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Elliptic balance solution to two degree of freedom, undamped, homogeneous systems having cubic nonlinearities

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Abstract

The application of the elliptic balance method to the solution of undamped, two degree of freedom homogeneous nonlinear systems is described. This method uses Jacobian elliptic functions in the balance and is based on the concept of averaging with respect to complete elliptic integrals of the first kind. To assess the accuracy of the approximate solution thus obtained, we consider the motion of a linear vibration absorber attached to a rigid body that is supported symmetrically by incompressible, homogeneous and isotropic hyperelastic shear blocks. It is shown that the amplitude–time response of the model system is well predicted by the elliptic balance method solution even for relatively large parameter values.

1. Introduction

Many researchers have investigated various approximate solutions for homogeneous nonlinear, second order, ordinary differential equations by use of perturbation methods involving elliptic functions [1–19]. The methods include, for example, elliptic harmonic balance [5], elliptic Krylov–Bogoliubov averaging [1,7,10], and certain other elliptic averaging procedures [9]. Of course, each of these diverse methods has its own advantages and exhibits its own analytical complexities exposed in the aforementioned sampling of papers. Most of these works are focused on only first-order approximate solutions of the perturbed elliptic function variety. Nevertheless, it is generally acknowledged among experts that the accuracy of elliptic function perturbation methods in the solution of such equations, particularly those of the perturbed Duffing oscillator type, is greater than that based on parallel methods that use circular functions.

Specifically, Barkham and Soudack [1,2] were the first to use the elliptic function perturbation method as an extension of the Krylov–Bogoliubov (K–B) averaging method [20] in which Jacobian elliptic functions are used to approximate the solution for a perturbed Duffing oscillator equation. They find improved accuracy of

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the solution phase for both weak and strong nonlinear homogeneous systems in comparison with the usual Krylov–Bogoliubov method. The same approximation technique was extended in Refs. [3,4] to allow the modulus of the Jacobian elliptic function to depend on time. This led to increased accuracy of the method for certain perturbed Duffing type oscillators. The extension of the application of Jacobian elliptic function methods to obtain approximate solutions of several classes of nonlinear problems, including certain wave propagation problems, subsequently was studied by a number of other workers [5–27].

It is also well known that the homogeneous Duffing equation has an exact solution in terms of Jacobian elliptic functions, whereas a coupled, two degree of freedom homogeneous system having cubic nonlinearities of the Duffing type has no known exact solution. For corresponding coupled linear oscillators, on the other hand, we recall that for each unknown function one adopts certain familiar sinusoidal solutions and then by balance finds relations between the modal frequencies and amplitudes in reaching a closed form result. Similarly, therefore, to obtain a solution of the more complex problem of coupled oscillators with cubic nonlinearities, it is useful to consider a similar solution in terms of Jacobian elliptic functions. With this idea and familiar balancing techniques for linear oscillators in mind, but now applied to Jacobian elliptic functions, the method of elliptic balance is applied here in a perturbation process that ultimately involves averaging over complete elliptic integrals. This elliptic balance method or procedure has been demonstrated on single degree of freedom systems in several papers noted in the references. Furthermore, Elías-Zúñiga obtained the approximate solution of a damped, nonlinear two degree of freedom system by using Jacobian elliptic functions in conjunction with the method of averaging where the undetermined parameters of the proposed solution were assumed to be slowly varying functions of time and for null initial velocity data [28]. Of course, results for the undamped case are included as a special case when damping is absent. In this work, however, we adopt a different approximation for which the parameters are independent of time and we consider general initial data. The results are illustrated in the important, though somewhat easier problem of a linear vibration absorber controlling the free, undamped motion of a load supported by nonlinear shear supports of the Duffing type [29]. It is found that this method yields very accurate solutions, virtually indistinguishable from the numerical solution of the system of equations. Needless to say, other perturbation techniques that use multiple scales or the Krylov–Bogoliubov averaging method, among others, are useful, perhaps somewhat simpler perturbation techniques, but these do not admit the exact solution for the homogeneous Duffing equation as a special reduced case when either one of the oscillators is fixed and the other is released from rest, for example. The elliptic balance method captures this special exact solution of the homogeneous Duffing type oscillator. For additional comparison, however, we also exhibit approximate results based on the method of multiple scales. Of course, the quality of comparison with different procedures often depends on the range of values assigned to the various problem parameters, not all of which need be small. We find that the amplitude–time response of the model system is nicely predicted by the elliptic balance method solution even for relatively large parameter values.

2. Equations of motion

It is well-known that governing equations of motion with cubic nonlinearities arise in many physical systems such as the vibrations of strings, beams, membranes, plates with significant stretching, dynamic vibration-isolation systems, dynamic vibration absorbers, and so on, see, for example, Refs. [30,31]. The general motion of these physical systems having cubic nonlinearities and negligible damping is governed by a system of two homogeneous, ordinary differential equations, namely

$$\left\{\begin{matrix} \ddot{u}_1 \\ \ddot{u}_2 \end{matrix}\right\} + \left\{\begin{matrix} \omega_{n1}^2 & 0 \\ 0 & \omega_{n2}^2 \end{matrix}\right\}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \varepsilon\left\{\begin{matrix} \varphi_1 u_1^3 + \varphi_2 u_1^2 u_2 + \varphi_3 u_1 u_2^2 + \varphi_4 u_2^3 \\ \varphi_5 u_1^3 + \varphi_6 u_1^2 u_2 + \varphi_7 u_1 u_2^2 + \varphi_8 u_2^3 \end{matrix}\right\}, \tag{1}$$

where a superposed dot denotes the derivative with respect to the time τ, u_1 and u_2 are finite modal amplitudes, ω_{n1} and ω_{n2} are natural frequencies of the system, ε is a small, positive dimensionless parameter, and $\varphi_1, \ldots, \varphi_8$ are certain parameters related to the physics of the system. We show that the elliptic balance method developed here delivers a general approximate solution for physical systems characterized by nonlinear equations of the canonical form shown in Eq. (1).

