Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 303 (2007) 343–356
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Geometrical design method of multi-degree-of-freedom dynamic vibration absorbers

Seon J. Jang, Yong J. Choi*

School of Mechanical Engineering, Yonsei University, 134 Shinchon-dong, Seodaemun-gu, Seoul 120-749, Republic of Korea

Received 23 May 2006; received in revised form 25 October 2006; accepted 19 January 2007
Available online 21 March 2007

Abstract

This paper presents a new geometrical design method of multi-degree-of-freedom (mdof) dynamic vibration absorbers that reduce multiple modes of vibration. The design of an mdof vibration absorber involves, in general, the complexity of the equations and large numbers of design variables. For this reason, previous researches mainly focused on finding optimized stiffness and damping values that minimize the vibration responses. In this paper, we introduce a simple geometrical design method in which the sets of three mutually orthogonal line springs are used to first simplify the stiffness matrix. The dynamic equations of a main body and an absorber are then decoupled to obtain the geometric design rules for an mdof absorber. Numerical examples are used to illustrate the new design method.

1. Introduction

A typical dynamic vibration absorber has a single degree-of-freedom (dof). It has a tuned spring and mass arranged in the direction of the excitation force. However, many practical vibration systems are regarded as mdof vibration systems that have multiple vibration modes and resonant frequencies. In order to deal with multiple vibration modes, some research works have been focused on the use of a single vibration absorber or a set of single dof vibration absorbers. Vakakis and Paipetis [1] investigated the effect of a single dof vibration absorber mounted on an mdof system. Sadek [2] estimated the optimal stiffness and damping constants of multiple single dof vibration absorbers attached to an mdof system. A single rigid body suspended by springs has 6dof in space. This implies that a single rigid body can be used as a 6dof vibration absorber. In light of this nature, Zuo and Nayfeh [3] showed a single body vibration absorber, which can diminish multiple vibration modes. They regarded this as an optimization problem solving for optimal spring and damping constants set up in the estimated positions and directions.

The design of an mdof vibration absorber using a single body can be greatly simplified when an adequate geometrical approach to the design is taken. To make use of every dof of the vibration absorber, the dynamic equation of the vibration absorber is completely decoupled. By describing stiffness matrix and mass matrix via screw theory [4], one can easily obtain decoupling conditions of the dynamic equation. From the decoupled

*Corresponding author. +82 2 2123 2826; fax: +82 2 362 2736.
E-mail address: yjchoi@yonsei.ac.kr (Y.J. Choi).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2007.01.017

Nomenclature

a_i, b_i, c_i	Cartesian coordinates of $\mathbf{p}_i$
c_d	damping constant
$\bar{c}_r$	modal damping of rth mode
C	mass center of rigid body
$\bar{\mathbf{C}}$	proportional damping matrix
$\mathbf{d}$	position vector from mass center of body 2 (absorber) to mass center of body 1
$\mathbf{E}$	parallel axis congruent transformation matrix
I_{xx}, I_{yy}, I_{zz}	mass moment of inertias with respect to x-, y-, z-axis
k	spring constant
$\bar{k}_r$	modal stiffness of rth mode
K_i	ith diagonal element of diagonalized stiffness matrix
$\mathbf{K}$	stiffness matrix
$\bar{\mathbf{K}}$	global stiffness matrix
m	mass
$\bar{m}_r$	modal mass of rth mode
$\mathbf{M}$	mass matrix
$\bar{\mathbf{M}}$	global mass matrix
$\mathbf{p}_i$	position vector from origin to the coincident point of ith orthogonal three-spring set
$\hat{\mathbf{w}}$	externally applied wrench
$\hat{\mathbf{X}}$	small twist
β	proportional percent damping ratio
δ	translational displacement
ϕ	rotational displacement
$\mathbf{\Phi}_j^i$	jth mode vector of ith body
$\mathbf{\Psi}$	modal matrix
ω	excitation frequency
Ω	natural frequency
Ω_i	ith suppression frequency

Subscripts

1	rigid body 1
2	rigid body 2 (absorber)
C_1	mass center of rigid body 1
C_2	mass center of rigid body 2

dynamic equation, six stiffness and mass ratios for each direction can be determined. They can be used as six suppression frequencies of an mdof vibration absorber. When the dynamic equation of a rigid body is completely decoupled and it is used as an mdof vibration absorber, it reduces multiple modes under some working conditions.

In summary, this paper presents a new geometric design method of mdof vibration absorbers in which the geometrical conditions that decouple the dynamic equation of an mdof vibration absorber are derived and thereby the new design rules for an mdof vibration absorber are suggested. Numerical examples are presented for both spatial and planar design cases.

2. Preliminary

2.1. Stiffness matrix

Consider a rigid body which is supported by n line springs acting only along their axial directions. When a small wrench $\hat{\mathbf{w}}$ is externally applied on the rigid body, the relation between the wrench and the small twist $\hat{\mathbf{X}}$ of a body can be expressed by

$$\hat{\mathbf{w}} = \mathbf{K}\hat{\mathbf{X}}, \tag{1}$$

where $\hat{\mathbf{w}} = \begin{bmatrix} \mathbf{f}^{\mathrm{T}} & \mathbf{m}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ are the Plücker's ray coordinates of the wrench and $\hat{\mathbf{X}} = \begin{bmatrix} \boldsymbol{\delta}^{\mathrm{T}} & \boldsymbol{\phi}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ are the Plücker's axis coordinates of the twist. The vectors $\mathbf{f}$ and $\mathbf{m}$ are, respectively, the force and the moment. The vectors $\boldsymbol{\delta}$ and $\boldsymbol{\varphi}$ are, respectively, the translational and the rotational displacements. For small oscillations of a body, the stiffness matrix in Eq. (1) can be expressed as [5]

$$\mathbf{K} = \mathbf{j}[k]\mathbf{j}^{\mathrm{T}}, \tag{2}$$

where $\mathbf{j}$ is the $6 \times n$ Jacobian matrix expressed in the form $\mathbf{j} = [\hat{\mathbf{s}}_1, \ldots, \hat{\mathbf{s}}_n]$ and $\hat{\mathbf{s}}_i$ is the line vector of the ith line spring. $[k]$ is the diagonal matrix whose diagonal elements are the spring constants $k_i (i = 1,\ldots,n)$. The line vector $\hat{\mathbf{s}}_i$ is expressed in the Plücker's ray coordinates as

$$\hat{\mathbf{s}}_i = \begin{bmatrix} \mathbf{s}_i \\ \mathbf{s}_{Oi} \end{bmatrix} = \begin{bmatrix} \mathbf{s}_i \\ \mathbf{r}_i \times \mathbf{s}_i \end{bmatrix}, \tag{3}$$

where the vectors $\mathbf{s}_i$ and $\mathbf{r}_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}^{\mathrm{T}}$ denote, respectively, the unit direction vector of the line and the position vector from the origin to the line. Now, substituting Eq. (3) into Eq. (2) yields

$$\mathbf{K} = \sum_{i=1}^{n} k_i \begin{bmatrix} \mathbf{s}_i\mathbf{s}_i^{\mathrm{T}} & -\mathbf{s}_i\mathbf{s}_i^{\mathrm{T}}\mathbf{R}_i \\ \mathbf{R}_i\mathbf{s}_i\mathbf{s}_i^{\mathrm{T}} & -\mathbf{R}_i\mathbf{s}_i\mathbf{s}_i^{\mathrm{T}}\mathbf{R}_i \end{bmatrix}, \tag{4}$$

where

$$\mathbf{R}_i = \mathbf{r}_i \times = \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}.$$

