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Modal analysis of structures with uncertain-but-bounded parameters via interval analysis

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Abstract

In this paper, the modal interval analysis method to estimate modal parameters, frequency response function (FRF), and mode shapes of structures with uncertain-but-bounded is presented. Although the system parameters or properties are uncertain in many engineering problems, but their probable range of values i.e. upper and lower bounds, can be provided from practical experience and engineering knowledge. Moreover, to avoid the resonance of a structure and to consider the dynamic response of an uncertain one, and also for reliability and stability analysis, we often need the bounds of the ranges of structural characteristic parameters such as natural frequency and normal mode. By using modal analysis and interval calculus, we investigate the method of computing upper and lower bounds of paramenters such as, natural frequencies, modal shapes, and FRFs. Theoretically, it is possible to analyze the uncertain-but-bounded of a structure by using modal analysis and interval calculus. They can be estimated by modal interval analysis. On the basis of the estimated intervals, the engineering structure parameters can be applied into engineering design. A numerical example is presented for a tower structure, and the results illustrate that the proposed method is effective. A comparison of the modal interval method with the results of Monte Carlo simulation serves to validate the solutions and to identify the bounded ranges of parameters.

1. Introduction

Modal analysis method, as an engineering tool, is used to calculate the natural frequencies and mode shapes of a structure. This method is familiar for determining the dynamic response of complicated structural dynamic problems. In general, applications of modal analysis today cover a broad range of objectives: identification and evaluation of vibration phenomena, validation, structural integrity assessment, structural modification, and damage detection. It had been used on mechanical systems, transportation systems, and large civil engineering structures—anything that is subject to dynamic motions or vibration. Modal analysis method had been introduced by many scientists in many books, e.g. [1–3].

Interval analysis method for a system with interval parameters had been used in uncertain structural analysis. Interval calculus is a tool to evaluate a mathematical expression for ranges of values of its

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parameters. Since Moore established the basic theory for the interval analysis in 1966, it has become a tool in many fields in recent years, e.g. [4–6]. In engineering design, it is important to calculate the response quantities such as the displacement, stress, and vibration frequencies, buckling loads and mode shapes to assess the integrity of a proposed structure against a given set of design parameters. In many practical engineering applications, the design parameters may be uncertain because of e.g. manufacturing errors and errors in observation. No surprise that the concept of uncertainty plays an important role in investigation of various engineering problems.

These uncertain structural parameters had been modeled as probabilistic models by a number of authors [7–9]. However, uncertainty could be described by interval vector [10–18] or ellipsoid [19–21], rather than a probabilistic modeling. Based on the invariance properties of eigenvector entries, Deif [10] developed a numerical method for the interval eigenvalue analysis. Because there exists no efficient criterion for judging invariance properties of signs of the components of the eigenvectors, the application of Deif's approach appears restricted.

In structural analysis, Chen et al. [14] had used interval set models to evaluate eigenvalue problems of structures with uncertain-but-bounded parameters. Qiu presented in the year 1994 itself, interval analysis for static response and eigenvalue problem of structures with uncertain parameters. Therefore, in itself practical engineering computation, the considered methods can be used successfully. By means of the stationary condition of Rayleigh quotient, the generalized eigenvalue problem of structures with bounded uncertain parameters can be transformed into two distinct generalized eigenvalue problems. The paper was also proposed for the problem of forced vibration with uncertain structures under interval loading. In their studies, several important results have been obtained. Dimarogonas [11] introduced interval modal analysis, and interval solution of the eigenvalue problem. He discussed the natural and forced vibration problems for interval rotor-bearing systems and solutions were developed using interval calculus.

Dynamic response of structures with uncertain-but-bounded parameters was studied via interval analysis by Qiu and Wang [17]. They did not assume extensive knowledge of the probabilistic characteristics of the uncertain parameters, adopted as a nonprobabilistic, set-theoretic approach to model uncertainty in the structural parameters that often are uncertain-but-bounded not certain.

However, in our previous works [14,18] only direct estimation methods for the interval of natural frequency were proposed. In this paper, interval estimation for the interval of modal parameters is also given. Hence, other interval eigensolutions can be derived using interval modal parameters such as interval natural frequency, frequency response function (FRF), etc. The interval mode shapes are also given.

This paper starts with a brief review of the interval mathematics and then, based on the modal analysis method, the interval stiffness and mass matrices for eigenvalue analysis of structures with interval parameters are derived. Modal interval analysis method is based on modal analysis method via interval analysis, and uncertain structure parameters will be expressed the interval stiffness matrix and interval mass matrix are developed directly from the interval parameters. The obtained generalized interval eigenvalues and corresponding eigenvectors are applied to analyze the range between lower and upper bounds of natural frequencies needed in engineering designs, and the method is extended to solve the modal dynamic response analysis. Then in Section 5, the mode shapes interval is given based on the first-order Taylor's series expansion. A numerical example is given to illustrate the application of our present method. The results obtained by the present method are compared with some in Ref. [19], p. 60–64.

2. Mathematical background of interval algebra conventions

In the following section, we will give a brief review of the definitions of the interval and interval operations. In the following, the field of real numbers is denoted by R and its members are denoted by lower case letters. A subset of R of the form

$$X^I = [\underline{x}, \bar{x}] = \{t | \underline{x} \leqslant t \leqslant \bar{x}, \;\; \underline{x}, \bar{x} \in R\} \tag{1}$$

will be called a (closed) interval; $\underline{x}$ is the lower bound, and $\bar{x}$ is the upper bound. The set of closed real intervals will be denoted by $I(R)$ and its members by upper case letters. Assume that $I(R)$, $I(R^n)$, and $I(R^{n\times n})$ denote the sets of all closed real interval numbers, n-dimensional real interval vectors, and $n \times n$ real interval matrices, respectively. An n-dimensional real interval vector $X^I \in I(R^n)$ can be written as

$$X^I = (X_1^I, X_2^I, \ldots, X_n^I)^{\mathrm{T}}, \tag{2}$$

X^I is a member of $I(R)$ and X^I can be usually written in the following form:

$$X^I = [X^c - \Delta X, X^c + \Delta X], \tag{3}$$

in which X^c and ΔX denote the mean (or midpoint) value of X^I and the radius (or the maximum width) in X^I, respectively. It follows that

$$X^c = \mathrm{mid}(X) = \frac{\underline{x} + \bar{x}}{2}, \quad \Delta X = \mathrm{rad}(X) = \frac{\bar{x} - \underline{x}}{2}. \tag{4}$$

In terms of the interval addition, Eq. (3) can be put into the more useful form

$$X^I = X^c + \Delta X^I, \quad \Delta X^I = [-\Delta X, \Delta X]. \tag{5}$$

Thus, the midpoint (or mean) value and the radius (or uncertainty) in X^I are

$$X^C = (X_1^c, X_2^c, \cdots, X_n^c)^{\mathrm{T}}, \quad \Delta X = (\Delta X_1, \Delta X_2, \cdots, \Delta X_n)^{\mathrm{T}}. \tag{6}$$

