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Parametric translation models for stationary non-Gaussian processes and fields

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Abstract

Parametric representations, that is, deterministic functions with time and/or space argument depending on finite families of random variables, are defined and used to describe approximately stationary non-Gaussian processes and fields specified partially by their marginal distribution and second-moment properties. The proposed parametric representations are memoryless transformations of sequences of parametric stationary Gaussian processes and fields, and are referred to as parametric translation models. Conditions are established for the convergence of statistics of sequences of parametric translation models to target statistics. Two numerical examples are presented to illustrate some properties of parametric translation models and demonstrate their use in random vibration.

1. Introduction

Parametric representations for stochastic processes and random fields, that is, deterministic functions of time and/or space depending on a finite number of random variables, are commonly used in applications since numerical calculations can only handle finite families of random variables, and stochastic processes and random fields generally consist of infinite families of random variables. There are many parametric representations providing satisfactory approximations for the second-moment properties of processes and fields but no information on their distribution, unless they are Gaussian. This is a severe limitation since (1) most physical parameters do not follow Gaussian distributions [1, Chapter 2], (2) the existence and uniqueness of the solution of partial differential equations with spatially varying random coefficients requires, for example, that the samples of these coefficients take values in some bounded intervals [2], and (3) second-moment properties are insufficient for generating samples of non-Gaussian processes and fields.

Our objective is to construct parametric representations for non-Gaussian stationary processes and fields. A two-step algorithm is developed. First, a target process or field specified partially by its second-moment properties and marginal distribution needs to be approximated by a translation function, that is, a memoryless transformation of a stationary Gaussian process or field, provided it exists. The resulting approximation is referred to as the target translation function. Second, parametric representations need to be constructed for

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the Gaussian image of the target translation function. Memoryless transformations of these representations can be used to define sequences of parametric non-Gaussian functions, referred to as parametric translation models. It is shown that the first two moments and the finite dimensional distributions of the sequence of parametric translation models converge to the corresponding properties of the target translation function under some conditions. Two numerical examples are presented to illustrate among other properties the convergence of the statistics of parametric translation models to those of the target translation function and potential use of parametric translation models in random vibration.

The use of parametric translation models in applications is not new, for example, these models have been used to generate samples of non-Gaussian functions [3, Section 5.3.3.1]. However, there has been no systematic study attempting to establish conditions under which statistics of sequences of parametric translation models converge to specified target statistics. Determination of such conditions is a major objective of this paper.

2. Translation processes and fields

Let $\mathbf{X}(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^{d'}$, be an $\mathbb{R}^d$-valued stationary non-Gaussian function with coordinates $X_i(\mathbf{t})$, $i = 1, \ldots, d$, of marginal distribution $F_i(\xi; \mathbf{t}) = P(X_i(\mathbf{t}) \leqslant \xi)$, mean 0, variance 1, and covariance functions $\zeta_{ij}(\boldsymbol{\tau}) = E[X_i(\mathbf{t} + \boldsymbol{\tau}) X_j(\mathbf{t})]$, $\boldsymbol{\tau} \in \mathbb{R}^{d'}$. The (d, d)-matrix with entries $\zeta_{ij}(\boldsymbol{\tau})$ is denoted by $\boldsymbol{\zeta}(\boldsymbol{\tau}) = E[\mathbf{X}(\mathbf{t} + \boldsymbol{\tau})\mathbf{X}(\mathbf{t})^{\mathrm{T}}]$. If $\mathbf{t}$ is space, then $\mathbf{X}$ is a homogeneous non-Gaussian field. If $\mathbf{t} = t$ $(d' = 1)$ is time, then $\mathbf{X}$ is a stationary non-Gaussian process.

Suppose that $\mathbf{X}$ is specified partially by its second-moment properties and the marginal distributions of its coordinates, that is, the functions ζ_{ij} and F_i, $i, j = 1, \ldots, d$. Our objective is to construct parametric models for $\mathbf{X}$, that is, deterministic functions depending on time and/or space arguments and finite families of random variables. The construction involves two steps. First, $\mathbf{X}$ needs to be approximated by a translation function $\mathbf{X}_T$ matching the specified statistics of $\mathbf{X}$. We refer to $\mathbf{X}_T$ as the target translation process, field, or function, and assume it exists. Second, we need to construct a sequence of models $\mathbf{X}_T^{(n)}$, $n = 1, 2, \ldots$, converging in some sense to $\mathbf{X}$ as $n \to \infty$. The members of this sequence are defined by memoryless transformations of parametric Gaussian functions, and are referred to as parametric translation models. We consider the above partial characterization for $\mathbf{X}$ since the first two moments and the marginal distributions of non-Gaussian functions are frequently known in applications, and information beyond these properties is rarely available.

In this section we define translation functions $\mathbf{X}_T$ associated with non-Gaussian stationary functions $\mathbf{X}$ specified by their first two moments and marginal distributions; review of some of their essential properties is also given. Parametric translation models $\mathbf{X}_T^{(n)}$ are introduced in the next section and the convergence of the first two moments is examined along with the convergence of the properties of the finite dimensional distributions of these models to the corresponding properties of the target translation model $\mathbf{X}_T$ associated with $\mathbf{X}$.

Let $\mathbf{X}_T(\mathbf{t}) \in \mathbb{R}^d$, $\mathbf{t} \in \mathbb{R}^{d'}$, be a translation random function defined by

$$X_{T,i}(\mathbf{t}) = F_i^{-1}(\Phi(G_i(\mathbf{t}))) = h_i(G_i(\mathbf{t})), \quad i = 1, \ldots, d, \tag{1}$$

where Φ is the distribution of the standard Gaussian variable $N(0, 1)$ and $\mathbf{G}(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^{d'}$, is an $\mathbb{R}^d$-valued stationary Gaussian function with coordinates $G_i(\mathbf{t})$, $i = 1, \ldots, d$, of mean 0, variance 1, and covariance functions $\rho_{ij}(\boldsymbol{\tau}) = E[G_i(\mathbf{t} + \boldsymbol{\tau})\, G_j(\mathbf{t})]$, $\boldsymbol{\tau} \in \mathbb{R}^{d'}$. The (d, d) matrix with entries $\rho_{ij}(\boldsymbol{\tau})$ is denoted by $\boldsymbol{\rho}(\boldsymbol{\tau}) = E[\mathbf{G}(\mathbf{t} + \boldsymbol{\tau})\, \mathbf{G}(\mathbf{t})^{\mathrm{T}}]$. It is assumed throughout the paper that the distribution functions F_i, $i = 1, \ldots, d$, are absolutely continuous.

Property 1. *The translation process/field* $\mathbf{X}_T$ *in Eq.* (1) *is stationary with marginal distributions* F_i, $i = 1, \ldots, d$.

