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Journal of Sound and Vibration 304 (2007) 263–271
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

# Optimal sliding mode control for linear time-delay systems with sinusoidal disturbances

Gong-You Tang*, Shan-Shan Lu, Rui Dong

*College of Information Science and Engineering, Ocean University of China, Qingdao 266100, China*

Received 8 October 2006; received in revised form 28 January 2007; accepted 3 March 2007

**Abstract**

This paper develops a successive approximation approach (SAA) of optimal sliding mode control (SMC) for linear time-delay systems with sinusoidal disturbances. A sequence of two-point boundary value (TPBV) problems with both time-delay and time-advance terms is derived from the optimal sliding mode design. According to the SAA, the original TPBV problems are transformed into a sequence of linear TPBV problems without delay and advance terms. The obtained SMC ensures that the state trajectories reach the sliding surface in finite time and remain on it thereafter. The stability of the sliding mode is proved. A numerical simulation is employed to verify the effectiveness of the proposed approach.


## 1. Introduction

Sliding mode control (SMC) is known to achieve high-performance robust control against external disturbances and unpredictable parameter variations in some certain conditions. The sliding mode occurs on a prescribed switching surface [1]. Consequently, designing the sliding surface is the same important as designing the controller. Many researchers have dedicated themselves to constructing the sliding surface. A time-varying sliding surface is designed for fast and robust tracking in a class of second-order uncertain systems [2]. For a class of uncertain dynamic systems with mismatched uncertainties, a new design method of linear sliding surfaces, which are linear to the state is developed based on the linear matrix inequality (LMI) approach [3]. The sliding surfaces for uncertain systems with single or multiple, constant or time-varying state delay are designed so to maximize the calculable set of admissible delays [4]. Based on the *Lyapunov* method, a controller guaranteeing convergence of the state trajectory to the sliding manifold is developed and then generalized to account for uncertainties in the delay for a class of uncertain time-delay systems [5]. A new nonlinear integral-type sliding surface, which incorporates a virtual nonlinear nominal control to achieve prescribed specifications is presented for both matched and unmatched uncertain systems [6]. Constructed from a *Riccati* inequality associated with quadratic stabilizability, a method to design robust sliding surfaces in the presence of mismatched parametric uncertainty is proposed [7].

*Corresponding author. Tel.: +86 532 66781230.
*E-mail address:* gtang@ouc.edu.cn (G.-Y. Tang).

0022-460X/$ - see front matter 
doi:10.1016/j.jsv.2007.03.006

The analysis and synthesis of the time-delay systems is one of the difficult and hot investigations in control theory and control engineering domain since time delay often occurs in various engineering systems. Because the presence of time delay often causes serious deterioration of the stability and performance of the system, considerable research has been devoted to the control of time-delay systems. SMC for time-delay systems with mismatched parametric uncertainties is developed [8]. A design method for the variable structure control of distributed time-delay systems is studied by constructing a finite dimensional system, which contains all the unstable poles of the original time-delay system [9]. Based on the LMI technique and the sliding mode variable structure control method, an SMC approach is proposed for a class of uncertain time-delay systems with mismatching uncertainties [10]. An observer-based SMC problem is studied for state-delay systems with immeasurable slates and nonlinear uncertainties [11].

Many practical control systems are frequently affected by sinusoidal disturbances. For example, the vibration control for offshore structures, where the sinusoidal disturbances are mainly from the ocean wave forces [12]. The flight attitude control under wind shear stresses, where the sinusoidal forcing terms arise from a model for wind shear based on harmonic oscillations [13]. Other applications include wave loads on ships sailing [14], noise reduction in vehicles and transformers [15], vibration damping for industrial machines and periodic disturbance reduction in disk drives [16]. Hence, the systems with sinusoidal disturbances have extensive engineering background.

The purpose of this paper is to design an optimal SMC for linear time-delay systems with sinusoidal disturbances based on the SAA [17]. First, some state variables of the given system are regarded as virtual control. The original TPBV problem with delay terms is transformed into a sequence of linear TPBV problems by using the SAA. The obtained optimal sliding mode consists of analytic terms and a compensation term, which is the limit of the adjoint vector sequence. Using a finite term of the adjoint vector sequence, the sliding mode is gotten. Then a new SMC is obtained, which drives the trajectories to reach the sliding surface in finite time. The stability of the sliding mode is analyzed. An example is presented to verify the validity of the proposed method.