Similar expressions exist for an $n \times n$ interval matrix

$$A^I = [\underline{A}, \bar{A}] \in I(R^{n \times n}), \quad A^I = A^c + \Delta A^I, \tag{7}$$

where $\Delta A^I = [-\Delta A,\ \Delta A]$. A^c and ΔA denote the mean matrix of A^I and the uncertainty (or the maximum width) is matrix A^I, respectively. It follows that

$$A^c = \frac{\bar{A} + \underline{A}}{2} \quad \text{or} \quad a_{ij}^c = \frac{\bar{a}_{ij} + \underline{a}_{ij}}{2}, \quad \Delta A = \frac{\bar{A} - \underline{A}}{2} \quad \text{or} \quad \Delta a_{ij} = \frac{\bar{a}_{ij} - \underline{a}_{ij}}{2}, \tag{8}$$

where $A^c = a_{ij}^c$ and $\Delta A = \Delta a_{ij}$. Let X^I, $Y^I \in I(R)$, $X^I = [\underline{x}, \bar{x}]$, $Y^I = [\underline{y}, \bar{y}]$. Then operations for $X^I + Y^I$, $X^I - Y^I$, $X^I \times Y^I$, X^I/Y^I are

$$X^I + Y^I = [\underline{x}, \bar{x}] + [\underline{y}, \bar{y}] = [\underline{x} + \underline{y}, \bar{x} + \bar{y}], \tag{9}$$

$$X^I - Y^I = [\underline{x}, \bar{x}] - [\underline{y}, \bar{y}] = [\underline{x} - \bar{y}, \bar{x} - \underline{y}], \tag{10}$$

$$\begin{aligned} X^I \times Y^I &= [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}] \\ &= \left[\min(\underline{x} \cdot \underline{y}, \underline{x} \cdot \bar{y}, \bar{x} \cdot \underline{y}, \bar{y} \cdot \bar{x}), \max(\underline{x} \cdot \underline{y}, \underline{x} \cdot \bar{y}, \bar{x} \cdot \underline{y}, \bar{y} \cdot \bar{x})\right], \end{aligned} \tag{11}$$

$$\frac{X^I}{Y^I} = \frac{[\underline{x}, \bar{x}]}{[\underline{y}, \bar{y}]} = [\underline{x}, \bar{x}]\left[\frac{1}{\bar{y}}, \frac{1}{\underline{y}}\right] = \left[\frac{\underline{x}}{\bar{y}}, \frac{\bar{x}}{\underline{y}}\right]. \tag{12}$$

Two sets $X^I = [\underline{x}, \bar{x}]$ and $Y^I = [\underline{y}, \bar{y}]$ are equal if $\underline{x} = \underline{y}$ and $\bar{x} = \bar{y}$. An interval of zero width $[x,x]$ will be called as the point interval, and it is a regular real number. It is apparent that division is not defined if $0 \in [\underline{x}, \bar{x}]$ and that $[\underline{x}, \bar{x}]c = [c\,\underline{x}, c\bar{x}]$ for $c>0$ and $[\underline{x}, \bar{x}]c = [c\bar{x}, c\,\underline{x}]$ for $c<0$.

3. Theoretical background of modal analysis

In this paper, real modal analysis is considered. The equation of motion for an undamped structure with multiple degrees of freedom (MDOF) is generally given by

$$M\ddot{x} + Kx = f, \tag{13}$$

where M is the mass matrix of the structure (M represents the inertia properties of the structure), K is the stiffness matrix of the structure, $\ddot{x}$ is the acceleration vector, $\dot{x}$ is the velocity vector, x is the displacement vector, and f is the external force vector.

The problem of free vibration requires that the force vector f be equal to zero in the formulations of the equations of motion. For the stiffness equation with $f = 0$, the motion of an undamped dynamic system in free vibration is governed by a homogeneous system of differential equations, which in matrix notation is

$$M\ddot{x} + Kx = 0. \tag{14}$$

For free vibrations of the undamped structure, we seek solutions of Eq. (14) in the form

$$x_i = \varphi_i \exp^{j\omega t}, \quad i = 1, 2, \cdots, n$$

or in vector notation

$$x = \varphi \exp^{i\omega t}, \quad \ddot{x} = -\omega^2 \varphi \exp^{i\omega t}, \tag{15}$$

where φ_i is the amplitude of motion of the ith coordinate and n is the number of degree of freedom. φ is constant vector, and ω is constant. The substitution of Eq. (15) into Eq. (14) gives

$$-\omega^2 M\varphi \exp^{i\omega t} + K\varphi \exp^{i\omega t} = 0, \tag{16}$$

or rearranging terms

$$(K - \omega^2 M)\varphi = 0, \tag{17}$$

which for the general case is a set of n homogeneous (right-hand side equal to zero) algebraic system of linear equations with n unknown displacements φ_i, and an unknown parameter ω^2. The formulation of Eq. (17) is an important mathematical problem known as an eigenproblem.

Its nontrivial solution, that is, the solution for which not all $\varphi_i = 0$, requires that the determinant of the matrix factor of φ be equal to zero. In this case,

$$\det(K - \omega^2 M) = 0 \quad \text{or} \quad \det(K - \lambda M) = 0. \tag{18}$$

The roots ω_i^2 of this equation provide the natural frequencies ω_i. It is then possible to solve for the unknowns φ_i in terms of relative values. The vectors φ_i corresponding to the roots ω_i^2 are the modal shapes (eigenvectors) of the dynamic system

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \tag{19}$$

$$\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_n), \;\; \varphi_r = (\varphi_{1r}, \varphi_{2r}, \ldots, \varphi_{nr})^{\mathrm{T}}, \;\; r = 1, 2, \ldots, n, \tag{20}$$

where $\lambda = \omega^2$ is a eigenvalue. If the structure has n dynamic degrees of freedom (degrees of freedom with mass), there are n number of ω's that are solutions of the eigenvalue problem. These ω's (ω_1, ω_2,…,ω_n) are the natural frequencies of the structure, also known as normal frequencies, characteristic frequencies, fundamental frequencies, or resonant frequencies. The eigenvector φ_i associated with the natural frequency ω_i is called normal mode or mode shape. The normal mode corresponds to deflected shape patterns of the structure. When a structure is vibrating, its shape at any given time is a linear combination of its normal modes.

We shall now introduce an important property of the normal modes, the orthogonality property. This property constitutes the basis of one of the most attractive methods for solving dynamic problems of MDOF systems. We begin by rewriting the equations of motion in free vibration, Eq. (17), as

$$K\varphi_r = \lambda_r M\varphi_r, \quad r = 1, 2, \ldots, n. \tag{21}$$

From the orthogonality relation between any two modal shapes of an MDOF system, for an n-DOF system in which the mass and stiffness matrix are diagonal, if the term φ_S^{T} multiplies into right- and left-hand side of Eq. (21), the orthogonality condition between any two modes s and r may be expressed as

$$\varphi_S^{\mathrm{T}} K\varphi_r = \lambda_r \varphi_S^{\mathrm{T}} M\varphi_r, \quad r = 1, 2, \ldots, n, \tag{22}$$

$$\varphi_S^{\mathrm{T}} M\varphi_r = \begin{cases} 0, & r \neq s, \\ m_r, & r = s, \end{cases} \quad \varphi_r^{\mathrm{T}} M\varphi_r = m_r, \tag{23}$$

$$\varphi_S^{\mathrm{T}} K \varphi_r = \begin{cases} 0, & r \neq s, \\ k_r, & r = s, \end{cases} \quad \varphi_r^{\mathrm{T}} K \varphi_r = k_r \tag{24}$$

in which φ_S and φ_r are any two modal vectors, M is the mass matrix, K is the stiffness matrix of the structure, m_r is the generalized mass or modal mass, and k_r is the generalized stiffness or modal stiffness. As mentioned above, the modal vectors satisfy the important condition of orthogonality. Thus,

$$\varphi_r^{\mathrm{T}} K \varphi_r = \lambda_r \varphi_r^{\mathrm{T}} M \varphi_r, \tag{25}$$

$$k_r = \lambda_r m_r, \quad \lambda_r = \omega_r^2 = k_r m_r^{-1}. \tag{26}$$

The normal modes may be conveniently arranged in the columns of a matrix known as the modal matrix of the system. For the general case of n-DOF, the modal matrix is written as

$$\Phi = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{pmatrix} \quad \text{or} \quad \Phi = (\varphi_1 \quad \varphi_2 \quad \cdots \quad \varphi_n) \tag{27}$$

and the orthogonality condition may then be expressed in general as

$$\Phi^{\mathrm{T}} M \Phi = I, \tag{28}$$

where Φ^{T} is the matrix transpose of Φ and M is the mass matrix of the system, and the amplitudes of vibration in a normal mode are only relative values which may be scaled or normalized to some extent as a matter of choice.

Using the matrix form, modal parameters be expressed that

$$\Phi^{\mathrm{T}} M \Phi = M_r = \mathrm{diag}(m_1, m_2, \ldots, m_n), \tag{29}$$

$$\Phi^{\mathrm{T}} K \Phi = K_r = \mathrm{diag}(k_1, k_2, \ldots, k_n), \tag{30}$$

$$\Lambda = K_r M_r^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \quad r = 1, 2, \ldots, n, \tag{31}$$

where M_r, K_r, Λ, and Φ are called as the modal mass matrix, modal stiffness matrix, modal frequency that is a diagonal matrix, and modal shape matrix, respectively, in the modal analysis.