The finite dimensional distributions of $\mathbf{X}_T$ can be calculated from

$$\begin{aligned} P(X_{T,1}(\mathbf{t}_{1,1}) \leqslant x_{1,1}, \ldots, X_{T,1}(\mathbf{t}_{1,m_1}) \leqslant x_{1,m_1}, \ldots, X_{T,d}(\mathbf{t}_{d,1}) \leqslant x_{d,1}, \ldots, X_{T,d}(\mathbf{t}_{d,m_d}) \leqslant x_{d,m_d} g) \\ = P(G_1(\mathbf{t}_{1,1}) \leqslant h_1^{-1}(x_{1,1}), \ldots, G_1(\mathbf{t}_{1,m_1}) \leqslant h_1^{-1}(x_{1,m_1}), \\ \ldots, G_d(\mathbf{t}_{d,1}) \leqslant h_d^{-1}(x_{d,1}), \ldots, G_d(\mathbf{t}_{d,m_d}) \leqslant h_d^{-1}(x_{d,m_d})) = \Phi(\boldsymbol{\xi}; \mathbf{r}), \end{aligned} \tag{2}$$

where $m_1 \geqslant 0, \ldots, m_d \geqslant 0$ are integers, $(\mathbf{t}_{1,1}, \ldots, \mathbf{t}_{d,m_d})$ are arbitrary arguments, $\Phi(\boldsymbol{\xi}; \mathbf{r})$ denotes the joint cumulative distribution of the Gaussian vector $(G_1(\mathbf{t}_{1,1}), \ldots, G_d(\mathbf{t}_{d,m_d}))$ with covariance matrix $\mathbf{r}$ obtained from the covariance functions $\rho_{ij}(\cdot)$ of the Gaussian process $\mathbf{G}$, and is calculated at the argument $\boldsymbol{\xi} = (h_1^{-1}(x_{1,1}), \ldots, h_d^{-1}(x_{d,m_d}))$. The equality in Eq. (2) holds since $h_i : \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1, \ldots, d$, are increasing functions, and shows that $\mathbf{X}_T$ is stationary since $\mathbf{G}$ is stationary. We also have

$$P(X_{T,i}(\mathbf{t}) \leqslant x_i) = P(G(\mathbf{t}) \leqslant h_i^{-1}(x_i)) = \Phi(h_i^{-1}(x_i)) = F_i(x) \tag{3}$$

from Eq. (2), so that the coordinates of $\mathbf{X}_T$ and $\mathbf{X}$ have the same marginal distributions.

Property 2. *Moments of any order of* $\mathbf{X}_T$ *can be calculated from*

$$\mu(k_1, \ldots, k_d; \mathbf{t}_1, \ldots, \mathbf{t}_d) = E\left[\prod_{i=1}^{d} X_{T,i}(\mathbf{t}_i)^{k_i}\right] = E\left[\prod_{i=1}^{d} h_i(G_i(\mathbf{t}_i))^{k_i}\right] \tag{4}$$

provided they exist, where $k_1 \geqslant 0, \ldots, k_d \geqslant 0$ *are arbitrary integers.*

The result in Eq. (4) follows from the definition of $\mathbf{X}_T$ in Eq. (1) and the fact that h_i are increasing functions. We note that $\mu(k_1, \ldots, k_d; \mathbf{t}_1, \ldots, \mathbf{t}_d)$ is equal to 0 and 1 for $(k_i = 1, k_j = 0, j \neq i)$ and $(k_i = 2, k_j = 0, j \neq i)$, respectively. The covariance functions of $\mathbf{X}_T$ are given by

$$\begin{aligned} \zeta_{ij}(\mathbf{t}_i - \mathbf{t}_j) &= E[X_{T,i}(\mathbf{t}_i)\, X_{T,j}(\mathbf{t}_j)] = E[h_i(G_i(\mathbf{t}_i))\, h_j(G_j(\mathbf{t}_j))] \\ &= \int_{\mathbb{R}^2} h_i(u) h_j(v) \phi(u, v; \rho_{ij}(\mathbf{t}_i - \mathbf{t}_j))\, \mathrm{d}u\, \mathrm{d}v, \end{aligned} \tag{5}$$

where $\phi(u, v; \rho) = \exp(-(u^2 + v^2 - 2\rho u v)/2)/(2\pi\sqrt{1 - \rho^2})$ denotes the joint density of a two-dimensional Gaussian vector with correlated $N(0, 1)$ coordinates and correlation coefficient ρ. The argument $\mathbf{t}_i - \mathbf{t}_j$ rather than $(\mathbf{t}_i, \mathbf{t}_j)$ is used in Eq. (5) since $\mathbf{X}_T$ is stationary. We note that $\zeta_{ij}(\mathbf{t}_i - \mathbf{t}_j)$ coincides with μ in Eq. (4) for $k_i = k_j = 1$ and $k_q = 0$ for $q \neq i, j$, and that ζ_{ij} is bounded, by the Cauchy–Schwarz inequality and properties of F_i.

Property 3. *If the functions* h_i, $i = 1, \ldots, d$, *are differentiable, then* $\zeta_{ij}(\boldsymbol{\tau})$ *is an increasing function of* $\rho_{ij}(\boldsymbol{\tau})$. *If* $\rho_{ij}(\boldsymbol{\tau}) = 0$, *then* $\zeta_{ij}(\boldsymbol{\tau}) = 0$. *The covariance function* $\zeta_{ij}(\boldsymbol{\tau})$ *takes values in the range* $[\zeta_{ij}^*, \zeta_{ij}^{**}]$, *where*

$$\begin{aligned} \zeta_{ij}^* &= E[h_i(N) h_j(-N)], \\ \zeta_{ij}^{**} &= E[h_i(N) h_j(N)] \end{aligned} \tag{6}$$

and N *denotes a* $N(0, 1)$ *variable.*

Since the functions h_i are differentiable, we have

$$\frac{\partial \zeta_{ij}(\boldsymbol{\tau})}{\partial \rho_{ij}(\boldsymbol{\tau})} = E[h_i'(G_i(\mathbf{t}_i))\, h_j'(G_j(\mathbf{t}_j))] \tag{7}$$

by Price's theorem [4, Section 2.3], where h_i', $i = 1, \ldots, d$, denote the derivatives of h_i. Since h_i are increasing functions, their derivatives are positive, so that the expectation on the right side of Eq. (7) is positive. Hence, $\zeta_{ij}(\boldsymbol{\tau})$ is an increasing function of $\rho_{ij}(\boldsymbol{\tau})$.

That $\zeta_{ij}(\boldsymbol{\tau}) = 0$ for $\rho_{ij}(\boldsymbol{\tau}) = 0$ follows from Eq. (5) since $\phi(u, v; 0) = \phi(u)\phi(v)$, where $\phi(u) = \exp(-u^2/2)\sqrt{2\pi}$. If $\rho_{ij}(\boldsymbol{\tau}) = \pm 1$, then $G_i(\mathbf{t} + \boldsymbol{\tau})$ and $\pm G_j(\mathbf{t})$ are equal to a standard Gaussian variable N in distribution. The range in Eq. (6) results since $\zeta_{ij}(\boldsymbol{\tau})$ increases with $\rho_{ij}(\boldsymbol{\tau})$ and $\rho_{ij}(\boldsymbol{\tau})$ can only take values in $[-1, 1]$.

In applications we may be given the functions (F_i, ζ_{ij}), so that we need to find ρ_{ij}. This inverse problem has no solution if ζ_{ij} takes values outside $[\zeta_{ij}^*, \zeta_{ij}^{**}]$. If ζ_{ij} takes values in $[\zeta_{ij}^*, \zeta_{ij}^{**}]$, we can calculate $\rho_{ij}(\boldsymbol{\tau})$ for each $\zeta_{ij}(\boldsymbol{\tau})$. However, the resulting functions ρ_{ij} are not necessarily covariance functions, for example, one or more functions ρ_{ii} may not be positive definite [1, Section 3.1.1].