2.1. The elliptic balance method

The approximate solution of Eq. (1), by use of the elliptic balance method is developed in terms of the Jacobian elliptic functions *cn*, *sn,* and *dn*. The approximate solution will be obtained by following the same idea used in the method of harmonic balance [30,32], but instead of balancing trigonometric functions, we now balance Jacobian elliptic functions. This will require use of several identities recalled as the need arises.

To study the solution of the system described by Eq. (1) by application of Jacobian elliptic functions, we shall assume the general approximate solution to be of the form

$$u_1 = a_1 cn(\omega_1\tau, k_1^2) + a_3 sn(\omega_3\tau, k_3^2), \tag{2}$$

$$u_2 = a_2 cn(\omega_2\tau, k_2^2) + a_4 sn(\omega_4\tau, k_4^2), \tag{3}$$

where ω_j, k_j, and a_j are constants to be found. Note that we have assumed that the moduli k_j of the Jacobian elliptic functions as well as the frequencies ω_j are independent of time. For simplicity in the notation, let us write the Jacobian elliptic functions in Eqs. (2) and (3) as $cn_j \equiv cn(\omega_j\tau, k_j^2)$, for $j = 1, 2, 3, 4$. Similar notation is defined for the functions sn_j and dn_j. Substitution of Eqs. (2) and (3) into Eq. (1) and introduction of the well-known relations [33]

$$sn_j^2 + cn_j^2 = 1, \quad dn_j^2 + k_j^2 sn_j^2 = 1 \tag{4}$$

yields

$$\begin{aligned} &a_1 cn_1[\Delta_1 - \varepsilon\varphi_3 a_4^2 sn_4^2 - 3\varepsilon\varphi_1 a_2^2 sn_2^2 - \varepsilon\varphi_3 a_3^2 cn_3^2] - a_1 cn_1^3(2\omega_1^2 k_1^2 + \varepsilon\varphi_1 a_1^2) \\ &\quad + a_2 sn_2[\omega_{n1}^2 - 3\varepsilon\varphi_1 a_1^2 cn_1^2 - \omega_2^2(1 + k_2^2) - \varepsilon\varphi_3 a_3^2 cn_3^2 - \varepsilon\varphi_3 a_4^2 sn_4^2] - \varepsilon\varphi_4 a_3^3 cn_3^3 \\ &\quad - \varepsilon a_3 cn_3[\varphi_2 a_2^2 sn_2^2 + 3\varphi_4 a_4^2 sn_4^2 + \varphi_2 a_1^2 cn_1^2] + a_2 sn_2^3[2k_2^2\omega_2^2 - \varepsilon\varphi_1 a_2^2] - \varepsilon\varphi_4 a_4^3 sn_4^3 \\ &\quad - 2\varepsilon\varphi_2 a_1 a_2 a_3 cn_1 cn_3 sn_2 - 2\varepsilon\varphi_3 a_1 a_3 a_4 cn_1 cn_3 sn_4 - 2\varepsilon\varphi_2 a_1 a_2 a_4 cn_1 sn_2 sn_4 \\ &\quad - 2\varepsilon\varphi_3 a_2 a_3 a_4 cn_3 sn_2 sn_4 - \varepsilon a_4 sn_4[\varphi_2 a_1^2 cn_1^2 + 3\varphi_4 a_3^2 cn_3^2 + \varphi_2 a_2^2 sn_2^2] = 0 \end{aligned} \tag{5}$$

and

$$\begin{aligned} &a_3 cn_3[\Delta_3 - \varepsilon\varphi_6 a_1^2 cn_1^2 - \varepsilon\varphi_6 a_2^2 sn_2^2 - 3\varepsilon\varphi_8 a_4^2 sn_4^2] - a_3 cn_3^3(2\omega_3^2 k_3^2 + \varepsilon\varphi_8 a_3^2) \\ &\quad - \varepsilon a_1 cn_1[\varphi_7 a_3^2 cn_3^2 + 3\varphi_5 a_2^2 sn_2^2 + \varphi_7 a_4^2 sn_4^2] - \varepsilon\varphi_5 a_1^3 cn_1^3 + a_4 sn_4^3[2k_4^2\omega_4^2 - \varepsilon\varphi_8 a_4^2] \\ &\quad - \varepsilon a_2 sn_2[3\varphi_5 a_1^2 cn_1^2 + \varphi_7 a_2^2 cn_3^2 + \varphi_7 a_4^2 sn_4^2] - \varepsilon\varphi_5 a_2^3 sn_2^3 - 2\varepsilon\varphi_6 a_1 a_2 a_3 cn_1 cn_3 sn_2 \\ &\quad - 2\varepsilon\varphi_7 a_1 a_3 a_4 cn_1 cn_3 sn_4 - 2\varepsilon\varphi_6 a_1 a_2 a_4 cn_1 sn_2 sn_4 - 2\varepsilon\varphi_7 a_2 a_3 a_4 cn_3 sn_2 sn_4 \\ &\quad - a_4 sn_4[-\omega_{n2}^2 + \varepsilon\varphi_6 a_1^2 cn_1^2 + 3\varepsilon\varphi_8 a_3^2 cn_3^2 + \omega_4^2(k_4^2 + 1) + \varepsilon\varphi_6 a_2^2 sn_2^2] = 0, \end{aligned} \tag{6}$$

in which

$$\Delta_1 \equiv \omega_{n1}^2 - \omega_1^2(1 - 2k_1^2), \tag{7}$$

$$\Delta_3 \equiv \omega_{n2}^2 - \omega_3^2(1 - 2k_3^2) \tag{8}$$

and we recall that ω_{n1} and ω_{n2} are the natural frequencies of the system introduced in Eq. (1). We next apply the averaging procedure proposed by Barkham and Soudack [1,2] to Eqs. (5) and (6) in turn. In this process, the arguments in Eqs. (2) and (3), namely, $\Psi_j \equiv \omega_j\tau$, are treated as independent variables.