2.2. *Stiffness matrix for orthogonal three-springs*

We consider a single rigid body supported by n sets of three line springs which are orthogonal each other and coincident at a point as shown in Fig. 1. Further, the line springs are assumed to be parallel to the axes of the coordinate system. In this case, the stiffness matrix $\mathbf{K}^i$ of the ith set of three line springs becomes significantly simplified and can be given by

$$\mathbf{K}^i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -c_i & b_i \\ c_i & 0 & -a_i \\ -b_i & a_i & 0 \end{bmatrix} \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & c_i & -b \\ 0 & 1 & 0 & -c_i & 0 & a_i \\ 0 & 0 & 1 & b_i & -a_i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} k_x & 0 & 0 & 0 & c_ik_x & -b_ik_x \\ & k_y & 0 & -c_ik_y & 0 & a_ik_y \\ & & k_z & b_ik_z & -a_ik_z & 0 \\ & & & (b_i)^2k_z + (c_i)^2k_y & -a_ib_ik_z & -a_ic_ik_y \\ \text{(symmetrical)} & & & & (a_i)^2k_z + (c_i)^2k_x & -b_ic_ik_x \\ & & & & & (a_i)^2k_y + (b_i)^2k_x \end{bmatrix}, \tag{5}$$

where the vector $\mathbf{p}_i = \begin{bmatrix} a_i & b_i & c_i \end{bmatrix}^{\mathrm{T}}$ denotes the position vector from the mass center C of a body to the coincident point of the springs. The constants k_x, k_y, and k_z are the common stiffness constants for all x-, y-, and z-directional spring, respectively. Such a spring system that the consisting three springs are mutually orthogonal and coincident at one point may be referred to here as the *orthogonal three-springs* (OTS). From a practical point of view, many elastically supporting mechanical components such as engine mounting materials can be modeled as the sets of OTS.

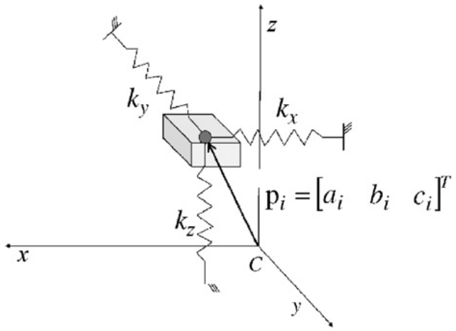

Fig. 1. A set of orthogonal three-springs (OTS).

When a rigid body is supported by n sets of OTS, the total stiffness matrix can be expressed as

$$\mathbf{K} = \sum_i \mathbf{K}^i$$
$$= \begin{bmatrix} nk_x & 0 & 0 & 0 & k_x\sum_i c_i & -k_x\sum_i b_i \\ & nk_y & 0 & -k_y\sum_i c_i & 0 & k_y\sum_i a_i \\ & & nk_z & k_z\sum_i b_i & -k_z\sum_i a_i & 0 \\ & & & k_z\sum_i (b_i)^2 + k_y\sum_i (c_i)^2 & -k_z\sum_i a_ib_i & -k_y\sum_i a_ic_i \\ \text{(symmetrical)} & & & & k_z\sum_i (a_i)^2 + k_x\sum_i (c_i)^2 & -k_x\sum_i b_ic_i \\ & & & & & k_y\sum_i (a_i)^2 + k_x\sum_i (b_i)^2 \end{bmatrix}. \tag{6}$$

Observation of the above stiffness matrix reveals that it can be diagonalized when the position vectors to the coincident points satisfy the following conditions:

$$\sum_i a_i = 0, \quad \sum_i b_i = 0, \quad \sum_i c_i = 0, \tag{7}$$

$$\sum_i a_ib_i = 0, \quad \sum_i b_ic_i = 0, \quad \sum_i a_ic_i = 0. \tag{8}$$

These conditions for diagonalization are always satisfied when the position vectors are arranged symmetrically with respect to each of the coordinate axes. When the conditions given by Eqs. (7) and (8) are satisfied, Eq. (6) becomes

$$\mathbf{K} = \mathrm{diag}\begin{pmatrix} K_1 & K_2 & K_3 & K_4 & K_5 & K_6 \end{pmatrix}, \tag{9}$$

where

$$K_1 = nk_x,\ \ K_2 = nk_y,\ K_3 = nk_z,\ K_4 = k_z\sum_i (b_i)^2 + k_y\sum_i (c_i)^2,$$
$$K_5 = k_z\sum_i (a_i)^2 + k_x\sum_i (c_i)^2,\ K_6 = k_y\sum_i (a_i)^2 + k_x\sum_i (b_i)^2.$$

If all the position vectors $\mathbf{p}_i$'s are arranged on a single coordinate axis, one or two $K_i (i = 4, 5, 6)$ values become zeros and the stiffness matrix $\mathbf{K}$ becomes singular.

3. Design of mdof vibration absorber

3.1. Decoupled dynamic equation of a rigid body

For an elastically supported single rigid body, the equation of motion for free vibration can be expressed at C by

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{0}, \tag{10}$$

where $\mathbf{M}$ is the 6×6 mass matrix. The small twist representing a general harmonic displacement in a three-dimensional space can be expressed by

$$\mathbf{X} = \hat{\mathbf{X}} e^{j\Omega t}, \tag{11}$$

where

$$\hat{\mathbf{X}} = \begin{bmatrix} \delta_x & \delta_y & \delta_z & \phi_x & \phi_y & \phi_z \end{bmatrix}^{\mathrm{T}} \tag{12}$$

and Ω denotes the natural frequency of the system. When the axes of the coordinate frames are chosen to be coincident with the principal axes of inertia, the mass matrix can be expressed by a diagonal matrix

$$\mathbf{M} = \mathrm{diag}\begin{pmatrix} m & m & m & I_{xx} & I_{yy} & I_{zz} \end{pmatrix}. \tag{13}$$