Property 4. *If* $\mathbf{G}$ *is continuous almost surely* (*a.s.*) *and the functions* h_i *are continuous*, *then* $\mathbf{X}_T$ *is continuous a.s.*

Since $\mathbf{G}$ is continuous a.s. its samples $G_i(\cdot,\omega)$ are continuous functions for $\omega \in \Omega\backslash\Omega_0$, where $(\Omega,\mathscr{F},P)$ is the probability space on which $\mathbf{G}$ is defined and $P(\Omega_0)=0$. Then $h_i \circ G_i(\cdot,\omega)$ is continuous for $\omega \in \Omega\backslash\Omega_0$ since h_i is continuous by assumption, so that $\mathbf{X}_T$ is continuous a.s.

Property 5. *Let* N_1 *and* N_2 *be independent* $N(0,1)$ *variables. If* $\mathbf{G}$ *is mean square* (*m.s.*) *continuous*, *the functions* h_i *are continuous*, *and* $E[h_i(|N_1|+|N_2|)^2]<\infty$, *then* $\mathbf{X}_T$ *is m.s. continuous.*

Consider an arbitrary coordinate $X_{T,i}(\mathbf{t}_i)=h_i(G_i(\mathbf{t}_i))$ of $\mathbf{X}_T$. We need to show that the m.s. continuity of G_i, that is, the convergence $\rho_{ii}(\boldsymbol{\tau})\to 1$ as $\|\boldsymbol{\tau}\|\to 0$, implies the convergence $\zeta_{ii}(\boldsymbol{\tau})\to 1$ as $\|\boldsymbol{\tau}\|\to 0$. If the limit $\|\boldsymbol{\tau}\|\to 0$ can be taken under the integral in Eq. (5), we have

$$\begin{aligned}\lim_{\|\boldsymbol{\tau}\|\to 0}\zeta_{ii}(\boldsymbol{\tau}) &= \int_{\mathbb{R}^2}\lim_{\|\boldsymbol{\tau}\|\to 0} h_i(\xi\rho_1+\xi_2\rho_2)h_i(\xi\rho_1-\xi_2\rho_2)\phi(\xi_1,\xi_2;0)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\\ &= \int_{\mathbb{R}^2} h_i(\xi_1)^2\phi(\xi_1,\xi_2;0)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 = \int_{\mathbb{R}} h_i(\xi_1)^2\phi(\xi_1)\,\mathrm{d}\xi_1\int_{\mathbb{R}}\phi(\xi_2)\,\mathrm{d}\xi_2 = E[h_i(N)^2]=1 \end{aligned}\tag{8}$$

by the change of variables $(u=\xi\rho_1+\xi_2\rho_2, v=\xi\rho_1-\xi_2\rho_2)$ with $\rho_1=\sqrt{(1+\rho_{ii}(\boldsymbol{\tau}))/2}$ and $\rho_2=\sqrt{(1-\rho_{ii}(\boldsymbol{\tau}))/2}$. The second and the last equalities in Eq. (8) follow by the continuity of h_i and the postulated properties of the distributions F_i, respectively.

That the interchange of the integral and the limit operations performed in Eq. (8) is valid results from the proof of Property 9 in the following section. The proof of this property also shows that the condition $E[h_i(|N_1|+|N_2|)^2]<\infty$ is needed.

3. Parametric translation processes and fields

Consider the sequence of parametric translation models

$$X_{T,i}^{(n)}(\mathbf{t}) = F_i^{-1}\circ\Phi(G_i^{(n)}(\mathbf{t})) = h_i(G_i^{(n)}(\mathbf{t})),\quad i=1,\ldots,d, \tag{9}$$

where $\mathbf{G}^{(n)}=\{G_i^{(n)}\}$, $n=1,2,\ldots,$ is a sequence of stationary Gaussian functions such that $E[G_i^{(n)}(\mathbf{t})]=0$, $E[G_i^{(n)}(\mathbf{t})^2]=1$, and $\rho_{ij}^{(n)}(\boldsymbol{\tau})=E[G_i^{(n)}(\mathbf{t}+\boldsymbol{\tau})G_j^{(n)}(\mathbf{t})]$. As previously, denote by $\boldsymbol{\rho}^{(n)}(\boldsymbol{\tau})=E[\mathbf{G}^{(n)}(\mathbf{t}+\boldsymbol{\tau})\mathbf{G}^{(n)}(\mathbf{t})^T]$ a (d,d)-matrix with entries $\rho_{ij}^{(n)}(\boldsymbol{\tau})$. Truncated Karhunen–Loève or other parametric representations for $\mathbf{G}$ depending on a finite number of random variables can be used to construct the sequence of processes $\mathbf{G}^{(n)}$. We assume the m.s. convergence of $\mathbf{G}^{(n)}$ to $\mathbf{G}$, which implies the convergence $\rho_{ij}^{(n)}(\boldsymbol{\tau})\to\rho_{ij}(\boldsymbol{\tau})$ as $n\to\infty$ for all $\boldsymbol{\tau}\in\mathbb{R}^{d'}$ and all $i,j=1,\ldots,d$.

Parametric translation models as in Eq. (9) have been used to generate samples of non-Gaussian functions [3, Section 5.3.3.1]. However, the relationship between statistics of $\mathbf{X}_T^{(n)}=\{X_{T,i}^{(n)}\}$ and $\mathbf{X}_T=\{X_{T,i}\}$ has not been examined systematically. One of the main objective of this paper is to study the convergence of statistics of $\mathbf{X}_T^{(n)}$ to statistics of $\mathbf{X}_T$ as $n\to\infty$.

We give some useful properties of the sequence of parametric translation processes $\mathbf{X}_T^{(n)}$ in Eq. (9), and establish conditions under which statistics of $\mathbf{X}_T^{(n)}$ converge to statistics of $\mathbf{X}_T$ as $n\to\infty$.

Property 6. *The members of the sequence of parametric translation processes* $\{\mathbf{X}_T^{(n)}\}$ *are stationary for each* n. *Moreover*, $\mathbf{X}_T^{(n)}$ *becomes a version of* $\mathbf{X}_T$ *as* $n\to\infty$.

The finite dimensional distributions of $\mathbf{X}_T^{(n)}$ can be calculated from Eq. (2) with $\mathbf{G}^{(n)}$ in place of $\mathbf{G}$, so that these distributions are equal to $\Phi(\boldsymbol{\xi};\mathbf{r}^{(n)})$, where $\mathbf{r}^{(n)}$ corresponds to the covariance functions $\rho_{ij}^{(n)}$ rather than ρ_{ij}.

The postulated convergence $\|\mathbf{r}^{(n)}-\mathbf{r}\|\to 0$, $n\to\infty$, implies that for $\varepsilon>0$ there exits $n_{ij}(\varepsilon)\geqslant 1$ such that $|r_{ij}^{(n)}-r_{ij}|<\varepsilon$ for $n\geqslant n_{ij}(\varepsilon)$, so that we have $|r_{ij}^{(n)}-r_{ij}|<\varepsilon$ for all i,j and $n\geqslant n(\varepsilon)=\max_{ij}\{n_{ij}(\varepsilon)\}$. Accordingly, $r_{ij}^{(n)}\in(r_{ij}-\varepsilon,r_{ij}+\varepsilon)$ for $n\geqslant n(\varepsilon)$ and some $\varepsilon>0$ such that $-1<r_{ij}-\varepsilon<r_{ij}+\varepsilon<1$ for all i,j. It is assumed without loss of generality that the off-diagonal entries of $\mathbf{r}$ are in the range $(-1,1)$. If an off-diagonal entry of $\mathbf{r}$

is ± 1, the coordinates of the Gaussian vector corresponding to this entry are perfectly correlated so that we can reduce the dimension of $\mathbf{r}$ to eliminate this redundancy.