Because the Jacobian elliptic functions *cn* and *sn* are of period $4K(k^2)$, we wish to average Eqs. (5) and (6) over this period. Notice, however, that these equations depend on Jacobian elliptic functions whose moduli are different. Therefore, we first compute the average of Eq. (5) with respect to $4K_3$, where $K_j = K(k_j^2)$ is the complete elliptic integral of the first kind for the modulus k_j. Notice that the corresponding average value of cn, sn , cn^3, and sn^3 is zero [33]. Thus, with Eq. (2) and Eq. (3) in mind, Eq. (5) becomes:

$$\begin{aligned}&a_1cn_1[\varDelta_1-\varepsilon\varphi_3a_4^2sn_4^2-3\varepsilon\varphi_1a_2^2sn_2^2-\varepsilon\varphi_3a_3^2I_3]-a_1cn_1^3(2\omega_1^2k_1^2+\varepsilon\varphi_1a_1^2)\\&\quad+a_2sn_2[\omega_{n1}^2-3\varepsilon\varphi_1a_1^2cn_1^2-\omega_2^2(1+k_2^2)-\varepsilon\varphi_3a_3^2I_3-\varepsilon\varphi_3a_4^2sn_4^2]+a_2sn_2^3[2k_2^2\omega_2^2-\varepsilon\varphi_1a_2^2]\\&\quad-2\varepsilon\varphi_2a_1a_2a_4cn_1sn_2sn_4-\varepsilon a_4sn_4[\varphi_2a_1^2cn_1^2+3\varphi_4a_3^2I_3+\varphi_2a_2^2sn_2^2]-\varepsilon\varphi_4a_4^3sn_4^3=0,\end{aligned}\tag{9}$$

where, from Ref. [33]

$$I_j\equiv\frac{1}{4K_j}\int_0^{4K_j}cn_j^2\,\mathrm{d}\Psi_j=1-\frac{1}{k_j^2}\left(1-\frac{E_j}{K_j}\right),\tag{10}$$

in which E_j is the complete elliptic integral of the second kind for the modulus k_j, and, as noted above

$$\Psi_j\equiv\omega_j\tau.\tag{11}$$

The solution of Eq. (9) is not known; therefore, we will attempt to simplify it by computing its average with respect to $4K_4$, this yields:

$$\begin{aligned}&a_1cn_1[\varDelta_1-\varepsilon\varphi_3a_4^2(1-I_4)-3\varepsilon\varphi_1a_2^2sn_2^2-\varepsilon\varphi_3a_3^2I_3]-a_1cn_1^3(2\omega_1^2k_1^2+\varepsilon\varphi_1a_1^2)\\&\quad+a_2sn_2[\omega_{n1}^2-3\varepsilon\varphi_1a_1^2cn_1^2-\omega_2^2(1+k_2^2)-\varepsilon\varphi_3a_3^2I_3-\varepsilon\varphi_3a_4^2(1-I_4)]\\&\quad+a_2sn_2^3[2k_2^2\omega_2^2-\varepsilon\varphi_1a_2^2]=0.\end{aligned}\tag{12}$$

To further simplify the solution of Eq. (12), we may compute its the average with respect to $4K_2$ to obtain

$$a_1cn_1[\varDelta_1-\varepsilon\varphi_3a_4^2(1-I_4)-3\varepsilon\varphi_1a_2^2(1-I_2)-\varepsilon\varphi_3a_3^2I_3]-a_1cn_1^3(2\omega_1^2k_1^2+\varepsilon\varphi_1a_1^2)=0.\tag{13}$$

We then average this result with respect to $4K_1$ to obtain

$$\begin{aligned}&a_2sn_2[\omega_{n1}^2-3\varepsilon\varphi_1a_1^2I_1-\omega_2^2(1+k_2^2)-\varepsilon\varphi_3a_3^2I_3-\varepsilon\varphi_3a_4^2(1-I_4)]\\&\quad+a_2sn_2^3[2k_2^2\omega_2^2-\varepsilon\varphi_1a_2^2]=0.\end{aligned}\tag{14}$$