Substituting Eq. (11) into Eq. (10) yields

$$(\mathbf{K} - \Omega^2 \mathbf{M})\hat{\mathbf{X}} = \mathbf{0}. \tag{14}$$

When the coordinate axes are coincident with respective principal axes of inertia and the body is supported by n sets of OTS that satisfy the conditions given by Eqs. (7) and (8), the dynamic equation of a rigid body is completely decoupled. Expanding Eq. (14) gives the following 6 independent equations:

$$\begin{aligned} &\left(\frac{K_1}{m} - \Omega^2\right)\delta_x = 0, \quad \left(\frac{K_2}{m} - \Omega^2\right)\delta_y = 0, \quad \left(\frac{K_3}{m} - \Omega^2\right)\delta_z = 0, \\ &\left(\frac{K_4}{I_{xx}} - \Omega^2\right)\phi_x = 0, \quad \left(\frac{K_5}{I_{yy}} - \Omega^2\right)\phi_y = 0, \quad \left(\frac{K_6}{I_{zz}} - \Omega^2\right)\phi_z = 0. \end{aligned} \tag{15}$$

From Eq. (15), the six frequencies are determined by the square root ratios of stiffness to mass or inertia for each direction:

$$\begin{aligned} &\Omega_1 = \sqrt{\frac{K_1}{m}}, \quad \Omega_2 = \sqrt{\frac{K_2}{m}}, \quad \Omega_3 = \sqrt{\frac{K_3}{m}}, \\ &\Omega_4 = \sqrt{\frac{K_4}{I_{xx}}}, \quad \Omega_5 = \sqrt{\frac{K_5}{I_{yy}}}, \quad \Omega_6 = \sqrt{\frac{K_6}{I_{zz}}}. \end{aligned} \tag{16}$$

In the next section, these frequencies are used as the suppression frequencies of an mdof vibration absorber which can be adjusted by changing stiffness constants, configuration of spring sets, or mass properties.

3.2. Working conditions of mdof vibration absorber

Consider the system shown in Fig. 2 where body 1 is excited by an external wrench $\mathbf{w}(t)$ and body 2 is connected in series to body 1 by in-parallel linear springs. It is assumed that the origin of the coordinates frame is placed at the mass center C_1 of body 1. For small displacements of the bodies, the equations of motions can be expressed by

$$\mathbf{M}_1\ddot{\mathbf{X}}_1 + \mathbf{K}_1\mathbf{X}_1 + \mathbf{K}_2(\mathbf{X}_1 - \mathbf{X}_2) = \mathbf{w}(t), \tag{17a}$$

$$\mathbf{M}_2\ddot{\mathbf{X}}_2 + \mathbf{K}_2(\mathbf{X}_2 - \mathbf{X}_1) = \mathbf{0}. \tag{17b}$$

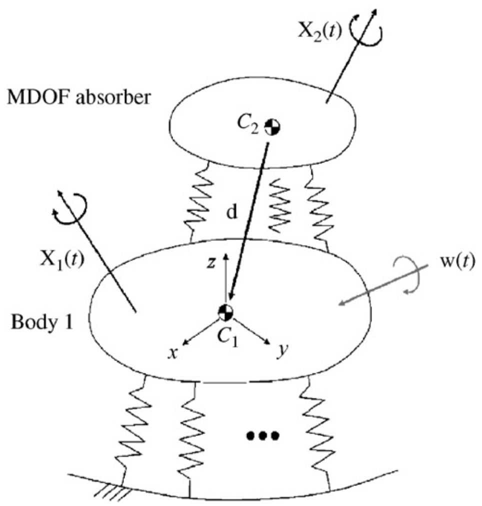

Fig. 2. Two-body system.

On the assumption that forced harmonic vibration takes place, the harmonic excitation force can be expressed as

$$\mathbf{w}(t) = \hat{\mathbf{w}}e^{j\omega t}, \tag{18}$$

where $\hat{\mathbf{w}} = [f_x \quad f_y \quad f_z \quad m_x \quad m_y \quad m_z]^{\mathrm{T}}$ is the time-independent applied wrench and ω is the excitation frequency. The small twist representing a general harmonic displacement can be expressed by

$$\mathbf{X} = \hat{\mathbf{X}}e^{j\omega t}. \tag{19}$$

Substituting Eqs. (18) and (19) into Eq. (17) yields

$$\mathbf{K}_1\hat{\mathbf{X}}_1 + \mathbf{K}_2\left(\hat{\mathbf{X}}_1 - \hat{\mathbf{X}}_2\right) - \omega^2\mathbf{M}_1\hat{\mathbf{X}}_1 = \hat{\mathbf{w}}, \tag{20a}$$

$$\mathbf{K}_2\left(\hat{\mathbf{X}}_2 - \hat{\mathbf{X}}_1\right) - \omega^2\mathbf{M}_2\hat{\mathbf{X}}_2 = \mathbf{0}. \tag{20b}$$

Eliminating $\hat{\mathbf{X}}_2$ from Eqs. (20a) and (20b) gives

$$\left\{\mathbf{K}_1 + \mathbf{K}_2 - \mathbf{K}_2\left(\mathbf{K}_2 - \omega^2\mathbf{M}_2\right)^{-1}\mathbf{K}_2 - \omega^2\mathbf{M}_1\right\}\hat{\mathbf{X}}_1 = \hat{\mathbf{w}}. \tag{21}$$

Using the parallel axis congruence transformation from the mass center C_2 of body 2 to C_1, the matrices $\mathbf{K}_2$ and $\mathbf{M}_2$ written at C_1 in Eq. (21) can be expressed in terms of $\mathbf{K}_{2C_2}$ and $\mathbf{M}_{2C_2}$ at C_2 as

$$\mathbf{K}_2 = \mathbf{E}_{C_2C_1}^{\mathrm{T}}\mathbf{K}_{2C_2}\mathbf{E}_{C_2C_1} \quad \text{and} \quad \mathbf{M}_2 = \mathbf{E}_{C_2C_1}^{\mathrm{T}}\mathbf{M}_{2C_2}\mathbf{E}_{C_2C_1}, \tag{22}$$

where $\mathbf{E}_{C_2C_1} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{d}\times \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}$ and $\mathbf{d} \equiv \overrightarrow{C_2C_1} = \begin{bmatrix} d_x & d_y & d_z \end{bmatrix}^{\mathrm{T}}$. Substituting Eq. (22) into Eq. (21) yields

$$\left(\mathbf{K}_1 - \mathbf{E}_{C_2C_1}^{\mathrm{T}}\mathbf{H}\mathbf{E}_{C_2C_1} - \omega^2\mathbf{M}_1\right)\hat{\mathbf{X}}_1 = \hat{\mathbf{w}}, \tag{23}$$

where

$$\mathbf{H} = \mathbf{K}_{2C_2}\left(\mathbf{K}_{2C_2} - \omega^2\mathbf{M}_{2C_2}\right)^{-1}\mathbf{K}_{2C_2} - \mathbf{K}_{2C_2}. \tag{24}$$