The matrices $\mathbf{r}^{+} = \{r_{ij} + \varepsilon\}$ and $\mathbf{r}^{-} = \{r_{ij} - \varepsilon\}$ are positive definite provided ε is sufficiently small, so that they are valid covariance matrices. Let $\Phi_c(\boldsymbol{\xi}; \mathbf{r})$ denote the probability that the coordinates of a standard Gaussian vector with covariance matrix $\mathbf{r}$ are larger than the coordinates of $\boldsymbol{\xi}$, for example, $\Phi_c(\boldsymbol{\xi}; \mathbf{r}) = P(N_1 > \xi_1, N_2 > \xi_2)$ for a bivariate standard Gaussian vector (N_1, N_2) with correlation coefficient r_{12}. The inequalities

$$\Phi_c(\boldsymbol{\xi}; \mathbf{r}^{+}) \geqslant \Phi_c(\boldsymbol{\xi}; \mathbf{r}^{(n)}) \geqslant \Phi_c(\boldsymbol{\xi}; \mathbf{r}^{-}) \tag{10}$$

hold by a theorem by Slepian [5]. Since $\varepsilon > 0$ can be made arbitrarily small, we have the convergence $\Phi_c(\boldsymbol{\xi}; \mathbf{r}^{(n)}) \longrightarrow \Phi_c(\boldsymbol{\xi}; \mathbf{r})$ as $n \to \infty$.

Arguments used to obtain the result in Eq. (3) can be used to show that the functions F_i, $i = 1, \ldots, d$, are the marginal distributions of the co-ordinates $X_{T,i}^{(n)}$ of $\mathbf{X}_T^{(n)}$. Hence, the coordinates of $\mathbf{X}_T$ and $\mathbf{X}_T^{(n)}$ have the same marginal distributions for each $n = 1, 2, \ldots$, and the finite dimensional distributions of $\mathbf{X}_T^{(n)}$ converge to those of $\mathbf{X}_T$ as $n \to \infty$, that is, $\mathbf{X}_T^{(n)}$ becomes a version of $\mathbf{X}_T$ for large n.

Property 7. *Moments $\mu^{(n)}(k_1, \ldots, k_d; \mathbf{t}_1, \ldots, \mathbf{t}_d)$ of any order of $\mathbf{X}_T^{(n)}$ can be calculated from Eq.* (4) *with $\mathbf{G}^{(n)}$ in place of $\mathbf{G}$.*

This statement follows directly from the definition of $\mathbf{X}_T^{(n)}$. The covariance functions of $\mathbf{X}_T^{(n)}$ are the moments $\mu^{(n)}(k_1, \ldots, k_d; \mathbf{t}_1, \ldots, \mathbf{t}_d)$ for $k_i = k_j = 1$ and $k_q = 0$ for $q \neq i, j$, and are given by (Eq. (5))

$$\begin{aligned} \zeta_{ij}^{(n)}(\mathbf{t}_i - \mathbf{t}_j) &= E[X_{T,i}^{(n)}(\mathbf{t}_i) X_{T,j}^{(n)}(\mathbf{t}_j)] = E[h_i(G_i^{(n)}(\mathbf{t}_i)) h_j(G_j^{(n)}(\mathbf{t}_j))] \\ &= \int_{\mathbb{R}^2} h_i(u) h_j(v) \phi(u, v; \rho_{ij}^{(n)}(\mathbf{t}_i - \mathbf{t}_j))\, \mathrm{d}u\, \mathrm{d}v. \end{aligned} \tag{11}$$

The expression of $\zeta_{ij}^{(n)}$ shows that $X_{T,i}^{(n)}$ is weakly stationary, in agreement with the previous property.

Property 8. *If the functions h_i, $i = 1, \ldots, d$, are differentiable, then $\zeta_{ij}^{(n)}$ are increasing functions of $\rho_{ij}^{(n)}$, and cannot take values outside the range $[\zeta_{ij}^{*}, \zeta_{ij}^{**}]$ given by Eq.* (6).

The arguments used to prove the third property in the previous section apply directly here since Eq. (7) is valid with $\mathbf{G}^{(n)}$ in place of $\mathbf{G}$.

Property 9. *Let N_1 and N_2 be independent $N(0,1)$ variables. If $E[|h_i(|N_1| + |N_2|) h_j(|N_1| + |N_2|)|] < \infty$, $i, j = 1, \ldots, d$, and h_i are continuous functions, then the covariance functions $\zeta_{ij}^{(n)}(\boldsymbol{\tau})$ of $\mathbf{X}_T^{(n)}$ converge to the covariance functions $\zeta_{ij}(\boldsymbol{\tau})$ of $\mathbf{X}_T$ as $n \to \infty$ for all $\boldsymbol{\tau} \in \mathbb{R}^{d'}$ and all $i, j = 1, \ldots, d$.*

A direct consequence of this result is that the second-moment properties of $\mathbf{X}_T$ can be approximated by the second-moment properties of $\mathbf{X}_T^{(n)}$ for a sufficiently large n.

The covariance functions $\zeta_{ij}^{(n)}$ in Eq. (11) can be expressed in the form

$$\begin{aligned} \zeta_{ij}^{(n)}(\boldsymbol{\tau}) &= \int_{\mathbb{R}^2} h_i(\xi_1 \beta_1^{(n)} + \xi_2 \beta_2^{(n)}) h_j(\xi_1 \beta_1^{(n)} - \xi_2 \beta_2^{(n)}) \phi(\xi_1, \xi_2; 0)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \\ &= \int_{\mathbb{R}^2} g_{ij}^{(n)}(\xi_1, \xi_2)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \end{aligned} \tag{12}$$

by using the change of variables

$$\begin{aligned} u &= \xi_1 \sqrt{(1 + \rho_{ij}^{(n)})/2} + \xi_2 \sqrt{(1 - \rho_{ij}^{(n)})/2} = \xi_1 \beta_1^{(n)} + \xi_2 \beta_2^{(n)}, \\ v &= \xi_1 \sqrt{(1 + \rho_{ij}^{(n)})/2} - \xi_2 \sqrt{(1 - \rho_{ij}^{(n)})/2} = \xi_1 \beta_1^{(n)} - \xi_2 \beta_2^{(n)}. \end{aligned} \tag{13}$$

Since h_i are increasing functions, $\beta_k^{(n)} \in [0,1]$, and $\xi_1\beta_1^{(n)} \pm \xi_2\beta_2^{(n)} \leqslant |\xi_1| + |\xi_2|$, we have $h_i(\beta_1^{(n)}\xi_1 \pm \beta_2^{(n)}\xi_2) \leqslant |h_i(|\xi_1| + |\xi_2|)|$, so that

$$|g_{ij}^{(n)}(\xi_1, \xi_2)| \leqslant |h_i(|\xi_1| + |\xi_2|)h_j(|\xi_1| + |\xi_2|)|\phi(\xi_1, \xi_2; 0) = q_{ij}(\xi_1, \xi_2), \quad n \geqslant 1. \tag{14}$$