Note that the resulting Eqs. (13) and (14) hold for all time τ if and only if

$$\varDelta_1=3\varepsilon\varphi_1a_2^2(1-I_2)+\varepsilon\varphi_3a_4^2(1-I_4)+\varepsilon\varphi_3a_3^2I_3,\tag{15}$$

$$k_1^2=-\frac{\varepsilon\varphi_1a_1^2}{2\omega_1^2},\tag{16}$$

$$\omega_{n1}^2=\omega_2^2(1+k_2^2)+3\varepsilon\varphi_1a_1^2I_1+\varepsilon\varphi_3a_3^2I_3+\varepsilon\varphi_3a_4^2(1-I_4),\tag{17}$$

$$k_2^2=\frac{\varepsilon\varphi_1a_2^2}{2\omega_2^2}.\tag{18}$$

Following the same procedure as in Eq. (5), we now compute the average of Eq. (6) with respect to $4K_1$, and then use this equation and compute its average with respect to $4K_2$. This yields

$$\begin{aligned}&a_3cn_3[\varDelta_3-\varepsilon\varphi_6a_1^2I_1-\varepsilon\varphi_6a_2^2(1-I_2)-3\varepsilon\varphi_8a_4^2sn_4^2]\\&\quad-a_3cn_3^3(2\omega_3^2k_3^2+\varepsilon\varphi_8a_3^2)+a_4sn_4^3[2k_4^2\omega_4^2-\varepsilon\varphi_8a_4^2]\\&\quad-a_4sn_4[-\omega_{n2}^2+\varepsilon\varphi_6a_1^2I_1+3\varepsilon\varphi_8a_3^2cn_3^2+\omega_4^2(k_4^2+1)+\varepsilon\varphi_6a_2^2(1-I_2)]=0.\end{aligned}\tag{19}$$

Eq. (19) can be simplified if we compute its average with respect to $4K_4$

$$\begin{aligned}&a_3cn_3[\varDelta_3-\varepsilon\varphi_6a_1^2I_1-\varepsilon\varphi_6a_2^2(1-I_2)-3\varepsilon\varphi_8a_4^2(1-I_4)]\\&\quad-a_3cn_3^3(2\omega_3^2k_3^2+\varepsilon\varphi_8a_3^2)=0.\end{aligned}\tag{20}$$

Finally, we find the average of this result with respect to $4K_3$ to reach

$$\begin{aligned} &- a_4 sn_4[-\omega_{n2}^2 + \varepsilon\varphi_6 a_1^2 I_1 + 3\varepsilon\varphi_8 a_3^2 I_3 + \omega_4^2(k_4^2+1) + \varepsilon\varphi_6 a_2^2(1-I_2)] \\ &+ a_4 sn_4^3[2k_4^2\omega_4^2 - \varepsilon\varphi_8 a_4^2] = 0. \end{aligned} \tag{21}$$

Eqs. (20) and (21) hold for all time τ if and only if

$$\Delta_3 = \varepsilon\varphi_6 a_1^2 I_1 + \varepsilon\varphi_6 a_2^2(1-I_2) + 3\varepsilon\varphi_8 a_4^2(1-I_4), \tag{22}$$

$$k_3^2 = -\frac{\varepsilon\varphi_8 a_3^2}{2\omega_3^2}, \tag{23}$$

$$\omega_{n2}^2 = \varepsilon\varphi_6 a_1^2 I_1 + 3\varepsilon\varphi_8 a_3^2 I_3 + \omega_4^2(k_4^2+1) + \varepsilon\varphi_6 a_2^2(1-I_2), \tag{24}$$

$$k_4^2 = \frac{\varepsilon\varphi_8 a_4^2}{2\omega_4^2}. \tag{25}$$

Use of Eqs. (7), (8), and the equations that define k_j^2, Eqs. (15), (17), (22), and (24) deliver the following exact relations for ω_j^2:

$$\omega_1^2 = \omega_{n1}^2 - \varepsilon a_1^2\varphi_1 - 3\varepsilon\varphi_1 a_2^2(1-I_2) - \varepsilon\varphi_3 a_4^2(1-I_4) - \varepsilon\varphi_3 a_3^2 I_3, \tag{26}$$

$$\omega_2^2 = \omega_{n1}^2 - 3\varepsilon\varphi_1 a_1^2 I_1 - \varepsilon\varphi_3 a_3^2 I_3 - \varepsilon\varphi_3 a_4^2(1-I_4) - \frac{\varepsilon\varphi_1 a_2^2}{2}, \tag{27}$$

$$\omega_3^2 = \omega_{n2}^2 - \varepsilon a_3^2\varphi_8 - \varepsilon\varphi_6 a_1^2 I_1 - \varepsilon\varphi_6 a_2^2(1-I_2) - 3\varepsilon\varphi_8 a_4^2(1-I_4), \tag{28}$$

$$\omega_4^2 = \omega_{n2}^2 - \varepsilon\varphi_6 a_1^2 I_1 - 3\varepsilon\varphi_8 a_3^2 I_3 - \varepsilon\varphi_6 a_2^2(1-I_2) - \frac{\varepsilon\varphi_8 a_4^2}{2}. \tag{29}$$

Eqs. (16), (18), (23), (25), and (26)–(29) provide a system of eight equations for the parameters ω_j and k_j.

The constants a_1, a_2, a_3, and a_4 may be evaluated from the assigned initial conditions that are assumed to be given by the general relations

$$u_1(0) = u_{10}, \quad u_2(0) = u_{20}, \quad \dot{u}_1(0) = \dot{u}_{10}, \quad \dot{u}_2(0) = \dot{u}_{20}. \tag{30}$$

Thus, the approximate solution of Eqs. (1) by the EBM is given by the following relations:

$$u_1 = u_{10} cn(\omega_1\tau, k_1^2) + \frac{\dot{u}_{10}}{\omega_2} sn(\omega_2\tau, k_2^2), \tag{31}$$

$$u_2 = u_{20} cn(\omega_3\tau, k_3^2) + \frac{\dot{u}_{20}}{\omega_4} sn(\omega_4\tau, k_4^2). \tag{32}$$

It is interesting to note that when $u_2 \equiv 0$ and the load is released from rest, Eq. (1) reduces to the well-known undamped, Duffing equation whose exact solution is then given by Eq. (31) for appropriate values of ω_1, and k_1; and similarly for u_2.