When the body 2 is supported by a number of sets of OTS and dynamically decoupled, $\mathbf{H}$ can be expressed in the form of a diagonal matrix

$$\mathbf{H} = \frac{1}{\Delta}\mathrm{diag}\left(K_1\Delta_1 \quad K_2\Delta_2 \quad K_3\Delta_3 \quad K_4\Delta_4 \quad K_5\Delta_5 \quad K_6\Delta_6\right) \equiv \frac{1}{\Delta}\tilde{\mathbf{H}}, \tag{25}$$

where

$$\Delta = \prod_{i=1}^{6} \frac{\Omega_i^2 - \omega^2}{\omega^2}$$

and

$$\Delta_j = \prod_{\substack{i=1 \\ i \neq j}}^{6} \Omega \frac{\Omega_i^2 - \omega^2}{\omega^2}.$$

Substituting Eq. (25) into Eq. (23) yields

$$\left(\mathbf{K}_1 - \frac{1}{\Delta} \mathbf{E}_{C_2C_1}^{\mathrm{T}} \tilde{\mathbf{H}} \, \mathbf{E}_{C_2C_1} - \omega^2 \mathbf{M}_1\right) \hat{\mathbf{X}}_1 = \hat{\mathbf{w}}. \tag{26}$$

When the excitation frequency ω gets closer to any one of Ω_i, Δ approaches zero and Eq. (26) can be rewritten as greatly simplified form

$$\mathbf{E}_{C_2C_1}^{\mathrm{T}} \tilde{\mathbf{H}} \, \mathbf{E}_{C_2C_1} \hat{\mathbf{X}}_1 = \mathbf{0}. \tag{27}$$

The fully expanded form of Eq. (27) may be given by

$$\begin{bmatrix} K_1\Delta_1 & 0 & 0 & 0 & -K_1\Delta_1 d_z & K_1\Delta_1 d_y \\ & K_2\Delta_2 & 0 & K_2\Delta_2 d_z & 0 & -K_2\Delta_2 d_x \\ & & K_3\Delta_3 & -K_3\Delta_3 d_y & K_3\Delta_3 d_x & 0 \\ & & & \begin{matrix} K_4\Delta_4 + K_3\Delta_3 d_y^2 \\ +K_2\Delta_2 d_z^2 \end{matrix} & -K_3\Delta_3 d_x d_y & -K_2\Delta_2 d_x d_z \\ \text{(symmetrical)} & & & & \begin{matrix} K_5\Delta_5 + K_3\Delta_3 d_x^2 \\ +K_1\Delta_1 d_z^2 \end{matrix} & -K_1\Delta_1 d_y d_z \\ & & & & & \begin{matrix} K_6\Delta_6 + K_2\Delta_2 d_x^2 \\ +K_1\Delta_1 d_y^2 \end{matrix} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_z \\ \phi_x \\ \phi_y \\ \phi_z \end{bmatrix}_1 = \mathbf{0}. \tag{28}$$

Now, at $\omega = \Omega_i (i = 1,\ldots,6)$, expanding the ith row of Eq. (28) gives the following relations:

$$\delta_x - d_z\phi_y + d_y\phi_z = 0 \quad \text{at } \omega = \Omega_1, \tag{29a}$$

$$\delta_y + d_z\phi_x - d_x\phi_z = 0 \quad \text{at } \omega = \Omega_2, \tag{29b}$$

$$\delta_z - d_y\phi_x + d_x\phi_y = 0 \quad \text{at } \omega = \Omega_3, \tag{29c}$$

$$\phi_x = 0 \quad \text{at } \omega = \Omega_4, \tag{29d}$$

$$\phi_y = 0 \quad \text{at } \omega = \Omega_5, \tag{29e}$$

$$\phi_z = 0 \quad \text{at } \omega = \Omega_6. \tag{29f}$$

From Eqs. (29d) to (29f), it is clear that when the excitation frequency ω approaches any one of Ω_4, Ω_5, or Ω_6, the corresponding rotational displacement ϕ_x, ϕ_y, or ϕ_z vanishes. This implies that the rigid body 2 can be

Table 1
Design rules for MDOF vibration absorber

$\mathbf{d}\left(=\overrightarrow{C_2C_1}\right)$	Frequency condition(s) for			Design rule
	$\delta_x=0$ (A)[a]	$\delta_y=0$ (B)[a]	$\delta_z=0$ (C)[a]	$\cap$
—	$\Omega_1=\Omega_5=\Omega_6$	$\Omega_2=\Omega_4=\Omega_6$	$\Omega_3=\Omega_4=\Omega_5$	N
$d_x=0$	$\Omega_1=\Omega_5=\Omega_6$	$\Omega_2=\Omega_4$	$\Omega_3=\Omega_4$	DX
$d_y=0$	$\Omega_1=\Omega_5$	$\Omega_2=\Omega_4=\Omega_6$	$\Omega_3=\Omega_5$	DY
$d_z=0$	$\Omega_1=\Omega_6$	$\Omega_2=\Omega_6$	$\Omega_3=\Omega_4=\Omega_5$	DZ
$d_x=d_y=0$	$\Omega_1=\Omega_5$	$\Omega_2=\Omega_4$	—	DXY
$d_y=d_z=0$	—	$\Omega_2=\Omega_6$	$\Omega_3=\Omega_5$	DYZ
$d_x=d_z=0$	$\Omega_1=\Omega_6$	—	$\Omega_3=\Omega_4$	DXZ
$d_x=d_y=d_z=0$	—	—	—	DXYZ

[a]The symbols used for classification of design rules.

utilized as an mdof vibration absorber. If, for example, two rotational displacements ϕ_x and ϕ_y are to be eliminated at ω, then both Ω_4 and Ω_5 have to be made equal to ω using Eq. (16).

On the other hand, when ω approaches any one of Ω_2, Ω_2, or Ω_3, the corresponding translational displacement δ_x, δ_y, or δ_z does not simply vanish. From Eqs. (29a) to (29c), there may be various design choices of elimination. For example, If δ_x is to be eliminated at ω, then, firstly, Ω_1 has to be made equal to ω using Eq. (16) so that Eq. (29a) can be used. Now, from Eq. (29a), δ_x can be eliminated by one of the following conditions: (1) $\phi_y=\phi_z=0$, (2) $d_y=d_z=0$, (3) $d_z=\phi_z=0$, or (4) $\phi_y=d_y=0$. It is clear, from Eqs. (29e) and (29f), that the condition (1) can be realized by making $\Omega_5=\Omega_6=\omega$. Further, if the body 2 is designed in such a way that $\Omega_1=\Omega_5=\Omega_6=\omega$, then the displacements δ_x, ϕ_y, and ϕ_z will not appear in response at ω. The other design conditions are self-explanatory.

The design rules for the elimination of each translational displacement are enumerated in Table 1. The design rule to eliminate any two or three translational displacements can be obtained by simultaneously applying the corresponding rules. For example, if both δ_x and δ_y are to be eliminated, then we may choose one condition of $\mathbf{d}\left(=\overrightarrow{C_2C_1}\right)$ from the first column of Table 1. If the condition of $d_x=d_y=0$ is chosen for any reason, then the final design rule will be DXY $\cap$ (A) $\cap$ (B) from Table 1.