We note that the functions $q_{ij}(\xi_1, \xi_2)$ are Lebesgue integrable in $\mathbb{R}^2$ by assumption. The inequality in Eq. (14) implies that the functions $g_{ij}^{(n)}(\xi_1, \xi_2)$ are also Lebesgue integrable in $\mathbb{R}^2$. Let $g_{ij}(\xi_1, \xi_2)$ be the function $g_{ij}^{(n)}(\xi_1, \xi_2)$ with $\rho_{ij}(\boldsymbol{\tau})$ in place of $\rho_{ij}^{(n)}(\boldsymbol{\tau})$. The continuity of the functions h_i and the convergence $\rho_{ij}^{(n)}(\boldsymbol{\tau}) \to \rho_{ij}(\boldsymbol{\tau})$ imply the convergence $g_{ij}^{(n)}(\xi_1, \xi_2) \to g_{ij}(\xi_1, \xi_2)$ as $n \to \infty$. These properties yield

$$\zeta_{ij}^{(n)}(\boldsymbol{\tau}) = \int_{\mathbb{R}^2} g_{ij}^{(n)}(\xi_1, \xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \longrightarrow \int_{\mathbb{R}^2} g_{ij}(\xi_1, \xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 = \zeta_{ij}(\boldsymbol{\tau}), \quad n \to \infty, \tag{15}$$

by Lebesgue's dominated convergence theorem [6, Theorem 4.3.5].

Property 10. *If the processes* $\mathbf{G}^{(n)}$ *are a.s. and m.s. continuous and the functions* h_i *are continuous, then* $\mathbf{X}_T^{(n)}$ *are a.s. and m.s. continuous, respectively.*

Arguments used to prove the fourth and fifth properties in the previous section can be applied directly to prove this property.

4. Monte Carlo algorithm

Let $\mathbf{X}_T$ be the translation random function in Eq. (1) specified completely by the marginal distributions F_i, $i = 1, \ldots, d$, and the covariance functions ρ_{ij} of its Gaussian image $\mathbf{G}$. Let $\mathbf{X}_T^{(n)}$, $n = 1, 2, \ldots$, be the sequence of parametric translation models in Eq. (9), where $\mathbf{G}^{(n)}$ converges in mean square to $\mathbf{G}$. We have seen that the second-moment properties and the finite dimensional distribution of $\mathbf{X}_T^{(n)}$ converge to those of $\mathbf{X}_T$ as $n \to \infty$.

Our objective is to generate samples of $\mathbf{X}_T$. Since it is not possible to generate samples of $\mathbf{X}_T$ but samples of $\mathbf{X}_T^{(n)}$ can be generated, we use samples of $\mathbf{X}_T^{(n)}$ as a substitute for samples of $\mathbf{X}_T$. Theoretical considerations in the previous sections show that statistics of $\mathbf{X}_T$ can be approximate by corresponding statistics of $\mathbf{X}_T^{(n)}$, which can be estimated from samples of $\mathbf{X}_T^{(n)}$ for a sufficiently large n.

The following two-step Monte Carlo algorithm can be used to generate samples of $\mathbf{X}_T^{(n)}$. The algorithm is based on the definition on $\mathbf{X}_T^{(n)}$ in Eq. (9).

Step 1: Generate samples $\mathbf{G}^{(n)}(\mathbf{t}, \omega)$ of the Gaussian image $\mathbf{G}^{(n)}$ of $\mathbf{X}_T^{(n)}$. There are many algorithms for generating samples of stationary Gaussian functions [3, Section 5.3.1]. An algorithm based on the spectral representation method is used in Example 1 of the following section to generate samples of $\mathbf{G}^{(n)}$ taking values in $\mathbb{R}^2$ (Eq. (21)).

Step 2: The mapping of the samples of $\mathbf{G}^{(n)}$ generated in Step 1 into samples of $\mathbf{X}_T^{(n)}$ is given by Eq. (9). For example, let $\mathbf{G}^{(n)}(\mathbf{t}, \omega)$ be a sample of $\mathbf{G}^{(n)}$. The corresponding sample of $\mathbf{X}_T^{(n)}$ can be calculated from

$$X_{T,i}^{(n)}(\mathbf{t}, \omega) = F_i^{-1} \circ \Phi(\mathbf{G}^{(n)}(\mathbf{t}, \omega)) = h_i(\mathbf{G}^{(n)}(\mathbf{t}, \omega)). \tag{16}$$

Generally, the transformation in Eq. (16) is not available analytically, so that it has to be performed numerically. Two MATLAB functions can be used to calculate samples of $X_{T,i}^{(n)}$ from samples of $\mathbf{G}^{(n)}$:

- The **cdf** MATLAB function,

  $$\Phi(G_i^{(n)}(\mathbf{t}, \omega)) = \mathrm{cdf}(\text{'normal'}, G_i^{(n)}(\mathbf{t}, \omega), 0, 1),$$

  performs the mapping $\mathbf{G}^{(n)}(\mathbf{t}, \omega) \mapsto \Phi(\mathbf{G}^{(n)}(\mathbf{t}, \omega))$.
- The **icdf** MATLAB function,

  $$X_{T,i}^{(n)}(\mathbf{t}, \omega) = F_i^{-1} \circ \Phi(\mathbf{G}^{(n)}(\mathbf{t}, \omega)) = \mathrm{icdf}(\text{'name'}, \Phi(G_i^{(n)}(\mathbf{t}, \omega)), a, b, \ldots),$$

performs the mapping $\mathbf{G}^{(n)}(\mathbf{t},\omega)\mapsto X_{T,i}^{(n)}(\mathbf{t},\omega)$. The use of these MATLAB functions is demonstrated in Example 1 of the following section.

5. Numerical examples

Two examples are presented. The first examples illustrates the convergence of the covariance functions of a sequence of parametric translation processes to those of a target translation process. The second example examines the convergence of responses of a linear oscillator to a sequence of parametric Gaussian and translation input processes.

Example 1. Let $\mathbf{X}_T$ be an $\mathbb{R}^2$-valued translation process with coordinates $X_{T,1}$ and $X_{T,2}$ following lognormal and beta distributions of mean 0 and variance 1, that is,

$$X_{T,1}(t) = -1.272 + \exp(0.6937 G_1(t)),$$

$$X_{T,2}(t) = -3 + 5F^{-1} \circ \Phi(G_2(t)), \tag{17}$$

where F is a beta distribution with range $(-3,2)$ and parameters $(\gamma=3, \eta=2)$ [7, Chapter 24], and the image $\mathbf{G}$ of $\mathbf{X}_T$ is an $\mathbb{R}^2$-valued stationary Gaussian process with coordinates (G_1, G_2) of mean 0 and variance 1. The covariance and one-sided spectral density functions of $\mathbf{G}$ are

$$\rho_{kl}(\tau) = E[G_k(t+\tau)G_l(t)] = (1-\lambda)\frac{\sin(\bar{v}_k\tau)}{\bar{v}_k\tau}\delta_{kl} + \lambda\frac{\sin(\bar{v}\tau)}{\bar{v}\tau}, \quad k,l=1,2 \tag{18}$$

and

$$g_{kl}(v) = \frac{1-\lambda}{\bar{v}_k}\delta_{kl}1(0<v<\bar{v}_k) + \frac{\lambda}{\bar{v}}1(0<v<\bar{v}), \tag{19}$$

respectively, where $\bar{v}_k, \bar{v}>0$ and $\lambda \in [0,1]$ are some constants, $1(A)=1$ or 0 if A is, respectively, true or false, and $\delta_{kl}=1$ and 0 for $k=l$ and $k\neq l$, respectively. The covariance functions ζ_{kl}, $k,l=1,2$, of $\mathbf{X}_T$ can be calculated from Eqs. (5), (17) and (18).