For a dynamic system with only a few degrees of freedom, natural frequencies and modal shapes may be determined expanding the determinant and calculating the roots of the resulting characteristic equation. However, for a system with a large number of degrees of freedom, this direct method of solution becomes impractical. It is then necessary to resort to other numerical methods which usually require an iteration process. The FRF can be expressed by the following relationship:

$$X = \sum_{r=1}^{n} \frac{\varphi_r \varphi_r^{\mathrm{T}}}{-\omega^2 m_r + k_r} F, \tag{32}$$

where $X = (X_1, X_2, \ldots, X_n)^{\mathrm{T}}$ is the displacement response vector that occurred in the external force vector $F = (F_1, F_2, \ldots, F_n)^{\mathrm{T}}$, φ_r is the rth-mode shape, m_r is the modal mass, and k_r is the modal stiffness. Any position response of an n-DOF system can be expressed as the superposition of n-number single DOF systems.

If we assume a force F_j at the j-point of structure, then i-point response

$$X_i = \sum_{r=1}^{n} \frac{\varphi_{ir} \varphi_{jr}}{-\omega^2 m_r + k_r} F_j. \tag{33}$$

Thus, the above equation can be rewritten as

$$H_{ij}(\omega) = \frac{X_i}{F_j} = \sum_{r=1}^{n} \frac{\varphi_{ir}\varphi_{jr}}{-\omega^2 m_r + k_r}. \tag{34}$$

This is the FRF between i-point and j-point. Otherwise, this is called frequency domain transfer function. From the reciprocity of linear system, $H_{ij} = H_{ji}$. From the definition of FRF, i-point deformed response X_i for loading F_j at j-point of system is

$$X_i = H_{ij}F_j. \tag{35}$$

If force is $F = (F_1, F_2, \ldots, F_n)^{\mathrm{T}}$, from the linear superposition principle,

$$X_i = H_{i1}F_1 + H_{i2}F_2 + \cdots + H_{in}F_n = (H_{i1}\ H_{i2}\ \cdots\ H_{in})(F_1\ F_2\ \cdots\ F_n)^{\mathrm{T}}. \tag{36}$$

Hence,

$$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1n} \\ H_{21} & H_{22} & \cdots & H_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n1} & H_{n2} & \cdots & H_{nn} \end{pmatrix} \begin{pmatrix} F_2 \\ F_2 \\ \vdots \\ F_2 \end{pmatrix} = HF, \tag{37}$$

in which

$$H = \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1n} \\ H_{21} & H_{22} & \cdots & H_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n1} & H_{n2} & \cdots & H_{nn} \end{pmatrix}. \tag{38}$$

This is called the diagonal FRF matrix, $H_{ij} = H_{ji}$.

The relationship between the FRF matrix and modal parameters is

$$X = \sum_{r=1}^{n} \frac{\varphi_r \varphi_r^{\mathrm{T}}}{-\omega^2 m_r + k_r} F. \tag{39}$$

If we compare Eqs. (37) and (39), then the FRF matrix is as follows:

$$H = \sum_{r=1}^{n} \frac{\varphi_r \varphi_r^{\mathrm{T}}}{-\omega^2 m_r + k_r}, \quad Y_r = \frac{1}{-\omega^2 m_r + k_r}, \tag{40}$$

$$H = \sum_{r=1}^{n} {}_rH = \sum_{r=1}^{n} \frac{1}{-\omega^2 m_r + k_r} \left(\begin{pmatrix} \varphi_{1r} \\ \varphi_{2r} \\ \vdots \\ \varphi_{nr} \end{pmatrix} (\varphi_{1r} \quad \varphi_{2r} \quad \cdots \quad \varphi_{nr}) \right), \tag{41}$$

where ${}_rH$ is rth-mode FRF matrix that contributes the rth-mode to H, and Y_r is called the rth modal conductivity

$$H = \sum_{r=1}^{n} {}_rH = \sum_{r=1}^{n} Y_r \begin{pmatrix} \varphi_{1r}\varphi_{1r} & \varphi_{1r}\varphi_{2r} & \cdots & \varphi_{1r}\varphi_{nr} \\ \varphi_{2r}\varphi_{1r} & \varphi_{2r}\varphi_{2r} & \cdots & \varphi_{2r}\varphi_{nr} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{nr}\varphi_{1r} & \varphi_{nr}\varphi_{2r} & \cdots & \varphi_{nr}\varphi_{nr} \end{pmatrix} = \sum_{r=1}^{n} Y_r \begin{pmatrix} \varphi_{1r}\varphi_r^{\mathrm{T}} \\ \varphi_{2r}\varphi_r^{\mathrm{T}} \\ \vdots \\ \varphi_{nr}\varphi_r^{\mathrm{T}} \end{pmatrix} = \sum_{r=1}^{n} Y_r \begin{pmatrix} \varphi_r\varphi_{1r} \\ \varphi_r\varphi_{2r} \\ \vdots \\ \varphi_r\varphi_{nr} \end{pmatrix}^{\mathrm{T}}. \tag{42}$$

The shape of FRF is drawn by the following method. From the FRF expressive equation,

$$H_{ij} = \sum_{r=1}^{n} \frac{1}{-\omega^2 m_r + k_r} \varphi_{ir}\varphi_{jr} = \sum_{r=1}^{n} Y_r \varphi_{ir}\varphi_{jr} = \sum_{r=1}^{n} {}_rH_{ij}, \tag{43}$$

$${}_rH_{ij} = Y_r \varphi_{ir}\varphi_{jr} = \frac{1}{-\omega^2 m_r + k_r} \varphi_{ir}\varphi_{jr}. \tag{44}$$

Herein, the shape of FRF can be drawn easily as the linear superposition making the transfer function curves of a single DOF system shown in Refs. [1–3].

4. Modal interval analysis method

The generalized interval eigenvalue problem is expressed as follows:

$$Ku = \lambda Mu. \tag{45}$$

In a variety of applications it is often desirable to obtain solutions to the eigenvalue problem in which K and M are affected by uncertainties as subjected to

$$\underline{K} \leqslant K \leqslant \bar{K} \quad \text{or} \quad \underline{k}_{ij} \leqslant k_{ij} \leqslant \bar{k}_{ij}, \quad i,j = 1,2,\ldots,n, \tag{46}$$

$$\underline{M} \leqslant M \leqslant \bar{M} \quad \text{or} \quad \underline{m}_{ij} \leqslant m_{ij} \leqslant \bar{m}_{ij}, \quad i,j = 1,2,\ldots,n, \tag{47}$$

where $K = (k_{ij})$ is stiffness matrix, $M = (m_{ij})$ is the mass matrix, u is the mode shape and λ is the square of the frequency of free vibration. $\underline{K} = (\underline{k}_{ij})$ and $\bar{K} = (\bar{k}_{ij})$ are the minimum and maximum allowable stiffness matrices of system, $\underline{M} = (\underline{m}_{ij})$ and $\bar{M} = (\bar{m}_{ij})$ are the minimum and maximum allowable mass matrices of system. With the use of the interval matrix notation, the above equations can be rewritten as

$$K \in K^I, \quad K^I = [\underline{K}, \bar{K}] = [K^C - \Delta K, K^C + \Delta K], \tag{48}$$

$$M \in M^I, \quad M^I = [\underline{M}, \bar{M}] = [M^C - \Delta M, M^C + \Delta M] \tag{49}$$

in which $K^I = [\underline{K}, \bar{K}]$ is a positive-semidefinite symmetric interval matrix and $M^I = [\underline{M}, \bar{M}]$ is a positive-definite symmetric interval matrix. Then K^C and M^C are centered interval stiffness and mass, ΔK and ΔM are respectively radius interval stiffness and mass. They are given by

$$K^C = \frac{\underline{K} + \bar{K}}{2}, \quad M^C = \frac{\underline{M} + \bar{M}}{2}, \quad \Delta K = \frac{\bar{K} - \underline{K}}{2}, \quad \Delta M = \frac{\bar{M} - \underline{M}}{2}.$$

