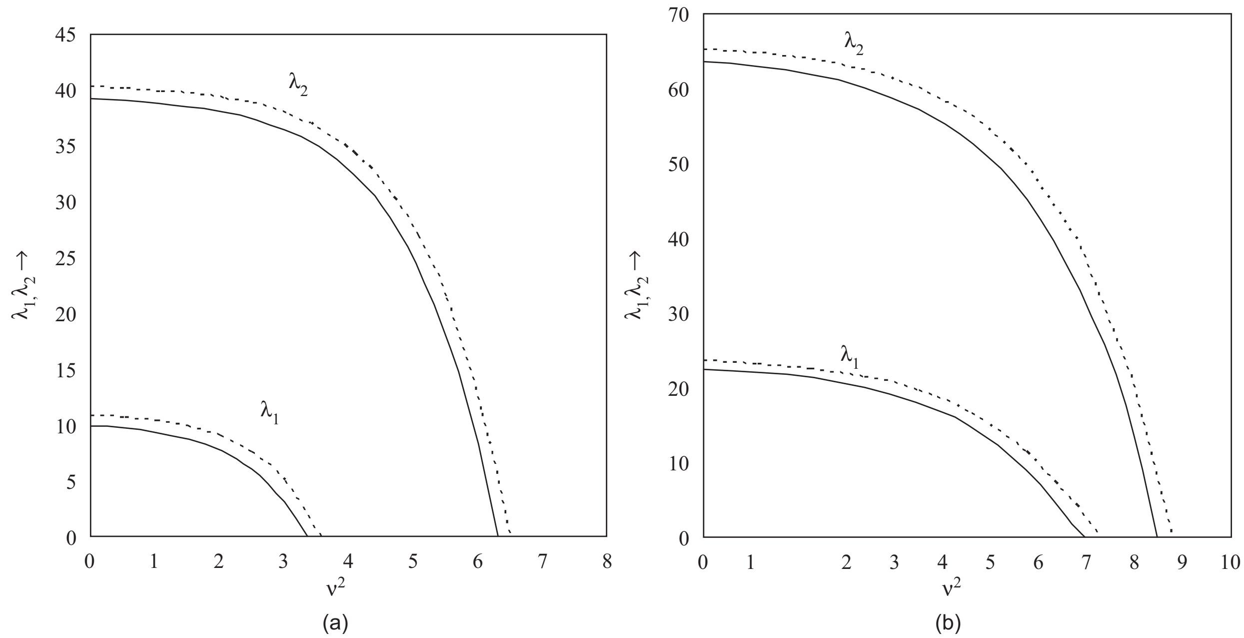

Fig. 3. Effect of moving speed on the first two natural frequencies. ———— $p^2 = 0$; - - - - - - - $p^2 = 4$: (a) simply supported and (b) fixed–fixed.

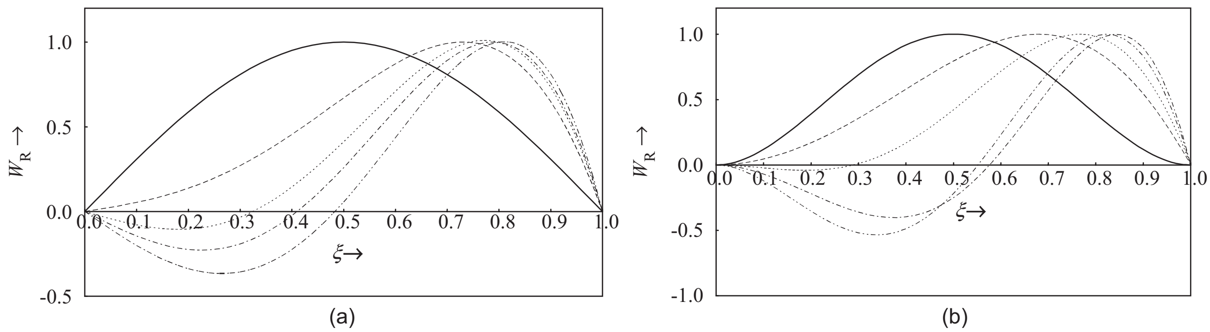

Fig. 4. Effect of moving speed on the fundamental mode shape of the moving beam. ———— $v^2 = 0.0$; - - - - - - - $v^2 = 0.25$; .... $v^2 = 0.375$; - · - · - · - · - · - · - $v^2 = 0.5$; — · — · — · — · — · — · $v^2 = 1$: (a) simply supported (b) fixed–fixed.

indicative step towards capturing the instability. The presence of an axially applied tensile load increases the critical speed and the fixed–fixed boundary conditions have resulted in higher critical speeds than the simply supported ones, as expected.

The final set of results shows the effect of the moving speed on the mode shapes of the moving beam. When the axial load is zero, Figs. 4(a) and (b) show the normalised (i.e. the maximum displacement is set to unity) fundamental mode of the simply supported and fixed–fixed moving beam, respectively. In this figure the real part of the bending displacement (W_R) is plotted along the length. As the moving speed increases the maximum bending displacement moves gradually from the centre of the beam to the right for both simply supported and fixed–fixed boundary conditions—a trend similar to the one observed by other investigators [3,4,7,9,16].

5. Conclusions

Using classical theory, the dynamic stiffness matrix of a moving beam is developed to investigate its free vibration characteristics. Natural frequencies for both simply supported and fixed–fixed boundary conditions are presented for a range of moving speeds and axially applied tensile loads, and wherever possible, results are compared with published ones. The variation of the fundamental mode with moving speed is illustrated. The critical speed is ascertained from the projected value of the linear theory, but for a deeper understanding, a nonlinear analysis might be helpful.

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