We note that the processes (G_1, G_2) with the above properties can be given by

$$G_k(t) = \sqrt{1-\lambda}Z_k(t) + \sqrt{\lambda}Z(t), \quad k=1,2, \tag{20}$$

where Z_k and Z are independent band limited Gaussian white noise processes with mean 0, variance 1, and frequency band $(0,\bar{v}_k)$ and $(0,\bar{v})$, respectively.

We construct a sequence of parametric translation processes $\mathbf{X}_T^{(n)}$, $n=1,2,\ldots$, converging to $\mathbf{X}_T$ by using the spectral representation theorem for weakly stationary stochastic processes [3, Section 5.3.1.1]. Let

$$Z_k^{(n)}(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(A_{k,i}^{(n)}\cos(v_{k,i}^{(n)}t) + B_{k,i}^{(n)}\sin(v_{k,i}^{(n)}t)),$$

$$Z^{(n)}(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(A_i^{(n)}\cos(v_i^{(n)}t) + B_i^{(n)}\sin(v_i^{(n)}t)), \tag{21}$$

be approximations of Z_k and Z corresponding to partitions of the frequency ranges $(0,\bar{v}_k)$ and $(0,\bar{v})$ in n equal intervals, where $A_{k,i}^{(n)}$, $B_{k,i}^{(n)}$, $A_i^{(n)}$, and $B_i^{(n)}$ are independent $N(0,1)$ variables, $v_{k,i}=(i-1/2)\bar{v}_k/n$, $i=1,\ldots,n$, and $v_i=(i-1/2)\bar{v}/n$, $i=1,\ldots,n$. Let

$$G_k^{(n)}(t) = \sqrt{1-\lambda}Z_k^{(n)}(t) + \sqrt{\lambda}Z^{(n)}(t), \quad k=1,2, \tag{22}$$

define a sequence $\mathbf{G}^{(n)} = (G_1^{(n)}, G_2^{(n)})$ of parametric stationary Gaussian processes. The limit of $\mathbf{G}^{(n)}$ as $n\to\infty$ is a process with the same probability law as $\mathbf{G}$, so that $\mathbf{G}^{(n)}$ becomes a version of $\mathbf{G}$ as $n\to\infty$ [3, Section 5.3.1.1]. The corresponding sequence of parametric translation models $\mathbf{X}_T^{(n)}$ is given by Eq. (17) with $\mathbf{G}^{(n)}$ in place of $\mathbf{G}$.

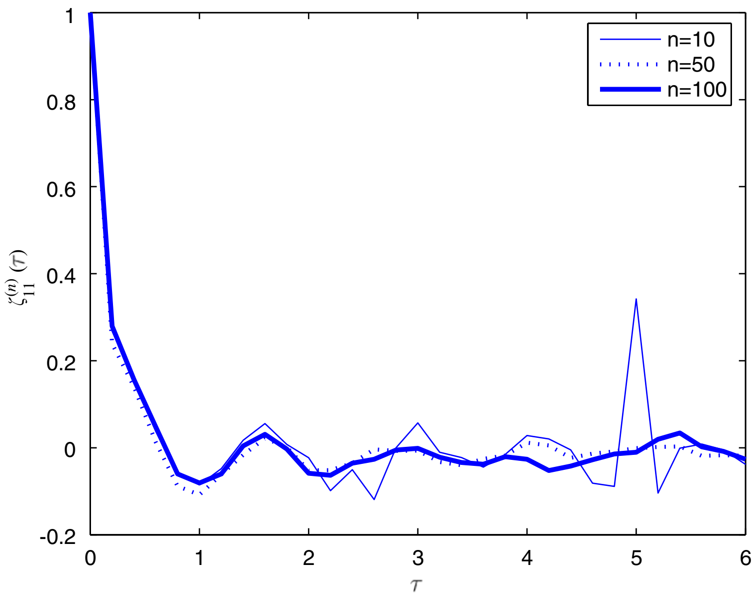

Fig. 1. Estimates of the covariance functions for $\zeta_{11}^{(n)}$ for $n = 10$ (——), $n = 50$ (········) and $n = 100$ (▬▬).

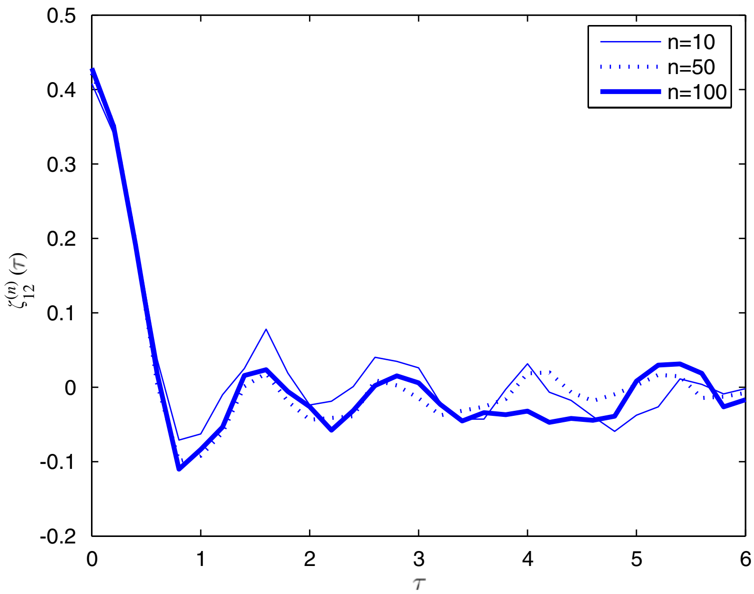

Fig. 2. Estimates of the covariance functions for $\zeta_{12}^{(n)}$ for $n = 10$ (——), $n = 50$ (········) and $n = 100$ (▬▬).

We note that the coordinates of $\mathbf{X}_T^{(n)}$ and $\mathbf{X}_T$ have the same marginal distributions by construction. In Figs. 1–3, are shown estimates of the covariance functions $\zeta_{kl}^{(n)}$, $k, l = 1, 2$, of $\mathbf{X}_T^{(n)}$ obtained from 500 independent samples of this process for $\bar{v}_k = 25$, $\bar{v} = 5$, and $\lambda = 0.3$. There are notable differences between the estimates of $\zeta_{kl}^{(n)}$ corresponding to $n = 10$ and 100. Estimates of $\zeta_{kl}^{(n)}$ for $n \geqslant 100$ are practically indistinguishable from those for $n = 100$ indicating the convergence $\zeta_{kl}^{(n)}(\tau) \to \zeta_{kl}(\tau)$ as $n \to \infty$ in agreement with a Property 9 proved in a previous section for parametric translation models. The results plotted in Figs. 1–3 show that the parametric translation models $\mathbf{X}_T^{(n)}$ with $n \geqslant 100$ considered in this example approximate satisfactorily the second-moment properties of the target translation process $\mathbf{X}_T$. The results also provide useful information on