4. Design examples

In this section, a spatial 3dof and a planar 2dof design cases are examined. The former illustrates the design method presented in the previous section, while the latter is used for a comparative study of vibration absorbers.

4.1. 3dof vibration absorber

A numerical example is used to illustrate the design method. In this example, rigid body 1 is placed on the elastic support whose stiffness matrix is arbitrarily given as the following:

$$\mathbf{K}_1=10^8\times\begin{bmatrix} 0.3715 & 0.1237 & -0.1212 & -0.4836 & -1.0686 & -0.8029 \\ 0.1237 & 0.1755 & -0.0574 & -0.2342 & -0.2774 & -0.0557 \\ -0.1212 & -0.0574 & 0.3900 & 0.4227 & 0.7864 & 0.7609 \\ -0.4836 & -0.2342 & 0.4227 & 1.1028 & 1.4286 & 1.3264 \\ -1.0686 & -0.2774 & 0.7864 & 1.4286 & 4.0424 & 3.0908 \\ -0.8029 & -0.0557 & 0.7609 & 1.3264 & 3.0908 & 2.9682 \end{bmatrix}. \tag{30}$$

Table 2
Mass properties of body 1

m	13.338 kg
I_{xx}	5.528×10^{-2} kg m
I_{yy}	3.314×10^{-2} kg m^2
I_{zz}	7.849×10^{-2} kg m^2

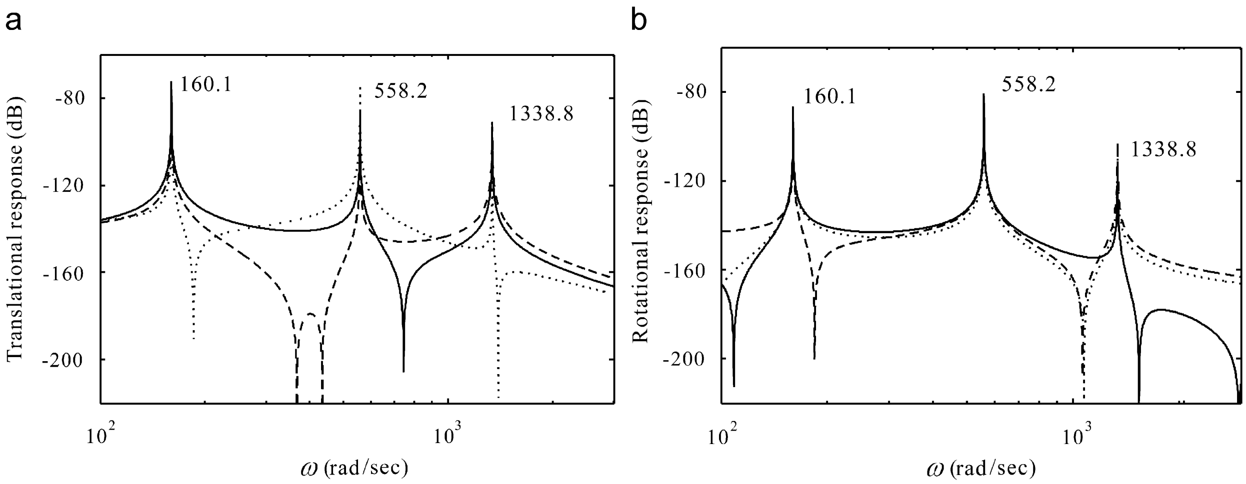

Fig. 3. Frequency responses of body 1: (a) translation in the x-axis (solid), y-axis (dotted), z-axis (dashed), (b) rotation on the x-axis (solid), y-axis (dotted), z-axis (dashed).

The mass properties of body 1 are listed in Table 2. When the coordinate axes are chosen to be coincident with the principal axes of inertia of body 1, the mass matrix of body 1 at its mass center is given by

$$\mathbf{M}_1 = \mathrm{diag}\begin{pmatrix} 13.338 & 13.338 & 13.338 & 0.05528 & 0.03314 & 0.07849 \end{pmatrix}. \tag{31}$$

The eigenvalue solution of the system with $\mathbf{K}_1$ and $\mathbf{M}_1$ given by Eqs. (30) and (31) produces the natural frequencies, 160.1, 558.2, 1,338.8, 22,270, 32,588, and 128,138 rad/s. Assuming that the harmonic excitation wrench $\hat{\mathbf{w}} = \begin{bmatrix} 100 & 100 & 200 & 55 & 49 & 98 \end{bmatrix}^{\mathrm{T}}$ is applied on body 1, the frequency response of body 1 are plotted in Fig. 3. In this figure, the first (at 160.1 rad/s), the second (at 558.2 rad/s), and the third (at 1338.8 rad/s) peaks are undesirable and will be reduced by use of an mdof vibration absorber. It is noted that the magnitude of the first peak is more influenced by δ_x than any other displacements. Likewise, δ_y at the second peak and δ_z at the third peak are more dominant than any other displacements. This implies that first, second, and third peak can be reduced most efficiently by eliminating translational displacement δ_x, δ_y, and δ_z, respectively, at the corresponding frequencies.

Since three translational displacements are to be eliminated, respectively, at three different frequencies of the resonant peaks, $\mathbf{d}$ has to have at least two zero elements and we apply the design rule DXY $\cap$ (A) $\cap$ (B) $\cap$ (C) from Table 1. For the elimination of δ_x at the first peak, Ω_1 and Ω_5 are tuned to the frequency of the first peak. Likewise, Ω_2 and Ω_4 are tuned to that of the second peak and Ω_3 is tuned to that of the third peak. The choice of the last undetermined suppression frequency Ω_6 is free. Here, we make Ω_6 equal to the frequency of the first peak.

In this example, four sets of OTS are used as shown in Fig. 4 and the position vectors to the coincident points of each set of OTS which satisfy Eqs. (7) and (8) are arranged on a plane ($c_i = 0$) symmetrically.

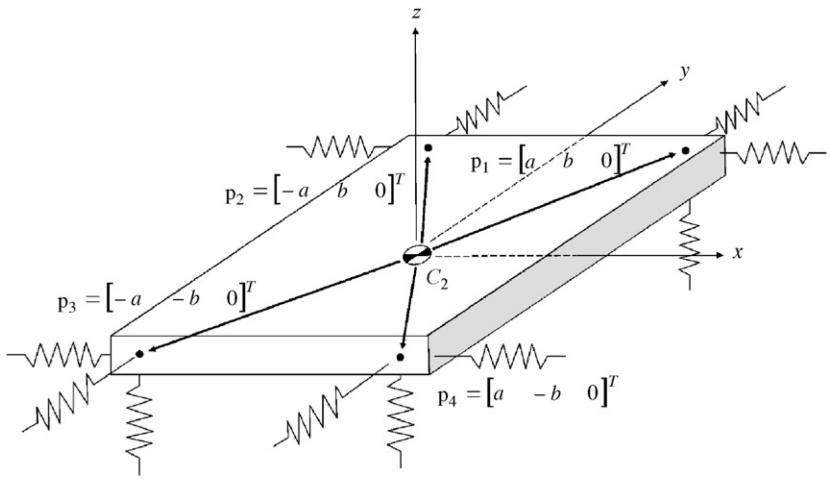

Fig. 4. Position vectors to each of supporting OTS of an mdof vibration absorber.