Since we have assumed that ε is a small, positive dimensionless parameter, we may derive explicit approximate expressions for ω_j and k_j^2. Utilizing the series expansion for $E_j(k_j^2)$ and $K_j(k_j^2)$ and noting that $|k_j^2| < 1$, we then see that Eq. (10) may be written as

$$I_j = \frac{1}{4K_j}\int_{\phi_j}^{4K_j+\phi_j} cn_j^2 \, \mathrm{d}\Psi_j = \frac{1}{2} - \frac{1}{16}k_j^2 - \frac{1}{32}k_j^4 - \frac{41}{2048}k_j^6 - \cdots \tag{33}$$

to an adequate degree of accuracy. Hence, use of Eq. (33) in Eqs. (26)–(29) yields explicit expressions for ω_j as functions of k_j; namely,

$$\omega_1^2 = \omega_{n1}^2 - a_1^2\varepsilon\varphi_1 - 3a_2^2\varepsilon\varphi_1\left(\frac{1}{2} + \frac{1}{16}k_2^2\right) - a_4^2\varepsilon\varphi_3\left(\frac{1}{2} + \frac{1}{16}k_4^2\right) - a_3^2\varepsilon\varphi_3\left(\frac{1}{2} - \frac{1}{16}k_3^2\right), \tag{34}$$

$$\omega_2^2 = \omega_{n1}^2 - 3\varepsilon\varphi_1 a_1^2\left(\frac{1}{2}-\frac{1}{16}k_1^2\right) - \varepsilon\varphi_3 a_3^2\left(\frac{1}{2}-\frac{1}{16}k_3^2\right) - \varepsilon\varphi_3 a_4^2\left(\frac{1}{2}+\frac{1}{16}k_4^2\right) - \frac{\varepsilon\varphi_1 a_2^2}{2}, \tag{35}$$

$$\omega_3^2 = \omega_{n2}^2 - \varepsilon a_3^2\varphi_8 - \varepsilon\varphi_6 a_1^2\left(\frac{1}{2}-\frac{1}{16}k_1^2\right) - \varepsilon\varphi_6 a_2^2\left(\frac{1}{2}+\frac{1}{16}k_2^2\right) - 3\varepsilon\varphi_8 a_4^2\left(\frac{1}{2}+\frac{1}{16}k_4^2\right), \tag{36}$$

$$\omega_4^2 = \omega_{n2}^2 - \varepsilon\varphi_6 a_1^2\left(\frac{1}{2}-\frac{1}{16}k_1^2\right) - 3\varepsilon\varphi_8 a_3^2\left(\frac{1}{2}-\frac{1}{16}k_3^2\right) - \varepsilon\varphi_6 a_2^2\left(\frac{1}{2}+\frac{1}{16}k_2^2\right) - \frac{\varepsilon\varphi_8 a_4^2}{2}. \tag{37}$$

Finally, substitution of Eqs. (16), (18), (23), and (25) in Eqs. (34)–(37) and neglecting terms of order ε^2 or higher, yields

$$\omega_1^2 = \omega_{n1}^2 - \varepsilon\varphi_1\left(a_1^2 + \frac{3}{2}a_2^2\right) - \frac{\varepsilon\varphi_3}{2}(a_3^2 + a_4^2), \tag{38}$$

$$\omega_2^2 = \omega_{n1}^2 - \frac{\varepsilon\varphi_1}{2}(3a_1^2 + a_2^2) - \frac{\varepsilon\varphi_3}{2}(a_3^2 + a_4^2), \tag{39}$$

$$\omega_3^2 = \omega_{n2}^2 - \frac{\varepsilon\varphi_6}{2}(a_1^2 + a_2^2) - \varepsilon\varphi_8\left(a_3^2 + \frac{3}{2}a_4^2\right), \tag{40}$$

$$\omega_4^2 = \omega_{n2}^2 - \frac{\varepsilon\varphi_6}{2}(a_1^2 + a_2^2) - \frac{\varepsilon\varphi_8}{2}(3a_3^2 + a_4^2). \tag{41}$$

Use of Eqs. (38)–(41) in Eqs. (16), (18), (23), and (25) provides the moduli. To the first order in ε, we have

$$k_1^2 = -\frac{\varepsilon\varphi_1 a_1^2}{2\omega_{n1}^2}, \quad k_2^2 = \frac{\varepsilon\varphi_1 a_2^2}{2\omega_{n1}^2}, \quad k_3^2 = -\frac{\varepsilon\varphi_8 a_3^2}{2\omega_{n2}^2}, \quad k_4^2 = \frac{\varepsilon\varphi_8 a_4^2}{2\omega_{n2}^2}. \tag{42}$$

If for the initial conditions assigned earlier $\dot{u}_1(0) = 0$ and $\dot{u}_2(0) = 0$, then the system modal solution is still given by Eqs. (31) and (32) in which

$$\omega_1^2 = \omega_{n1}^2 - \varepsilon\left(u_{10}^2\varphi_1 + \frac{u_{20}^2\varphi_3}{2}\right), \tag{43}$$

$$\omega_3^2 = \omega_{n2}^2 - \varepsilon\left(u_{20}^2\varphi_8 + \frac{u_{10}^2\varphi_6}{2}\right), \tag{44}$$

$$k_1^2 = \frac{-u_{10}^2\varepsilon\varphi_1}{2\omega_{n1}^2}, \tag{45}$$

$$k_3^2 = \frac{-u_{20}^2\varepsilon\varphi_8}{2\omega_{n2}^2}. \tag{46}$$

In this case, the periods T_1 and T_3 in τ for the two modes of oscillation u_1 and u_2 are given by the exact relations