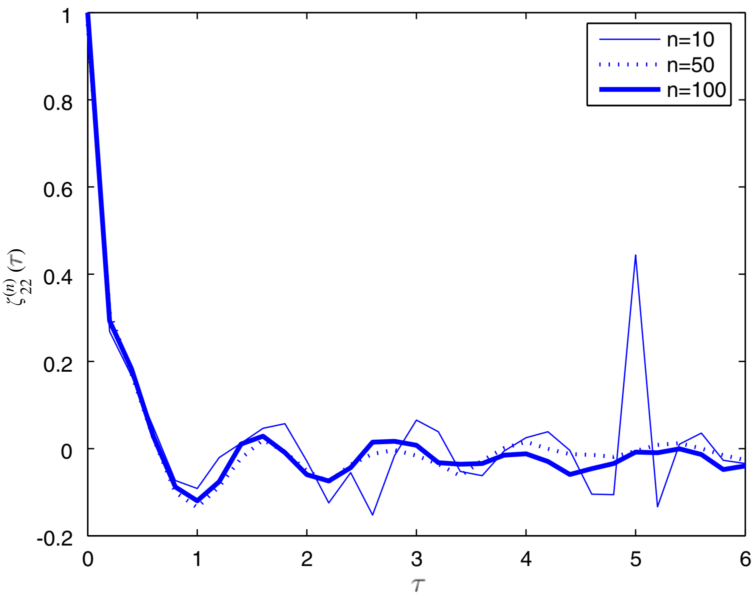

Fig. 3. Estimates of the covariance functions for $\zeta_{22}^{(n)}$ for $n=10$ (——), $n=50$ (········) and $n=100$ (▬▬).

the rate of convergence of the approximating sequence of processes $\mathbf{X}_T^{(n)}$. They show that the covariance functions $\zeta_{kl}^{(n)}(\tau)$ converge faster to the target covariance functions $\zeta_{kl}(\tau)$ for smaller rather than larger time lags, for example, $\zeta_{kl}^{(n)}(\tau) \simeq \zeta_{kl}(\tau)$ for $n=10$ and $\tau \leqslant 1$, but $\zeta_{kl}^{(n)}(\tau)$ with $n=10$ is an unsatisfactory approximation for $\zeta_{kl}(\tau)$ if $\tau > 1$.

The two-step Monte Carlo algorithm in the previous section has been applied to generate samples of $\mathbf{X}_T^{(n)}$. Samples of $\mathbf{G}^{(n)}$ have been calculated from Eqs. (21) and (22) with $A_{k,i}^{(n)}$, $B_{k,i}^{(n)}$, $A_i^{(n)}$, and $B_i^{(n)}$ replaced by independent samples of $N(0,1)$. The mapping in Eq. (17) has been used to obtain samples of $\mathbf{X}_T^{(n)}$ from samples of $\mathbf{G}^{(n)}$. The first equality in Eq. (17) is elementary and has been used directly. The second equality in this equation is not available in analytical form, so that the MATLAB function $-3+5$ icdf('beta',$\Phi(G_2^{(n)})(\mathbf{t},\omega)$, γ,η) was used to map samples of $G_2^{(n)}$ into samples of $X_{T,2}^{(n)}$.

Example 2. Let G_1 and $X_{T,1}$ be the processes in Eqs. (17) and (20) with $\lambda = 0$. Denote by D_G and D_T the displacement of a linear oscillator with natural frequency $v_0 = 3$ and damping ratio $\xi = 0.05$ subjected to G_1 and $X_{T,1}$, respectively. These displacement processes satisfy the differential equations

$$\ddot{D}_G(t) + 2\xi v_0 \dot{D}_G(t) + v_0^2 2 D_G(t) = G_1(t),$$

$$\ddot{D}_T(t) + 2\xi v_0 \dot{D}_T(t) + v_0^2 D_T(t) = X_{T,1}(t). \tag{23}$$

The stationary responses D_G and D_T have mean 0 but different covariance functions because the covariance functions of G_1 and $X_{T,1}$ do not coincide. Generally, the difference between the second-moment properties of $X_{T,1}$ and G_1 is not significant [1, Section 3.1.1], so that the first two moments of D_G and D_T are similar.

Consider first the process D_G, and let $D_G^{(n)}$ be the oscillator displacement to the sequence of parametric processes $G_1^{(n)}$ in Eq. (22) converging to G_1. The one-sided spectral density of the stationary response D_G is

$$g_{D_G}(v) = \frac{1/\bar{v}_1}{(v^2 - v_0^2)^2 + (2\xi v v_0)^2} 1(0 < v < \bar{v}_1) = |k(v)|^2 (1/\bar{v}_1) 1(0 < v < \bar{v}_1), \tag{24}$$

so that the variance σ_G^2 of D_G and the variance $\sigma_{\dot{D}_G}^2$ of $\dot{D}_G$ are

$$\sigma_{D_G}^2 = \int_0^{\bar{v}_1} g_{D_G}(v)\,\mathrm{d}v \simeq \frac{\pi/\bar{v}_1}{4\xi v_0^3},$$

$$\sigma_{\dot{D}_G}^2 = \int_0^{\bar{v}_1} v^2 g_{D_G}(v)\,\mathrm{d}v \simeq \frac{\pi/\bar{v}_1}{4\xi v_0}, \tag{25}$$

where the above approximations hold if $v_0 \ll \bar{v}_1$ and the damping ratio ξ is relatively small [8, Section 5.2.1]. The mean x-upcrossing rate of D_G can be calculated from

$$\mu_G(x) = \frac{\sigma_{\dot{D}_G}}{2\pi\sigma_{D_G}} \exp\left(-\frac{x^2}{2\sigma_{D_G}^2}\right). \tag{26}$$

The mean x-upcrossing rate of $D_G^{(n)}$ is

$$\mu_G^{(n)}(x) = \frac{\sigma_{\dot{D}_G}^{(n)}}{2\pi\sigma_{D_G^{(n)}}} \exp\left(-\frac{x^2}{2\sigma_{D_G^{(n)}}^2}\right), \tag{27}$$

where

$$\sigma_{D_G^{(n)}}^2 = \frac{1}{n}\sum_{i=1}^{n} |k(v_{1,i}^{(n)})|^2,$$

$$\sigma_{\dot{D}_G}^{(n)^2} = \frac{1}{n}\sum_{i=1}^{n} (v_{1,i}^{(n)})^2 |k(v_{1,i}^{(n)})|^2, \tag{28}$$

because the one-sided spectral density of $G_1^{(n)}$ is $g_{D_G^{(n)}}(v) = (1/n)\sum_{i=1}^{n}\delta(v - v_{1,i}^{(n)})$. In Fig. 4 are shown the mean x-upcrossing rates $\mu_G(x)$ and $\mu_G^{(n)}(x)$ for several values of n, where x is measured in standard deviation σ_{D_G} of D_G. The mean x-upcrossing rates $\mu^{(n)}(x)$ for $n = 10$ and 20 (bottom and top plots, respectively) differ significantly from $\mu_G(x)$, but they nearly coincide with $\mu(x)$ for $n \geqslant 100$. Since $\mu^{(n)}(x)$ can either overestimate or underestimate $\mu_G(x)$ significantly for small values of n, relatively large values of n are recommended.