Table 3
Design values determined in example 1

m	1.330 kg
I_{xx}	1.400×10^{-2} kg m^2
I_{yy}	1.870×10^{-2} kg m^2
I_{zz}	0.5684×10^{-2} kg m^2
k_x	8527 N/m
k_y	1036×10^2 N/m
k_z	5960×10^2 N/m
a	1.420×10^{-2} m
b	4.278×10^{-2} m

The stiffness matrix of body 2 is given by

$$\mathbf{K}_{2C_2} = \mathrm{diag}\Big(4k_x \quad 4k_y \quad 4k_z \quad 4k_z b^2 \quad 4k_z a^2 \quad 4\big(k_y a^2 + k_x b^2\big) \Big). \tag{32}$$

These diagonal constants of $\mathbf{K}_{2C_2}$ are substituted into Eq. (16) to get

$$\Omega_1 = \sqrt{\frac{4k_x}{m}} = 160.1, \quad \Omega_2 = \sqrt{\frac{4k_y}{m}} = 558.2, \quad \Omega_3 = \sqrt{\frac{4k_z}{m}} = 1338.8,$$
$$\Omega_4 = \sqrt{\frac{4b^2k_z}{I_{xx}}} = 558.2, \quad \Omega_5 = \sqrt{\frac{4a^2k_z}{I_{yy}}} = 160.1, \quad \Omega_6 = \sqrt{\frac{4a^2k_y + 4b^2k_x}{I_{zz}}} = 160.1. \tag{33}$$

In Eq. (33), a designer can now choose three spring constants (k_x, k_y, and k_z), the positions of the coincident points (a and b), mass (m), and three moment of inertias (I_{xx}, I_{yy}, and I_{zz}). Table 3 shows the design values of the parameters chosen in this example.

The performance of the designed mdof vibration absorber is verified via modal analysis. Using the design values in Table 3, the mass and stiffness matrix of body 2 are computed as follows:

$$\mathbf{M}_{2C_2} = \mathrm{diag}\big(1.330 \quad 1.330 \quad 1.330 \quad 0.01400 \quad 0.01870 \quad 0.005684\big),$$
$$\mathbf{K}_{2C_2} = 10^3 \times \mathrm{diag}\big(34.11 \quad 414.4 \quad 2384 \quad 4.362 \quad 0.4796 \quad 0.1458\big). \tag{34}$$

Since the design rule DXY $\cap$ (A) $\cap$ (B) $\cap$ (C) is applied, the x- and y-directional element of $\mathbf{d}$ should be zeros and the mass center of body 2 (the absorber) is placed 0.1 m high over that of body 1, i.e. $\mathbf{d} \equiv \overrightarrow{C_2C_1} = \begin{bmatrix} 0 & 0 & -0.1 \end{bmatrix}^{\mathrm{T}}$. Now, using Eq. (22), these mass and stiffness matrices are transformed to the mass center of body 1 to get

$$\mathbf{M}_2 = \mathbf{E}^{\mathrm{T}}_{C_2C_1}\mathbf{M}_{2C_2}\mathbf{E}_{C_2C_1} = \begin{bmatrix} 1.330 & 0 & 0 & 0 & 0.1330 & 0 \\ 0 & 1.330 & 0 & -0.1330 & 0 & 0 \\ 0 & 0 & 1.330 & 0 & 0 & 0 \\ 0 & -0.1330 & 0 & 0.02730 & 0 & 0 \\ 0.1330 & 0 & 0 & 0 & 0.03200 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.005684 \end{bmatrix},$$

$$\mathbf{K}_2 = \mathbf{E}^{\mathrm{T}}_{C_2C_1}\mathbf{K}_{2C_2}\mathbf{E}_{C_2C_1} = 10^3 \times \begin{bmatrix} 34.11 & 0 & 0 & 0 & 3.411 & 0 \\ 0 & 414.4 & 0 & -41.44 & 0 & 0 \\ 0 & 0 & 2384 & 0 & 0 & 0 \\ 0 & -41.44 & 0 & 8.506 & 0 & 0 \\ 3.411 & 0 & 0 & 0 & 0.8207 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1458 \end{bmatrix}. \quad (35)$$

The modal analysis is performed with the mass and stiffness matrices of body 1 and body 2. The global mass and stiffness matrices are given by

$$\bar{\mathbf{M}} = \begin{bmatrix} \mathbf{M}_1 & 0 \\ 0 & \mathbf{M}_2 \end{bmatrix}, \quad \bar{\mathbf{K}} = \begin{bmatrix} \mathbf{K}_1 + \mathbf{K}_2 & -\mathbf{K}_2 \\ -\mathbf{K}_2 & \mathbf{K}_2 \end{bmatrix}. \quad (36)$$

The 12 mode vectors calculated from the matrices in Eq. (36) can be expressed as

$$\mathbf{\Psi} = \begin{bmatrix} \mathbf{\Phi}^1_1 & \cdots & \mathbf{\Phi}^1_{12} \\ \mathbf{\Phi}^2_1 & \cdots & \mathbf{\Phi}^2_{12} \end{bmatrix}, \quad (37)$$

where the superscript and subscript denote the number of the rigid body and the number of the mode, respectively. The externally applied wrench $\hat{\mathbf{w}} = \begin{bmatrix} \mathbf{f}^{\mathrm{T}} & \mathbf{m}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ can also be expressed as $\hat{\mathbf{w}} = f\begin{bmatrix} \mathbf{s}^{\mathrm{T}}_f & \mathbf{s}^{\mathrm{T}}_{\mathrm{O}} \end{bmatrix}^{\mathrm{T}} = f\hat{\mathbf{s}}_f$ where f is the magnitude of the force $\mathbf{f}$ while $\mathbf{s}_f$ is the unit direction vector along the line of action. Then, the forced responses of the system can be computed as

$$\frac{1}{f}\begin{bmatrix} \hat{\mathbf{X}}_1 \\ \hat{\mathbf{X}}_2 \end{bmatrix} = \sum_{r=1}^{12} \frac{\begin{bmatrix} \mathbf{\Phi}^1_r \\ \mathbf{\Phi}^2_r \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \hat{\mathbf{s}}_f \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}^1_r \\ \mathbf{\Phi}^2_r \end{bmatrix}}{\bar{k}_r - \omega^2 \bar{m}_r}, \quad (38)$$

where $\bar{k}_r$ and $\bar{m}_r$ are rth diagonal elements of $\mathbf{\Psi}^{\mathrm{T}}\bar{\mathbf{K}}\mathbf{\Psi}$ and $\mathbf{\Psi}^{\mathrm{T}}\bar{\mathbf{M}}\mathbf{\Psi}$, respectively.