For the sake of simplicity, they can be expressed by

$$K^I u = \lambda M^I u. \tag{50}$$

The above equation is called a generalized interval eigenvalue problem [15], and this analysis procedure was introduced by Qiu et al. [13]. Here, we consider modal interval analysis, based on this conclusion. First, the centered eigenvalue problem for the ith eigenvector φ_i associated with the eigenvalue λ_i having interval parameter K^C and M^C, is considered:

$$K^C \varphi_i = \lambda_i M^C \varphi_i. \tag{51}$$

The centered eigenvalues λ_i^C that denotes the natural frequencies and the evaluative centered eigenvectors φ_i^C that denotes the normal modes can now be obtained. Then, to compute the lower and the upper bounds on a particular λ_i, we can introduce Deif's assumption [11], i.e. signs of the components of the associated eigenvector φ_i^C remain unchanged, when matrices K and M range over the interval $K^I = [\underline{K}, \bar{K}]$ and $M^I = [\underline{M}, \bar{M}]$. Then we define

$$S^i = \mathrm{diag}(\mathrm{sgn}(\varphi_{1i}^C), \mathrm{sgn}(\varphi_{2i}^C), \ldots, \mathrm{sgn}(\varphi_{ni}^C)), \quad \varphi_{ji}^C \neq 0, \quad i,j = 1,2,\ldots,n, \tag{52}$$

where S^i is a diagonal sign matrix expressed by the sign of row elements $(\varphi_{1i}^C, \varphi_{2i}^C, \ldots, \varphi_{ni}^C)^{\mathrm{T}}$ of the ith centered eigenvector ${\varphi_i}^C$. In the sign matrix S^i of the eigenvectors within the interval of the eigenvalues, the bounds found for the eigenvalues are exact. Using the sign matrix, the lower bound $\underline{\lambda}_i$ and the upper bound $\bar{\lambda}_i$ satisfy, respectively (see Qiu [18]):

$$(K^C - S^i\,\Delta K S^i)\underline{\underline{\varphi}}_i = \underline{\lambda}_i (M^C + S^i\,\Delta M S^i)\underline{\underline{\varphi}}_i \tag{53}$$

and

$$(K^C + S^i\,\Delta K S^i)\overline{\overline{\varphi}}_i = \bar{\lambda}_i (M^C - S^i\,\Delta M S^i)\overline{\overline{\varphi}}_i, \tag{54}$$

where $\overline{\overline{\varphi}}_i$ and $\underline{\underline{\varphi}}_i$ are the eigenvectors corresponding to the upper-bound eigenvalue and lower-bound eigenvalue, respectively, not the "real" upper- and lower bounds of eigenvectors, which will be given in Section 5. In the general case of a system with relatively narrow system interval matrices $K^I = [\underline{K}, \bar{K}]$ and $M^I = [\underline{M}, \bar{M}]$, the interval eigenvalue problem is equivalent to the optimization problems to find the minimum or maximum λ.

By means of the above conclusion and the orthogonality demands of the modal parameters such as modal mass (or generalized mass) and modal stiffness (or generalized stiffness), modal interval parameters can be written as

$$k_r^I = [\underline{k}_r, \bar{k}_r] = [\underline{\underline{\varphi}}_r^{\mathrm{T}}\,\underline{K}\,\underline{\underline{\varphi}}_r,\ \overline{\overline{\varphi}}_r^{\mathrm{T}}\bar{K}\overline{\overline{\varphi}}_r], \tag{55}$$

$$m_r^I = [\underline{m}_r, \bar{m}_r] = [\overline{\overline{\varphi}}_r^{\mathrm{T}}\,\underline{M}\,\overline{\overline{\varphi}}_r,\ \underline{\underline{\varphi}}_r^{\mathrm{T}}\bar{M}\underline{\underline{\varphi}}_r], \tag{56}$$

in which

$$\underline{k}_r = \underline{\underline{\varphi}}_r^{\mathrm{T}}\,\underline{K}\,\underline{\underline{\varphi}}_r = \underline{\underline{\varphi}}_r^{\mathrm{T}}(K^C - S^i\,\Delta K S^i)\underline{\underline{\varphi}}_r, \tag{57}$$

$$\bar{k}_r = \overline{\overline{\varphi}}_r^{\mathrm{T}}\bar{K}\overline{\overline{\varphi}}_r = \overline{\overline{\varphi}}_r^{\mathrm{T}}(K^C + S^i\,\Delta K S^i)\overline{\overline{\varphi}}_r, \tag{58}$$

$$\underline{m}_r = \overline{\overline{\varphi}}_r^{\mathrm{T}}\,\underline{M}\,\overline{\overline{\varphi}}_r = \overline{\overline{\varphi}}_r^{\mathrm{T}}(M^C - S^i\,\Delta M S^i)\overline{\overline{\varphi}}_r, \tag{59}$$

$$\bar{m}_r = \underline{\underline{\varphi}}_r^{\mathrm{T}}\bar{M}\underline{\underline{\varphi}}_r = \underline{\underline{\varphi}}_r^{\mathrm{T}}(M^C + S^i\,\Delta M S^i)\underline{\underline{\varphi}}_r. \tag{60}$$

From the relationship between the above modal parameters and modal frequencies, modal interval frequency can be expressed as

$$\lambda_r^I = [\underline{\lambda}_r, \bar{\lambda}_r] = \left[\frac{\underline{k}_r}{\bar{m}_r}, \frac{\bar{k}_r}{\underline{m}_r}\right] = \left[\frac{\underline{\underline{\varphi}}_r^{\mathrm{T}}\,\underline{K}\,\underline{\underline{\varphi}}_r}{\underline{\underline{\varphi}}_r^{\mathrm{T}}\bar{M}\underline{\underline{\varphi}}_r}, \frac{\overline{\overline{\varphi}}_r^{\mathrm{T}}\bar{K}\overline{\overline{\varphi}}_r}{\overline{\overline{\varphi}}_r^{\mathrm{T}}\,\underline{M}\,\overline{\overline{\varphi}}_r}\right], \tag{61}$$

where $\underline{\underline{\varphi}}_r$ is the associated eigenvector with $\underline{\lambda}_r$ and $\overline{\overline{\varphi}}_r$ is the associated eigenvector with $\bar{\lambda}_r$.

Otherwise, in order to draw the frequency domain response formally, the interval FRF H^I that is associated to the modal interval frequencies and shapes can be denoted:

$$H^I = [\bar{H},\ \underline{H}] = \left[\sum_{r=1}^{n} {}_r\underline{H}_{ij}, \sum_{r=1}^{n} {}_r\bar{H}_{ij}\right], \quad i,j = 1,2,\ldots,n \tag{62}$$

in which

$${}_r\bar{H}_{ij} = \frac{1}{-\omega^2\underline{m}_r + \bar{k}_r}\bar{\varphi}_{ir}\bar{\varphi}_{jr}, \quad {}_r\underline{H}_{ij} = \frac{1}{-\omega^2\bar{m}_r + \underline{k}_r}\underline{\varphi}_{ir}\underline{\varphi}_{jr}, \tag{63}$$

where $\bar{\varphi}_r$ and $\underline{\varphi}_r$ are the real upper and lower bounds of mode shape given in Section 5.

Applying to normalized condition, modal interval mass and stiffness can be written as

$$\underline{k}_r = \underline{\underline{\varphi}}_r^{\mathrm{T}}\,\underline{K}\,\underline{\underline{\varphi}}_r = \underline{\lambda}_r, \quad \bar{k}_r = \overline{\overline{\varphi}}_r^{\mathrm{T}}\bar{K}\overline{\overline{\varphi}}_r = \bar{\lambda}_r \tag{64}$$

and

$$\underline{m}_r = \overline{\overline{\varphi}}_r^{\mathrm{T}} \underline{M}\, \overline{\overline{\varphi}}_r = 1, \quad \bar{m}_r = \underline{\underline{\varphi}}_r^{\mathrm{T}} \bar{M} \underline{\underline{\varphi}}_r = 1. \tag{65}$$

Thus we can see the interval FRF to satisfy

$$_r\underline{H}_{ij} = \frac{1}{-\omega^2 + \underline{\lambda}_r} \underline{\varphi}_{ir} \underline{\varphi}_{jr}, \quad _r\bar{H}_{ij} = \frac{1}{-\omega^2 + \bar{\lambda}_r} \bar{\varphi}_{ir} \bar{\varphi}_{jr}, \tag{66}$$

where interval FRF can be written by means of associated interval parameters.