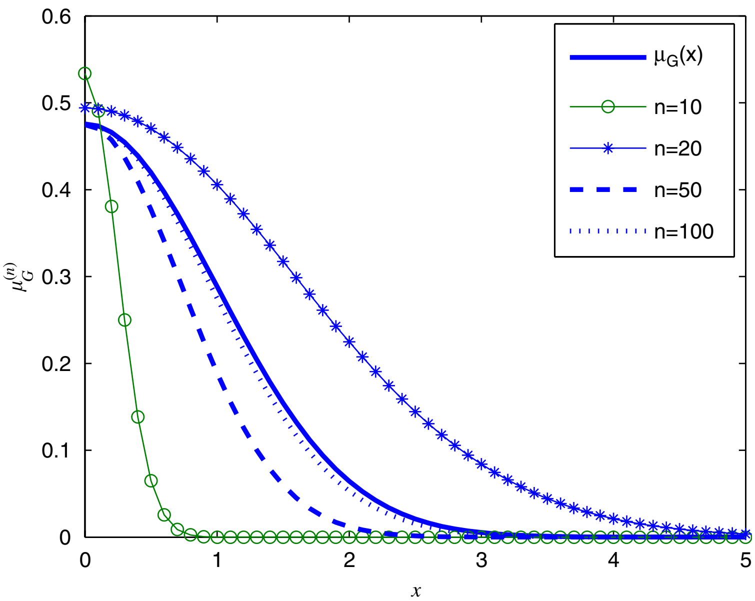

Fig. 4. Mean x-upcrossing rates μ_G (▬▬) of D_G and $\mu_G^{(n)}$ of $D_G^{(n)}$ for $n = 10$ (—o—), $n = 20$ (—✳—), $n = 50$ (▬ ▬) and $n = 100$ (·········).

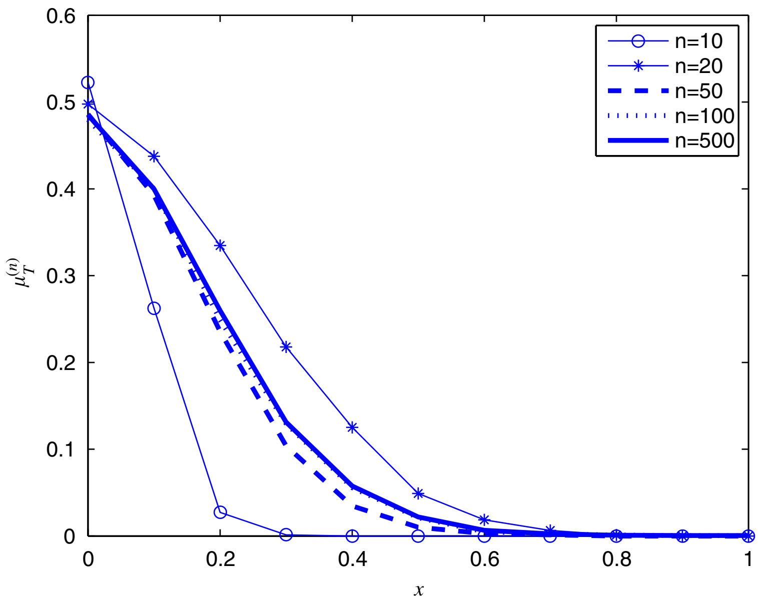

Fig. 5. Estimates of the mean x-upcrossing rates of $\mu_T^{(n)}$ of $D_T^{(n)}$ for $n = 10$ (—o—), $n = 20$ (—*—), $n = 50$ (▬ ▬), $n = 100$ (·········) and $n = 500$ (▬▬).

Consider now the process D_T, and let $D_T^{(n)}$ be the oscillator displacement to the sequence of parametric translation models $X_{T,1}^{(n)}$ converging to $X_{T,1}$. It is not possible to calculate the mean x-upcrossing rates $\mu_T(x)$ and $\mu_T^{(n)}(x)$ for D_T and $D_T^{(n)}$ analytically. However, estimates of $\mu_T^{(n)}(x)$ can be obtained simply from samples of $D_T^{(n)}$. In Fig. 5 is shown the estimates of $\mu_T^{(n)}(x)$ for $n = 10$, 20, 50, 100, and 500 derived from 500 independent samples of $D_T^{(n)}$ each of length 50. The results are plotted against the actual threshold x because the variance of D_T is not known exactly. The estimates of $\mu_T^{(n)}(x)$ exhibit similar behavior to the mean upcrossing rates shown in Fig. 4. They differ significantly for small values of n, for example, $\mu_T^{(n)}(x)$ for $n = 10$ and 20 (bottom and top plots, respectively). The estimates of $\mu_T^{(n)}(x)$ become stable for $n \geqslant 50$. These results indicate the need to use a sufficiently large n in applications to avoid unreliable approximations for the mean x-upcrossing rate μ_T of D_T.

We note that the mean square (m.s.) convergence of $D_G^{(n)}$ and $D_T^{(n)}$ to D_G and D_T, respectively, as $n \to \infty$ follows from the m.s. convergence $G^{(n)} \xrightarrow{\text{m.s.}} G$ and $X_{T,1}^{(n)} \xrightarrow{\text{m.s.}} X_{T,1}$ because the oscillator displacement is the image of the input by a linear and bounded operator $\mathscr{L} : L_2(\Omega, \mathscr{F}, P) \to L_2(\Omega, \mathscr{F}, P)$, for example, $D_T^{(n)} = \mathscr{L}[X_{T,1}^{(n)}]$. Accordingly, we have

$$\|D_T^{(n)} - D_T\| = \|\mathscr{L}[X_{T,1}^{(n)}] - \mathscr{L}[X_{T,1}]\| \leqslant \|\mathscr{L}\| \|X_{T,1}^{(n)} - X_{T,1}\|, \tag{29}$$

where $\|\mathscr{L}\|$ is the norm of $\mathscr{L}$ and $\|\cdot\|$ denotes the norm in $L_2(\Omega, \mathscr{F}, P)$. Hence, the convergence $X_{T,1}^{(n)} \xrightarrow{\text{m.s.}} X_{T,1}$ implies the m.s. convergence of $D_T^{(n)}$ to D_T. A similar result holds for the responses $D_G^{(n)}$ and D_G to the Gaussian input processes $G^{(n)}$ and G. Moreover, we also have the convergence of the finite dimensional distributions of $D_G^{(n)}$ to those of D_G since Gaussian processes are completely specified by their second-moment properties. The results in Fig. 5 showing that the mean x-upcrossing rate $\mu_T^{(n)}(x)$ approaches a limit value suggest that the convergence of $D_T^{(n)}$ to D_T is stronger than the m.s. convergence established above based on the inequality in Eq. (29). This remark is beyond the scope of this study and was not examined.

6. Conclusions

Numerical calculations involving processes and fields cannot be performed directly since, generally, these random functions consist of infinite families of random functions. Processes and fields need to be approximated for calculations by deterministic functions depending on time and/or space arguments and finite families of random variables, referred to as parametric representations. There are numerous parametric representations for Gaussian functions defined, for example, by finite sums of deterministic functions of time and/or space with Gaussian coefficients. It is shown that parametric representations can also be developed for non-Gaussian stationary functions.

A two-step algorithm has been proposed for constructing parametric representations for non-Gaussian stationary functions, referred to as parametric translation models. First, a target non-Gaussian stationary function specified partially by its second-moment properties and marginal distribution needs to be approximated by a translation function, provided it exists. The translation function, referred to as target translation function, has the specified second-moment properties and marginal distribution. Second, sequences of parametric translation models approximating the target translation function need to be defined. It was shown that parametric translation models can match any marginal distribution. Conditions have been established for the convergence of the second-moment properties and finite dimensional distributions of the sequence of parametric translation models to the corresponding characteristics of the target translation function. Two numerical examples have been presented to illustrate some of the properties of parametric translation models, the convergence of moments and of other statistics of parametric translation models to target statistics, and the response statistics for the response of a linear oscillator subjected to parametric Gaussian and translation input models.

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