In this example, the damping of a system is regarded as being proportional to stiffness. The proportional damping matrix is given by

$$\bar{\mathbf{C}} = \left(\frac{\beta}{100}\right)\bar{\mathbf{K}}, \quad (39)$$

where β is the proportional percent damping ratio and is 0.1 in this example. The proportionally damped responses can be determined by

$$\frac{1}{f}\begin{bmatrix} \hat{\mathbf{X}}_1 \\ \hat{\mathbf{X}}_2 \end{bmatrix} = \sum_{r=1}^{12} \frac{\begin{bmatrix} \mathbf{\Phi}^1_r \\ \mathbf{\Phi}^2_r \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \hat{\mathbf{s}}_f \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}^1_r \\ \mathbf{\Phi}^2_r \end{bmatrix}}{\bar{k}_r + j\omega\bar{c}_r - \omega^2 \bar{m}_r}, \quad (40)$$

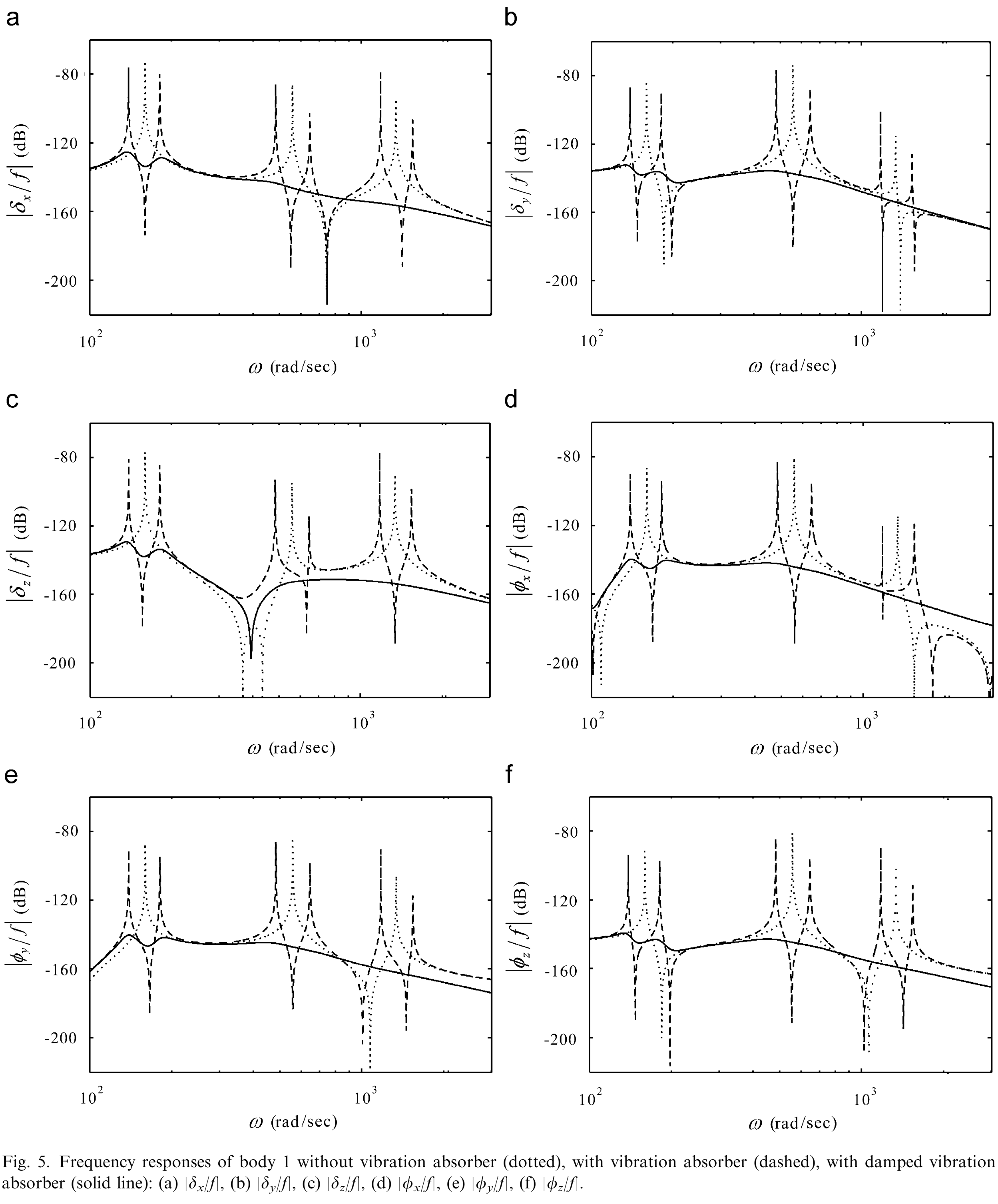

Fig. 5. Frequency responses of body 1 without vibration absorber (dotted), with vibration absorber (dashed), with damped vibration absorber (solid line): (a) $|\delta_x/f|$, (b) $|\delta_y/f|$, (c) $|\delta_z/f|$, (d) $|\phi_x/f|$, (e) $|\phi_y/f|$, (f) $|\phi_z/f|$.

where $\bar{c}_r$ is the rth diagonal element of $\mathbf{\Psi}^{\mathrm{T}}\bar{\mathbf{C}}\mathbf{\Psi}$. Fig. 5 shows three resonance peaks in every translational and rotational displacement. Attaching the mdof vibration absorber to body 1, the original resonant peak is disappeared and two new adjacent peaks are emerged. This is the same result as the single dof vibration absorber exhibits. When a 0.1% proportional damping is applied, the peaks are disappeared.

4.2. Planar 2dof vibration absorber: a comparative study

One possible method of designing an mdof vibration absorber is to utilize the optimization technique suggested by Zuo and Nayfeh [3]. Fig. 6 shows a 2dof system used for a comparative study of planar absorber designs. In Ref. [3], the mass properties and location of the absorber and the positions of the springs are specified as shown in Fig. 6(a). The mass and stiffness matrices of body 1 are given by

$$\mathbf{M}_1 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad \mathbf{K}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 130,000 & 3500 \\ 0 & 3500 & 6325 \end{bmatrix}. \tag{41}$$

This design example involves reducing the first peak at 159.2 rad/s and the second one at 252.8 rad/s appeared in the response of body 1 as shown in Fig. 7. The stiffness and damping values of a vibration absorber are determined by means of the minimax optimization algorithm as $k_1 = 6038.93$, $k_2 = 2679.96$ N/m, $c_{d1} = 11.74$, and $c_{d2} = 5.94$ N s/m. The graphs of the response of body 1 with the vibration absorber designed through optimization are plotted by dash–dot lines in Fig. 7.