As mentioned above, the bounded FRF can reflect not only the range of modal interval parameters for drawing frequency domain response, but also the width of dynamic response for modal dynamic response analysis of a large structure with uncertain-but-bounded parameters.

5. Interval mode shapes

In this section, the ‘‘real’’ upper and lower bounds for mode shapes interval will be given based on the first-order Taylor’ series expansion.

When physical properties and geometric variables of structures are taken as structural parameters, the stiffness matrix and the mass matrix are functions of the structural parameters:

$$K = K(b), \quad M = M(b), \tag{67}$$

where $b = (b_1, b_2, \ldots, b_m)^{\mathrm{T}}$ is the structural parameter vector. Thus, eigenvalue problem (45) can be written in the form

$$K(b)\varphi = \lambda M(b)\varphi \tag{68}$$

and

$$\varphi^{\mathrm{T}} M(b)\varphi = 1. \tag{69}$$

Consider eigenvalue problem (68) subject to the following structural parameter constraint condition:

$$\underline{b} \leqslant b \leqslant \overline{b} \quad \text{or} \quad \underline{b}_i \leqslant b_i \leqslant \overline{b}_i, \quad i = 1, 2, \ldots, m, \tag{70}$$

where $\overline{b} = (\overline{b}_i)$ and $\underline{b} = (\underline{b}_i)$ are respectively the upper- and lower-bound vectors of the structural parameter vector b.

In terms of the interval matrix notation in interval mathematics or interval analysis, the inequality condition (70) can be written as

$$b \in b^I \quad \text{or} \quad b_i \in b_i^I, \quad i = 1, 2, \ldots, m, \tag{71}$$

where

$$b^I = (b_i^I), \quad b_i^I = [\underline{b}_i, \overline{b}_i], \quad i = 1, 2, \ldots, m, \tag{72}$$

in which b^I is the interval vector and b_i^I, $i = 1, 2, \ldots, m$, is the component of the interval vector b^I. b^I is called the interval structural parameter.

Consider the eigenvector φ_i which is dependent on m structural parameters b_i, $i = 1, 2, \ldots, m$. This function is defined by

$$\varphi_i = \varphi_i(b_1, b_2, \ldots, b_m). \tag{73}$$

Let b_{ic}, $i = 1, 2, \ldots, m$, be a nominal value of the structural parameters. By Taylor’s series expansion, the eigenvector for the structural parameters $b_i = b_{ic} + \delta b_i$, $i = 1, 2, \ldots, m$, to the first order in $\delta b_i = b_i - b_{ic}$, $i = 1, 2, \ldots, m$, is given by

$$\varphi_i(b_1, b_2, \ldots, b_m) = \varphi_i(b_{1c}, b_{2c}, \ldots, b_{mc}) + \sum_{j=1}^{m} \frac{\partial \varphi_i(b_{1c}, b_{2c}, \ldots, b_{mc})}{\partial b_j}(b_j - b_{jc}). \tag{74}$$

It is understood that in the summation, the partial derivative of the eigenvector $\varphi_i(b_1, b_2, \ldots, b_m)$ is taken and then evaluated at the nominal value b_{ic}, $i = 1, 2, \ldots, m$, which can be obtained by the first-order perturbation theory [22] as follows:

$$\frac{\partial \varphi_i(b_c)}{\partial b_j} = \sum_{s=1}^{n} c_{is} \varphi_i(b_c), \tag{75}$$

where

$$c_{is} = \frac{1}{\lambda_{ic} - \lambda_{sc}} \left\{ \varphi_{sc}^{\mathrm{T}} \frac{\partial K(b_c)}{\partial b_j} \varphi_{ic} - \lambda_{ic} \varphi_{sc}^{\mathrm{T}} \frac{\partial M(b_c)}{\partial b_j} \varphi_{ic} \right\} \quad (s \neq i) \tag{76}$$

and

$$c_{ii} = -\frac{1}{2} \varphi_i^{\mathrm{T}} \frac{\partial M(b_c)}{\partial b_j} \varphi_i \quad (s = i). \tag{77}$$

For convenience of notation, let us define

$$g^{\mathrm{T}} = \left(\frac{\partial \varphi_i(b_c)}{\partial b_1}, \frac{\partial \varphi_i(b_c)}{\partial b_2}, \ldots, \frac{\partial \varphi_i(b_c)}{\partial b_m} \right) = \left(\frac{\partial \varphi_{ic}}{\partial b_1}, \frac{\partial \varphi_{ic}}{\partial b_2}, \ldots, \frac{\partial \varphi_{ic}}{\partial b_m} \right), \tag{78}$$

where the superscript T means vector transposition. Then Eq. (74) can be rewritten as

$$\varphi_i(b) = \varphi_i(b_c + \delta b) = \varphi_{ic} + g^{\mathrm{T}} \delta b, \tag{79}$$

where

$$\varphi_{ic} = \varphi_i(b_c) = \varphi_i(b_{1c}, b_{2c}, \ldots, b_{mc}), \quad \delta b^{\mathrm{T}} = (\delta b_1, \delta b_2, \ldots, \delta b_m). \tag{80}$$

The deviation from the nominal value $\delta b = (\delta b_i)$ of the structural parameter $b = (b_i)$ is assumed to vary in the following rectangular set:

$$\Delta b^I = [-\Delta b, \Delta b] = \{\delta b : -\Delta b \leqslant \delta b \leqslant \Delta b\}, \tag{81}$$

where $\Delta b^I = (\Delta b_i^I)$ is the interval parameter vector and can be expressed in a component form, i.e.

$$\Delta b_i^I = [-\Delta b_i, \Delta b_i] = \{\delta b_i : -\Delta b_i \leqslant \delta b_i \leqslant \Delta b_i\}, \quad i = 1, 2, \ldots, m, \tag{82}$$

where $\Delta b = (\Delta b_j)$ is a nonnegative constant vector. The objective now is to find the lower-bound eigenvector and the upper-bound eigenvector of the eigenvector for all possible structural parameters belonging to the set Δb^I by interval analysis method. Then the problem of finding upper or maximum and lower or minimum eigenvectors becomes the following extreme value problem:

$$\varphi_{\mathrm{iext}} = \underset{\delta b \in \Delta b^I = [-\Delta b, \Delta b]}{\mathrm{extremum}} \{\varphi_{ic} + g^{\mathrm{T}} \delta b\}. \tag{83}$$

Thus, we arrive at a way of determining upper and lower bounds of eigenvectors which can be applied to an ensemble of structures with uncertain-but-bounded Fourier coefficients.

From Eq. (83), since φ_i is a function of uncertain variable δb, by means of the natural interval extension, we have

$$\varphi_i^I = [\underline{\varphi}_i, \overline{\varphi}_i] = \varphi_{ic} + g^{\mathrm{T}} \Delta b^I. \tag{84}$$

By interval operations, Eq. (84) can be rewritten as follows:

$$\varphi_i^I = [\underline{\varphi}_i, \overline{\varphi}_i] = \varphi_{ic} + [-|g|^{\mathrm{T}} \Delta b, |g|^{\mathrm{T}} \Delta b] = [\varphi_{ic} - |g|^{\mathrm{T}} \Delta b, \varphi_{ic} + |g|^{\mathrm{T}} \Delta b]. \tag{85}$$

Using the necessary and sufficient condition of two intervals equality, from Eq. (85), we obtain

$$\underline{\varphi}_i = \varphi_{ic} - |g|^{\mathrm{T}} \Delta b \tag{86}$$

and

$$\overline{\varphi}_i = \varphi_{ic} + |g|^{\mathrm{T}} \Delta b. \tag{87}$$

Substitution of Eq. (78) into Eqs. (86) and (87), yields

$$\underline{\varphi}_i = \varphi_{ic} - \sum_{j=1}^{m} \left| \frac{\partial \varphi_{ic}}{\partial b_j} \right| \Delta b_j \tag{88}$$

and

$$\bar{\varphi}_i = \varphi_{ic} + \sum_{j=1}^{m} \left| \frac{\partial \varphi_{ic}}{\partial b_j} \right| \Delta b_j. \tag{89}$$

So from Eqs. (88) and (89), we can obtain the lower- and upper bounds of eigenvector.