$$T_1 = \frac{4K(k_1^2)}{\omega_1}, \tag{47}$$

$$T_3 = \frac{4K(k_3^2)}{\omega_3}. \tag{48}$$

Using Eqs. (43)–(46), and expanding $K(k_j^2)$ in powers of k_j^2 to the first order in ε, we obtain

$$T_1 = \frac{2\pi}{\omega_{n1}^3}\left[\omega_{n1}^2 + \varepsilon\left(\frac{3}{8}u_{10}^2\varphi_1 + \frac{1}{4}u_{20}^2\varphi_3\right)\right], \tag{49}$$

$$T_3 = \frac{2\pi}{\omega_{n2}^3}\left[\omega_{n2}^2 + \varepsilon\left(\frac{3}{8}u_{20}^2\varphi_8 + \frac{1}{4}u_{10}^2\varphi_6\right)\right]. \tag{50}$$

The typical dependance of the periods on the amplitudes of the oscillations is evident here.

2.2. Results

To assess the accuracy of the approximate solution of Eqs. (1) obtained by the EBM, we consider the familiar example [28,29] of motion of a linear undamped vibration absorber of stiffness k and mass m attached to a rigid load of mass M supported symmetrically by incompressible, homogeneous and isotropic hyperelastic shear mountings of original length L and cross-sectional area A as shown in Fig. 1. Shear mountings are used widely for engine mountings, bridge and building supports, chassis steel leaf spring suspension, and various kinds of packaging supports, see, for example Ref. [34]. To model these diverse kinds of applications, we consider shear blocks bonded to the load at one face and to parallel rigid supports at the other, as illustrated in Fig. 1. For simplicity, we suppose that both bodies are supported on a smooth horizontal bearing surface and parallel to the plane of shear. Inertia of the shear and absorber springs will be neglected, as usual, and effects due to symmetrical bending of the shear mounts will be ignored. We thus suppose that each shear block executes an ideal isochoric, time-dependent simple shear deformation of amount $\sigma(t) = \tan\alpha(t)$ from its initial, undeformed state. Practical applications usually limit the maximum value of the angular deflection $\alpha(t)$ to 45° [35]. For shear mounting materials whose shear response function is quadratic, namely, $\mu(\sigma) = \mu_0 + 2\mu_1\sigma^2 = \mu_0(1 + \varepsilon\sigma^2)$ where μ_0 represents the shear modulus of the shear mount material in the natural state, μ_1 represents the magnitude of the second-order modulus such that $\mu_1 \ll \mu_0$, and thus $\varepsilon \equiv 2\mu_1/\mu_0 \ll 1$. The details and derivation of the equations of motion for this hyperelastic spring–mass system are provided in Ref. [29] wherein the dimensionless governing equations of motion for this system are given by

$$\ddot{\sigma} + \sigma - \alpha_2^2 z = \varepsilon b_1\sigma^3, \tag{51}$$

$$\beta\ddot{z} + \alpha_2^2 z - \alpha_2^2\sigma = 0, \tag{52}$$

where, in terms of the aformentioned physical parameters,

$$p^2 \equiv \frac{k}{m}, \quad P^2 \equiv \frac{k}{M} = \beta p^2, \quad \beta \equiv \frac{m}{M}, \quad \Omega_0^2 \equiv P^2 + \Omega_1^2, \tag{53}$$

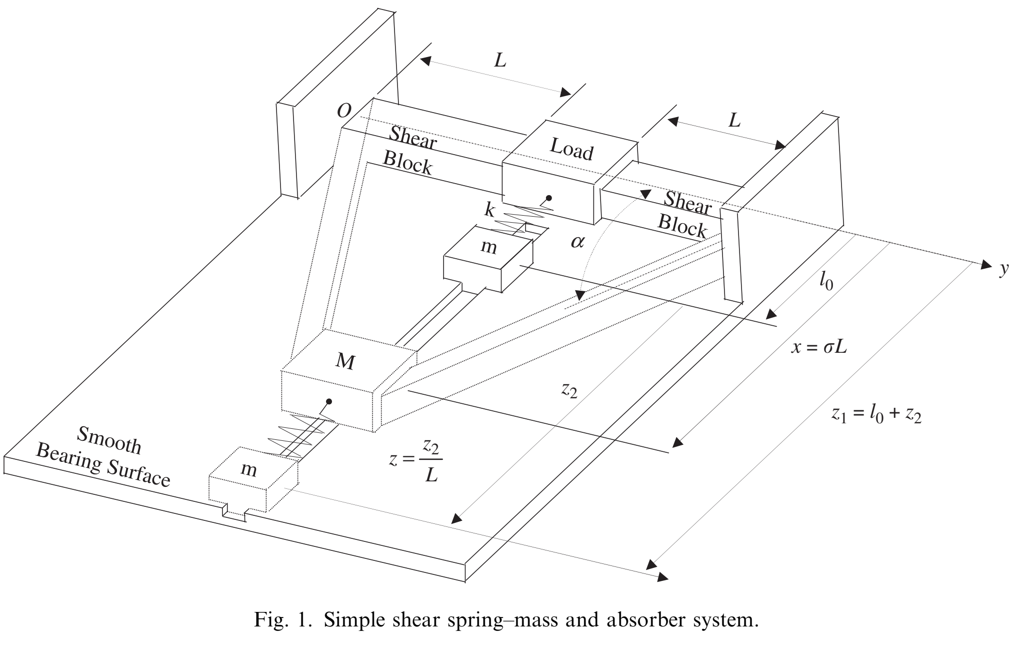

Fig. 1. Simple shear spring–mass and absorber system.

$$\alpha_2^2 \equiv \frac{P^2}{\Omega_0^2}, \quad \Omega_1^2 \equiv \frac{2A}{ML}\mu_0, \quad r_2 \equiv \frac{p}{\Omega_1}, \quad b_1 \equiv \alpha_2^2 - 1. \tag{54}$$

In accordance definitions given in Fig. 1, the dimensionless variables $\sigma \equiv x/L$ and $z \equiv z_2/L$ characterize, respectively, the motion of the load M and of the linear absorber system from their natural states; β is the mass ratio; ε is a small dimensionless material parameter mentioned earlier, so that $0<\varepsilon \ll 1$; and Ω_1 and p represent, respectively, the natural frequency of the main system and of the secondary, absorber system.