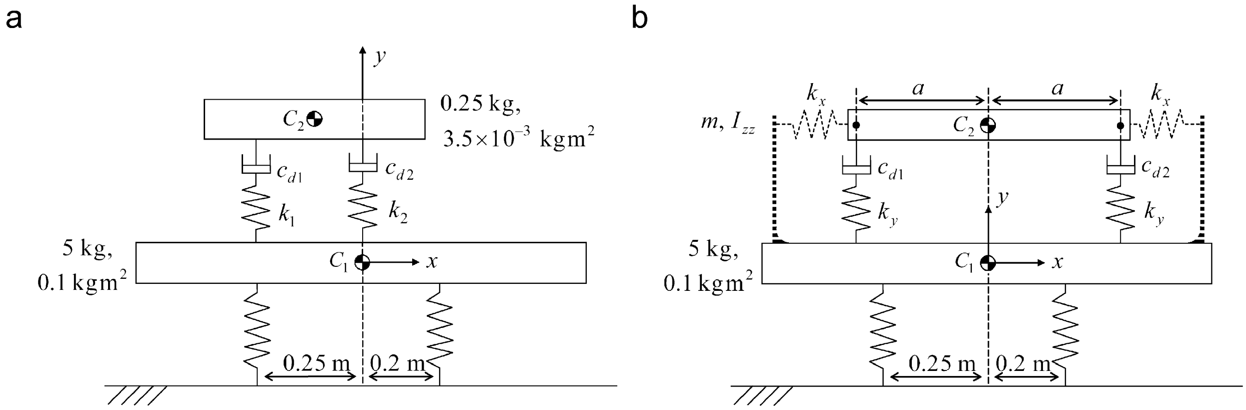

Fig. 6. Planar vibrating systems with 2dof vibration absorbers: (a) Ref. [3], (b) present work.

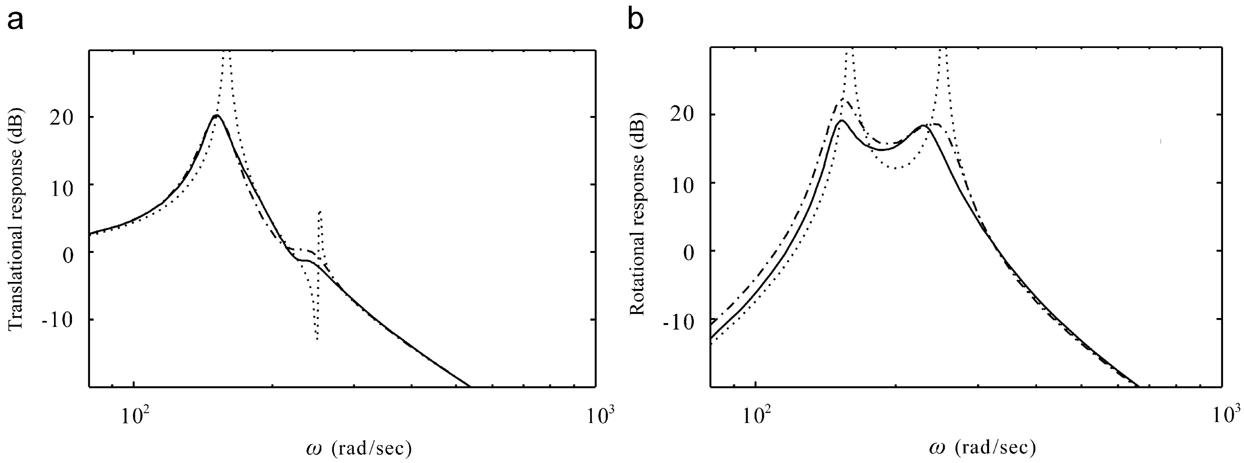

Fig. 7. Bode plots of transmission from ground vertical input to (a) translational displacement δ_y, (b) rotational displacement ϕ_z of rigid body 1 without vibration absorber (dotted), with vibration absorber designed by present method (solid), with vibration absorber designed by optimization (dash–dot).

Table 4
Design values determined in example 2

m	0.25 kg
I_{zz}	$1.22\times10^{-2}\,\mathrm{kg\,m^2}$
k_y	3169.3 N/m
a	0.35 m

In a geometrical design process, two peaks of original system can be reduced by eliminating δ_y and ϕ_z. For the elimination of δ_y, we apply the design rule DX $\cap$ (B) from Table 1. A vibration absorber is installed just over the mass center of body 1. Two sets of OTS are placed symmetrically at the same distance a from the mass center C_2 in x-direction so that Eqs. (7) and (8) can be satisfied. It is noted here that a planar OTS for this design has only k_y, i.e., $k_x = 0$ as shown in Fig. 6(b). The stiffness matrix of body 2 (an absorber) is given by

$$\mathbf{K}_{2C_2} = \mathrm{diag}\begin{pmatrix} 0 & 2k_y & 2a^2k_y \end{pmatrix}. \tag{42}$$

The diagonal elements in Eq. (42) are substituted into Eq. (16) to get the following relations:

$$\Omega_1 = 0, \quad \Omega_2 = \sqrt{\frac{2k_y}{m}} = 159.2, \quad \Omega_6 = \sqrt{\frac{2a^2k_y}{I_{zz}}} = 252.8. \tag{43}$$

In Eq. (43), Ω_2 is tuned to the frequency of the first peak for the elimination of δ_y at the first peak. In the same manner, Ω_6 is tuned to that of second peak. The design variables determined for the vibration absorber are listed in Table 4. Using the same damping values as those found in the optimization, the performance of the geometrically designed vibration absorber is simulated. The performances of two different designs are compared in Fig. 7 where the geometrically designed one shows a better performance in the rotational response.

5. Conclusion

We presented a new geometrical design method of an mdof vibration absorber for reduction of multiple modes of vibration, which does not utilize optimization technique. The design method involves the technique to decouple the dynamic equation of a rigid body supported by many sets of OTS. From the decoupled dynamic equation, we obtained the design rules for an mdof vibration absorber, which can be realized by adjusting the location of the absorber and suppression frequencies. Two numerical simulations of the proposed method were presented to show that a single mdof vibration absorber can successfully reduce the undesired multiple modes of an oscillating system.

Acknowledgments

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2005-041-D00056).

References

[1] A.F. Vakakis, S.A. Paipetis, The effect of a viscously damped dynamic absorbers on a linear multi-degree-of-freedom system, *Journal of Sound and Vibration* 105 (1986) 49–60.

[2] F. Sadek, B. Mohraz, A.W. Taylor, R.M. Chung, A method of estimating the parameters of tuned mass dampers for seismic applications, *Earthquake Engineering and Structural Dynamics* 26 (1997) 617–635.

[3] L. Zuo, S.A. Nayfeh, Minmax optimization of multi-degree-of-freedom tuned-mass dampers, *Journal of Sound and Vibration* 272 (2004) 893–908.

[4] R.S. Ball, *A Treatise on the Theory of Screws*, Cambridge University Press, Cambridge, 1900.

[5] M. Griffis, J. Duffy, Kinestatic control: a novel theory for simultaneously regulating force and displacement, *Journal of Mechanical Design* 113 (1991) 508–515.