6. Numerical example

By modal interval analysis method, we illustrate how to determine the range between lower and upper natural frequencies and mode shapes of a five-story frame structure with uncertain-but-bounded parameters shown in Fig. 1(a). Assume that the horizontal members are very rigid compared to the columns of the frame. This assumption reduces the system to only five degrees of freedom shown in Fig. 1(b), indicated by coordinates $X_1, X_2, \ldots, X_n$ in the figure, and this structure may be modeled by the five mass system. The mass of the structure, which is lumped at the floor levels, has interval values such as $M_1^I = [29, 31]$, $M_2^I = [26, 28]$, $M_3^I = [26, 28]$, $M_4^I = [24, 26]$, and $M_5^I = [17, 19]$. The total stiffness of each story has also interval values such as $K_1^I = [2000, 2020]$, $K_2^I = [1800, 1850]$, $K_3^I = [1600, 1630]$, $K_4^I = [1400, 1420]$, and $K_5^I = [1200, 1210]$, as indicated in Fig. 1. The interval physical parameters of the system are listed in Table 1.

To apply the modal analysis method, we assume a deformed shape of φ_1, φ_2, φ_3, φ_4, φ_5 given in Fig. 2.

In order to apply the present method for generalized interval eigenvalues, interval stiffness matrix and interval mass matrix of the system can be written as K^I and M^I, respectively. From interval calculus definition,

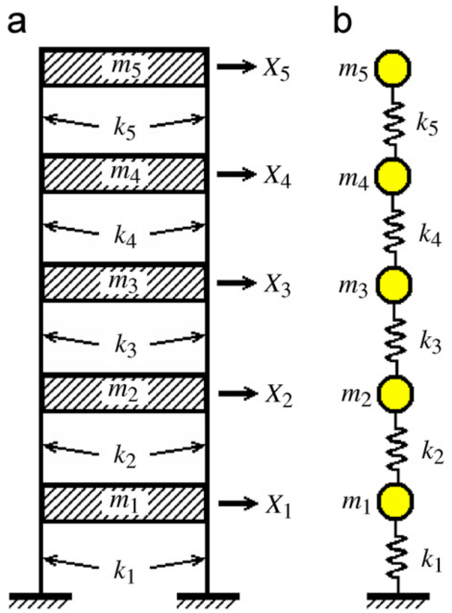

Fig. 1. (a) Five-story frame tower structure and (b) 5-DOF modeled system.

Table 1
Interval physical parameters of the system

Interval for each stiffness parameters (unit: N/m)			Interval for each mass parameter (unit: kg)		
Bounded value	Center	Radius	Bounded value	Center	Radius
$K_1^I = [2000, 2020]$	$K_1^C = 2010$	$\Delta K_1 = 10$	$M_1^I = [29, 31]$	$M_1^C = 30$	$\Delta M_1 = 1$
$K_2^I = [1800, 1850]$	$K_2^C = 1825$	$\Delta K_2 = 25$	$M_2^I = [26, 28]$	$M_2^C = 27$	$\Delta M_2 = 1$
$K_3^I = [1600, 1630]$	$K_3^C = 1615$	$\Delta K_3 = 15$	$M_3^I = [26, 28]$	$M_3^C = 27$	$\Delta M_3 = 1$
$K_4^I = [1400, 1420]$	$K_4^C = 1410$	$\Delta K_4 = 10$	$M_4^I = [24, 26]$	$M_4^C = 25$	$\Delta M_4 = 1$
$K_5^I = [1200, 1210]$	$K_5^C = 1205$	$\Delta K_5 = 5$	$M_5^I = [17, 19]$	$M_5^C = 18$	$\Delta M_5 = 1$

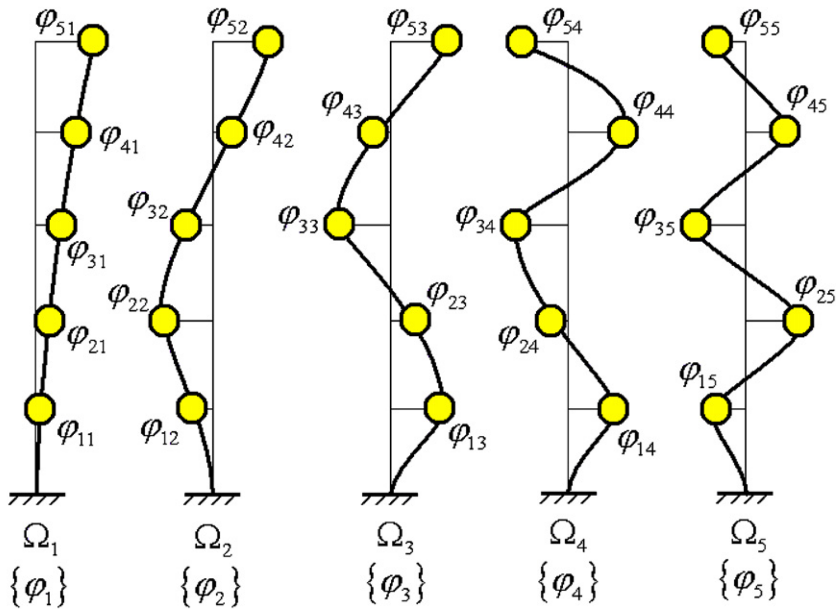

Fig. 2. Modal shapes of natural frequency of 5-DOF system.

we can obtain centered stiffness matrix K^C, derivative radius stiffness matrix ΔK, centered mass matrix M^C, and derivative radius mass matrix ΔM

$$K^I = \begin{pmatrix} [3800, 3870] & -[1800, 1850] & 0 & 0 & 0 \\ -[1800, 1850] & [3400, 3480] & -[1600, 1630] & 0 & 0 \\ 0 & -[1600, 1630] & [3000, 3050] & -[1400, 1420] & 0 \\ 0 & 0 & -[1400, 1420] & [2600, 2630] & -[1200, 1210] \\ 0 & 0 & 0 & -[1200, 1210] & [1200, 1210] \end{pmatrix},$$

$$K^C = \begin{pmatrix} 3835 & -1825 & 0 & 0 & 0 \\ -1825 & 3440 & -1615 & 0 & 0 \\ 0 & -1615 & 3025 & -1410 & 0 \\ 0 & 0 & -1410 & 2615 & -1205 \\ 0 & 0 & 0 & -1205 & 1205 \end{pmatrix}, \quad \Delta K = \begin{pmatrix} 35 & -25 & 0 & 0 & 0 \\ -25 & 40 & -15 & 0 & 0 \\ 0 & -15 & 25 & -10 & 0 \\ 0 & 0 & -10 & 15 & -5 \\ 0 & 0 & 0 & -5 & 5 \end{pmatrix},$$

$$M^I = \begin{pmatrix} [29,31] & 0 & 0 & 0 & 0 \\ 0 & [26,28] & 0 & 0 & 0 \\ 0 & 0 & [26,28] & 0 & 0 \\ 0 & 0 & 0 & [24,26] & 0 \\ 0 & 0 & 0 & 0 & [17,19] \end{pmatrix},$$

$$M^C = \mathrm{diag}(30 \quad 27 \quad 27 \quad 25 \quad 18),$$

$$\Delta M = \mathrm{diag}(1 \quad 1 \quad 1 \quad 1 \quad 1).$$

Hence, the eigensolutions for the system with centered parameters K^C and M^C can be solved. The results are listed in Table 2. the centered deformations for modal shapes are plotted by eigenvectors. they are denoted using solid line (black color) as a deformative modal shape for a centered eigenvalue in Figs. 3–7.