We now introduce the linear transformation

$$\begin{bmatrix} \sigma \\ z \end{bmatrix} = \begin{Bmatrix} R_1 & R_2 \\ R_1 f_1 & R_2 f_2 \end{Bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{55}$$

to obtain the canonical, normal mode form of the system of Eqs. (51) and (52), which is the same as Eqs. (1), see Ref. [28]. Note that the physical parameters of the system corresponding to Eqs. (1) are given by

$$\varphi_1 \equiv b_1 R_1^4, \quad \varphi_2 \equiv 3b_1 R_1^3 R_2, \quad \varphi_3 \equiv 3b_1 R_1^2 R_2^2, \quad \varphi_4 \equiv b_1 R_1 R_2^3, \tag{56}$$

$$\varphi_5 \equiv b_1 R_1^3 R_2, \quad \varphi_6 \equiv 3b_1 R_1^2 R_2^2, \quad \varphi_7 \equiv 3b_1 R_1 R_2^3; \quad \varphi_8 \equiv b_1 R_2^4, \tag{57}$$

$$R_1^2 = \frac{1}{(1+f_1^2\beta)}, \quad R_2^2 = \frac{1}{(1+f_2^2\beta)}, \quad f_1 = \frac{\alpha_2^2}{1-\beta\omega_{n1}^2}, \quad f_2 = \frac{\alpha_2^2}{1-\beta\omega_{n2}^2}, \tag{58}$$

and the characteristic values or eigenvalues ω_{nj}^2 that describe the natural mode frequencies of the system are provided by

$$\omega_{n1}^2 = \frac{1}{2}\left[\left(\frac{\alpha_2^2}{\beta}+1\right) - \sqrt{\left(\frac{\alpha_2^2}{\beta}+1\right)^2 + \frac{4}{\beta}\alpha_2^2 b_1}\right], \tag{59}$$

$$\omega_{n2}^2 = \frac{1}{2}\left[\left(\frac{\alpha_2^2}{\beta}+1\right) + \sqrt{\left(\frac{\alpha_2^2}{\beta}+1\right)^2 + \frac{4}{\beta}\alpha_2^2 b_1}\right]. \tag{60}$$

Thus, the approximate analytical EBM solution of the system of Eqs. (51) and (52) is provided by Eqs. (31), (32), and (43)–(46).

We next proceed with the numerical comparison of our elliptic balance method approximate solution with the multiple scales method solution derived by Nayfeh and Mook in [31] and with the numerical integration solution of Eqs. (1) obtained by applying the fourth–order Runge–Kutta method provided by the Mathematica symbolic package. The comparison has been made for a few cases shown graphically in Figs. 2–5 where the light solid line represents the numerical integration solution, the unfilled squares represent the multiple scales solution, and the dashed line represents our elliptic balance method approximate solution. The amplitude–time response curves for values of $\beta = 0.1, \varepsilon = 0.02, r_2 = 1.05$, a large initial shear deflection $\sigma(0) = 1$ (i.e. $\alpha = 45^\circ$), $z(0) = 0$, $\dot{\sigma}(0) = 0$, and $\dot{z}(0) = 0$, are shown in Fig. 2 for a reasonably large time $\tau \leqslant 100$, for illustration. It appears from Fig. 2 that all three solutions for this case are indistinguishable. The same conclusion may be drawn from results shown in Fig. 3 for moderate values of $\beta = 0.2, \varepsilon = 0.2, r_2 = 1.05$, $\sigma(0) = 1$, $z(0) = 0$, $\dot{\sigma}(0) = 0$, and $\dot{z}(0) = 1$. In Fig. 4 are shown the amplitude–time response curves for a large value of $\beta = 0.5$, a moderate value of $\varepsilon = 0.25$, and for $r_2 = 1.05, \sigma(0) = 1$, $z(0) = 1$, $\dot{\sigma}(0) = 0$, and $\dot{z}(0) = 0$. Again, the numerical integration, the multiple scales and elliptic balance method approximate solutions are virtually the same. However, it is seen in Fig. 5 that for unrealistically large initial conditions $\sigma(0) = 2$ (about 63°), $z(0) = 0$, $\dot{\sigma}(0) = 0$, and $\dot{z}(0) = 0$, there is a clear variation between the numerical integration, the multiple scales and elliptic balance method approximate solutions. In all practical cases and for the time interval shown, the accuracy of the elliptic balance method is evident and it appears to be closely related to the initial conditions and to the system parameter values as remarked earlier. Moreover, it is important to mention that the accuracy of our elliptic balance method approximate solution is expected to deteriorate after the dimensionless time τ progresses, a condition that is typical of averaging procedures [20].