Then, the modal interval parameters for the structure with uncertain-but-bounded parameters are calculated by using the modal interval analysis. The ranges between lower and upper bounds of natural frequencies and deformative shapes are obtained by applying its results. Bounded eigensolutions represented with the range between lower and upper bounds of natural frequencies and interval modal shapes are listed in Table 3. In Figs. 3–7 dashed line (blue color) as a deformative modal shape corresponds to the upper-bound eigenvalue and dotted line (red color) as a deformative modal shape corresponds to lower-bound eigenvalue.

Table 2
Centered eigensolutions for system with interval parameters

Centered values	Notate	1	2	3	4	5
Eigenvalues	λ^C	6.1662	44.078	103.57	165.59	219.42
Natural frequency	ω^C	2.4832	6.6391	10.177	12.868	14.813
Eigenvector or Normal mode	Φ^C	φ_1^C	φ_2^C	φ_3^C	φ_4^C	φ_5^C
		0.03181	−0.08099	0.10432	0.08900	−0.08342
		0.06362	−0.11150	0.04161	−0.05524	0.12560
		0.09301	−0.06382	−0.10130	−0.06531	−0.09893
		0.11568	0.04466	−0.06409	0.13025	0.05658
		0.12742	0.13075	0.11716	−0.08839	−0.02616

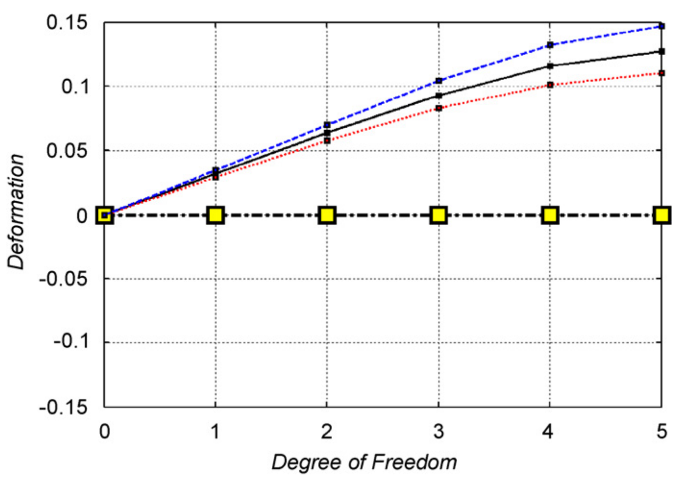

Fig. 3. Deformative first-mode shapes for bounded modal frequencies.

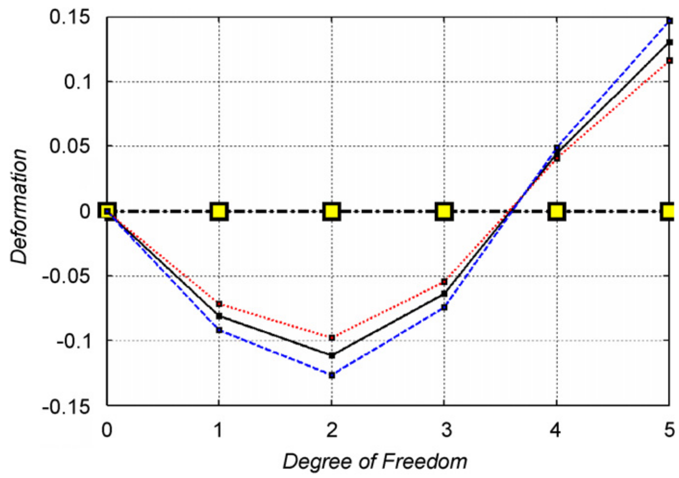

Fig. 4. Deformative second-mode shapes for bounded modal frequencies.

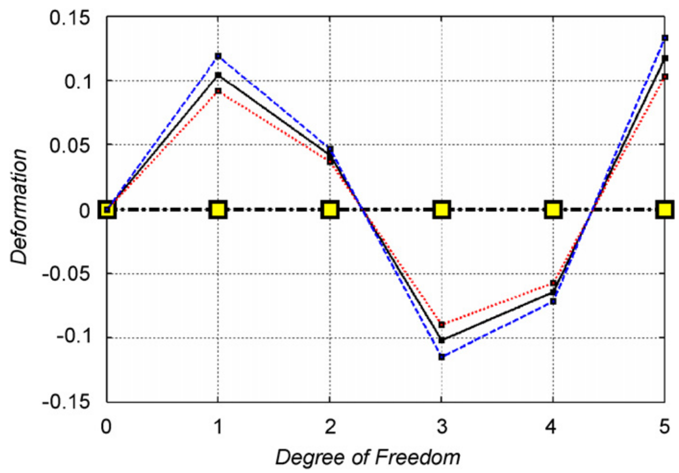

Fig. 5. Deformative third-mode shapes for bounded modal frequencies.

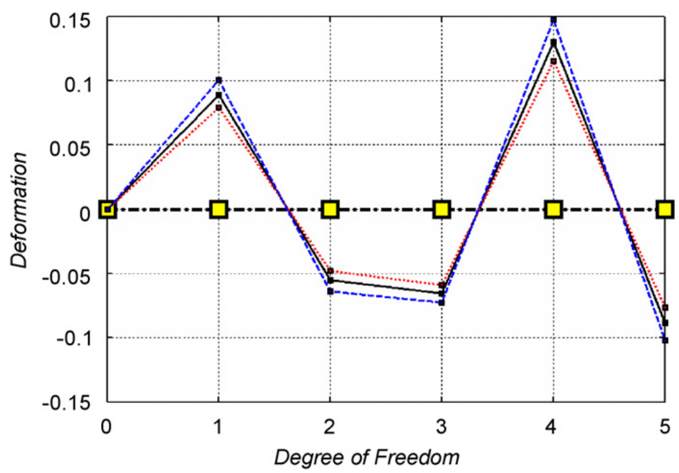

Fig. 6. Deformative fourth-mode shapes for bounded modal frequencies.

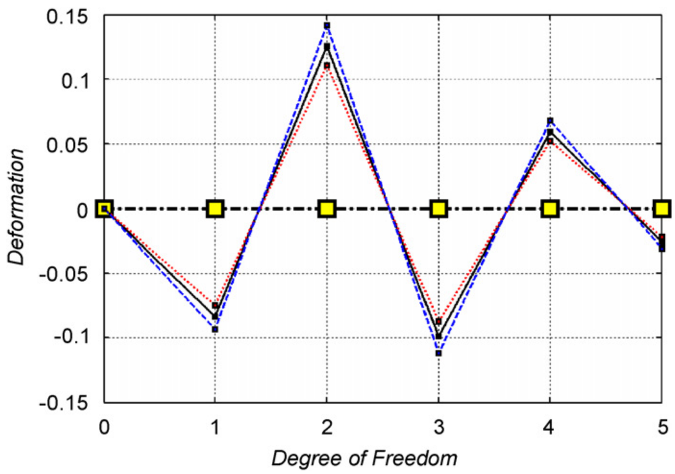

Fig. 7. Deformative fifth-mode shapes for bounded modal frequencies.