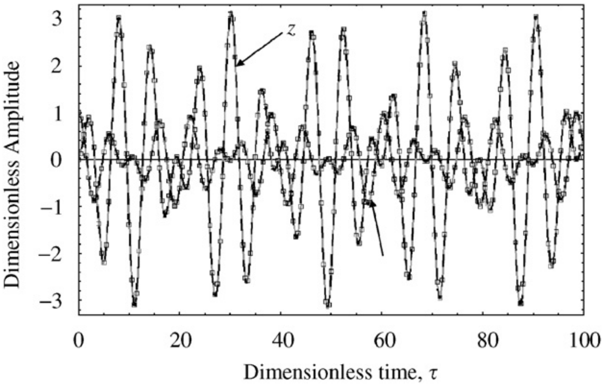

Fig. 2. Amplitude–time response curves for values of $\beta = 0.1, \varepsilon = 0.02, r_2 = 1.05, \sigma(0) = 1, z(0) = 0, \dot{\sigma}(0) = 0$, and $\dot{z}(0) = 0$. The light solid lines represent the numerical integration solution, the unfilled squares represent the multiple scales solution and the dashed lines represent the elliptic balance method solution for $\sigma(\tau)$ and $z(\tau)$.

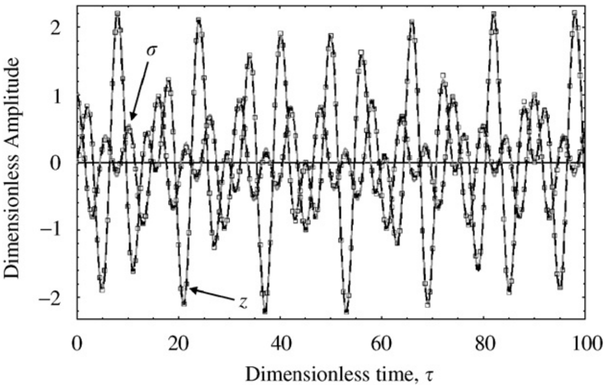

Fig. 3. Amplitude–time response curves for values of $\beta = 0.2, \varepsilon = 0.2, r_2 = 1.05, \sigma(0) = 1, z(0) = 0, \dot{\sigma}(0) = 0$, and $\dot{z}(0) = 1$. The light solid lines represent the numerical integration solution, the unfilled squares represent the multiple scales solution and the dashed lines represent the elliptic balance method solution for $\sigma(\tau)$ and $z(\tau)$.

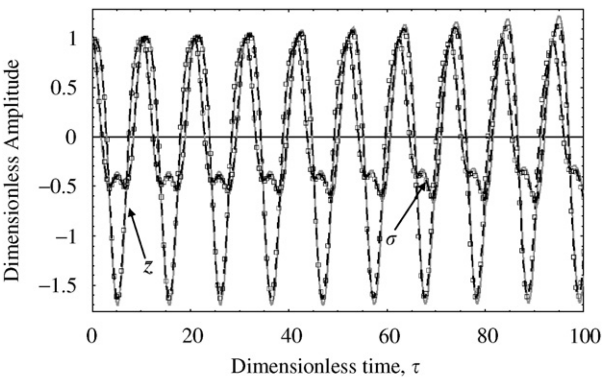

Fig. 4. Amplitude–time response curves for values of $\beta = 0.5, \varepsilon = 0.25, r_2 = 1.05, \sigma(0) = 1, z(0) = 1, \dot{\sigma}(0) = 0$, and $\dot{z}(0) = 0$. The light solid lines represent the numerical integration solution, the unfilled squares represent the multiple scales solution and the dashed lines represent the elliptic balance method solution for $\sigma(\tau)$ and $z(\tau)$.

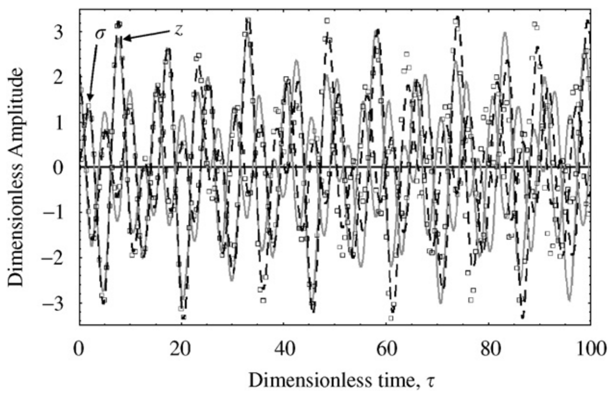

Fig. 5. Amplitude–time response curves for values of $\beta = 0.33, \varepsilon = 0.2$, $r_2 = 1$, $\sigma(0) = 2$, $z(0) = 0$, $\dot{\sigma}(0) = 0$, and $\dot{z}(0) = 0$. The light solid lines represent the numerical integration solution, the unfilled squares represent the multiple scales solution and the dashed lines represent the elliptic balance method solution for $\sigma(\tau)$ and $z(\tau)$.

3. Conclusions

This paper shows how the elliptic balance method may be applied to obtain the approximate solution of undamped, nonlinear multi-degree of freedom systems by assuming modal solutions whose undetermined parameters are time independent. It appears that the elliptic balance method, because of its general formulation and its high degree of accuracy, has considerable potential for applications that are described by Eq. (1). The accuracy can be improved by an appropriate choice of the parameter values and by supposing that the modulus of the Jacobian elliptic functions are time dependent, a complexity avoided in the foregoing analysis. Moreover, the elliptic balance method admits the exact solution for the homogeneous Duffing equation as a special reduced case when either one of the oscillators is fixed and the other is released from rest, something that is not possible by using multiple scales or other perturbation techniques. At present, we have also obtained, by our proposed elliptic balance method, the solution for two degree of freedom systems having cubic nonlinearities with a driving force of sinusoidal type. This work will be discussed elsewhere.

Acknowledgments

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