Table 3
Bounded eigensolutions between lower and upper bounds

	Bound	Notate	1	2	3	4	5
Eigenvalues	Lower	$\underline{\lambda}$	4.6166	40.754	98.572	158.86	211.51
	Upper	$\bar{\lambda}$	7.8303	47.702	108.99	172.89	227.95
Natural frequency	Lower	$\underline{\omega}$	2.1486	6.3839	9.9284	12.604	14.543
	Upper	$\bar{\omega}$	2.7983	6.9067	10.44	13.149	15.098
Eigenvector	Lower	$\underline{\Phi}$	$\underline{\varphi}_1$	$\underline{\varphi}_2$	$\underline{\varphi}_3$	$\underline{\varphi}_4$	$\underline{\varphi}_5$
			0.0311	−0.0858	0.0914	0.0740	−0.0982
			0.0621	−0.1181	0.0365	−0.0645	0.1034
			0.0908	−0.0676	−0.1138	−0.0763	−0.1164
			0.1129	0.0420	−0.0720	0.1083	0.0490
			0.1244	0.1230	0.1027	−0.1033	−0.0308
	Upper	$\bar{\Phi}$	$\bar{\varphi}_1$	$\bar{\varphi}_2$	$\bar{\varphi}_3$	$\bar{\varphi}_4$	$\bar{\varphi}_5$
			0.0326	−0.0762	0.1172	0.1040	−0.0687
			0.0651	−0.1049	0.0468	−0.0459	0.1478
			0.0952	−0.0600	−0.0888	−0.0543	−0.0814
			0.1184	0.0473	−0.0562	0.1522	0.0701
			0.1305	0.1385	0.1317	−0.0735	−0.0215

In order to evaluate the accuracy and effective bounded results of modal interval parameters, Monte Carlo simulation method is used to validate the results obtained from the presented modal interval analysis, and to observe the Monte Carlo solutions within bounded results, iterations of Monte Carlo simulation are used (10,000) in calculation, but only 100 iterations are used in the plots. Herein interval mass and stiffness are modeled as uniform distribution.Comparison of results of our analysis with those of Monte Carlo simulation is given in Fig. 8.

The ranges between lower- and upper-bound natural frequencies are represented by the FRF in Fig. 9.

7. Conclusions

The method presented in this paper provides a simple analytical tool for finding bounds of natural frequencies and normal modes for structures with uncertain-but-bounded parameters, by using interval

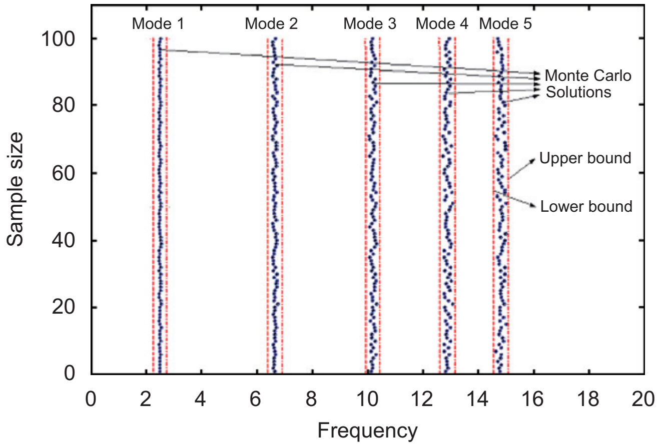

Fig. 8. Evaluation of bounded results compared with the modal interval analysis with Monte Carlo method.

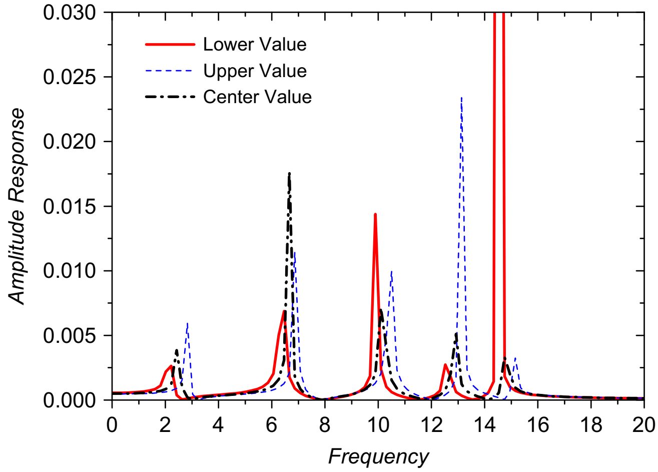

Fig. 9. Bounded natural frequency (rad/s) of the system.

analysis basis on modal analysis. The bounds of modal parameters obtained by the interval parameters are verified by the range of natural frequencies and the deformative shape of normal modes of uncertain structure and it is very important to estimate the stability or reliability of the structures. The numerical example is given to illustrate the effectiveness and correctness of the presented modal interval analysis method, and its results agree well with Monte Carlo simulation results. By modal analysis via interval calculus, the modal interval analysis method for structural dynamical response can determine the width between upper and lower bounds on structural dynamic response.

The modal interval analysis can be used not only in modal dynamic response analysis, but also in other areas of vibration analysis and diagnosis. For example, we can obtain the needed maximum and minimum values in dynamic response analysis, estimated the stability range that avoids the resonance of structures, and consider how to change the structural parameters in structural modification.

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References

[1] W. Heylen, S. lammens, P. Sas, *Modal Analysis Theory and Testing*, Katholieke Universiteit Levven, 1997.
[2] K.G. McConnell, *Vibration Testing Theory and Practice*, Wiley, New York, 1995.
[3] Jerry H. Ginsberg, *Mechanical and Structural Vibrations*, Wiley, New York, 2001.
[4] R.E. Moore, *Methods and Applications of Interval Analysis*, Prentice-Hall, Inc., London, 1979.
[5] G. Alefeld, J. Herzberger, *Introductions to Interval Computations*, Academic Press, New York, 1983.
[6] K. Nichel, *Interval Mathematics*, Academic Press, New York, 1980.
[7] I. Elishakoff, *Probabilistic Methods in the Theory of Structures*, Wiley, New York, 1983.
[8] S.H. Chen, Z.S. Liu, Z.F. Zhang, Random vibration analysis for large-scale structures with random parameters, *Computers & Structures* 43 (4) (1992) 681–685.
[9] M. Kleiber, H.D. Hien, *The Stochastic Finite Element Method: Basic Perturbation Technique and Computer Implement*, Wiley, New York, 1992.
[10] A. Deif, *Advanced Matrix Theory for Scientists and Engineers*, second ed., Abacus Press, Tunbridge Wells, UK, 1991.
[11] A.D. Dimarogonas, Interval analysis of vibrating systems, *Journal of Sound and Vibration* 183 (4) (1995) 739–749.
[12] S.H. Chen, H.D. Lian, X.W. Yang, Interval eigenvalue analysis for structures with interval parameters, *Finite Elements in Analysis and Design* 39 (2003) 419–431.
[13] S.H. Chen, Z.P. Qiu, D.T. Song, New method for computing the upper and lower bounds on frequencies of structures with interval parameters, *Mechanics Research Communications* 21 (6) (1994) 583–592.
[14] S.H. Chen, Z.P. Qiu, Z.S. Liu, A method for computing eigenvalue bounds in structural vibration systems with interval parameters, *Computers & Structures* 51 (3) (1994) 309–313.
[15] Z.P. Qiu, S.H. Chen, H.B. Jia, The Rayleigh quotient iteration method for computing eigenvalues bounds of structures with bounded uncertain parameters, *Computers & Structures* 55 (2) (1995) 221–227.
[16] Z.P. Qiu, Y.X. Gu, Interval parameter perturbation method for evaluation the bounds on natural frequencies of structures with interval parameters, *Acta Mechanics Solid Sinica* 11 (1) (1998) 56–62.
[17] Z.P. Qiu, X.J. Wang, Comparison of dynamic response of structures with uncertain-but-bounded parameters using non-probabilistic interval analysis method and probabilistic approach, *International Journal of Solids and Structures* 40 (20) (2003) 5423–5439.
[18] Z.P. Qiu, Interval Analysis for Static Response and Eigenvalue Problem of Structures with Uncertain Parameters. PhD Thesis, JiLin University of Technology, 1994 (in Chinese).
[19] Y. Ben-Haim, I. Elishakoff, *Convex Models of Uncertainty in Applied Mechanics*, Elsevier Science Publishers, Amsterdam, 1990.
[20] I. Elishakoff, P. Eliseeff, S. Glegg, Convex modeling of material uncertainty in vibrations of a viscoelastic structure, *AIAA Journal* 32 (1994) 843–849.
[21] Y. Ben-Haim, G. Chen, T.T. Soong, Maximum structural response using convex models, *ASCE Journal of Engineering Mechanics* 122 (4) (1996) 325–333.
[22] S.H. Chen, *Matrix Perturbation Theory in Structural Dynamics*, International Academic Publishers, Beijing, 1993.