## 2. Problem statement

Consider a class of linear time-delay systems with sinusoidal disturbances described by

$$\begin{aligned}&\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_1(t-\tau) + A_{13}x_2(t) + Dv(t),\\ &\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_1(t-\tau) + A_{23}x_2(t) + A_{24}x_2(t-\tau) + Bu(t) + d(x,t), \quad t>0,\\ &x_1(t) = \varphi_1(t), x_2(t) = \varphi_2(t), \quad -\tau \leqslant t \leqslant 0,\end{aligned} \tag{1}$$

where $x_i(t) \in R^{n_i}$; $x = [x_1^{\mathrm{T}} \quad x_2^{\mathrm{T}}]^{\mathrm{T}}$ and $u(t) \in R^{n_2}$ are the state vector and the control vector, respectively; $A_{1j}$, $A_{2j}$, $B$ and $D$ are constant matrices of appropriate dimensions, $\tau$ is a positive time delay, $\varphi_i(t)$ are the known continuous initial state vectors, $v(t) \in R^m$ is an external sinusoidal disturbance vector, $d(x,t) \in R^{n_2}$ is bounded matching perturbation and/or disturbance.

**Assumption 1.** The pair $(A_{11}, A_{13})$ is completely controllable.

**Assumption 2.** The dynamic characteristics of the external sinusoidal disturbance vector $v(t)$ can be expressed by

$$v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_m(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 \sin(\omega_1 t + \psi_1) \\ \alpha_2 \sin(\omega_2 t + \psi_2) \\ \vdots \\ \alpha_m \sin(\omega_m t + \psi_m) \end{bmatrix}, \tag{2}$$

where the frequency $\omega_i$ are known constant. The amplitude $\alpha_i$ and the phase $\psi_i$ may be unknown, but $v_i$ are measurable.

**Assumption 3.** There exists a known non-negative scalar function $\rho(x)$ such that

$$\|d(x,t)\| \leqslant \rho(x). \tag{3}$$

Regarding $x_2$ as virtual control of the first subsystem in system (1), one can choose the average quadratic performance index as

$$J = \lim_{T\to\infty} \frac{1}{T}\int_0^T [x_1^{\mathrm{T}}(t)Qx_1(t) + x_2^{\mathrm{T}}(t)Rx_2(t)]\,\mathrm{d}t, \tag{4}$$

where $Q$, $R$ are positive-definite matrices of appropriate dimensions. The optimal sliding surface design is to find a virtual control law $x_2^*(t)$, which minimizes quadratic performance index (4) subject to dynamic equality constraint (1).

According to the optimal control theory, the optimal sliding mode design may lead to a TPBV problem in which both time-delay and time-advance terms are involved:

$$\begin{aligned}
&\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_1(t-\tau) - S\lambda(t) + Dv(t),\\
&-\dot{\lambda}(t) = Qx_1(t) + A_{11}^{\mathrm{T}}\lambda(t) + A_{12}^{\mathrm{T}}\lambda(t+\tau),\\
&x_1(t) = \varphi_1(t), \quad -\tau \leqslant t \leqslant 0,\\
&\lambda(\infty) = 0,
\end{aligned} \tag{5}$$

where $S = A_{13}R^{-1}A_{13}^{\mathrm{T}}$ and the virtual control law can be written as

$$x_2^*(t) = -R^{-1}A_{13}^{\mathrm{T}}\lambda(t). \tag{6}$$

Choosing the optimal sliding surface as

$$s^*(t) = Rx_2^*(t) + A_{13}^{\mathrm{T}}\lambda(t) = 0 \tag{7}$$

and choosing a controller to force the trajectories to reach optimal sliding surface (7), the SMC law can be obtained such that the closed-loop system of system (1) is asymptotically stable. If one could solve TPBV problem (5), the design is finished. However, finding the analytical solution of TPBV problem (5) is very difficult. Hence, it is necessary to find an approximate approach to solve this kind of problem.

## 3. Optimal sliding surface design

### 3.1. Preliminaries

Consider the autonomous nonlinear system with time delay described by

$$\begin{aligned}
&\dot{x}(t) = G(t)x(t) + Hx(t-\tau) + f(t, x, v(t), \dot{v}(t)), \quad t > 0,\\
&x(t) = \phi(t), \quad -\tau \leqslant t \leqslant 0,
\end{aligned} \tag{8}$$

where $x(t) \in R^n$ is the state vector, $v(t) \in R^m$ the input vector, $\phi(t)$ the initial state vector, $G(t) \in R^{n\times n}$ is a continuous function matrix, $H \in R^{n\times n}$ is a constant matrix, $f$: $(R_+ \times R^n \times R^m \times R^m) \to R^n$.

**Lemma 1.** [18] *Define the state sequence* $\{x^{(i)}(t)\}$ *as*

$$\begin{aligned}
&x^{(0)}(t) = \Phi(t,0)\phi(0), \quad t > 0,\\
&x^{(i)}(t) = \Phi(t,0)\phi(0) + \int_0^t \Phi(t,r)[Hx^{(i-1)}(r-\tau) + f(r, x^{(i-1)}(r), v(r), \dot{v}(r))]\,\mathrm{d}r, \quad t > 0,\\
&x^{(i)}(t) = \phi(t), \quad -\tau \leqslant t \leqslant 0, \quad i = 1, 2, \ldots,
\end{aligned} \tag{9}$$

*where* $\Phi(t, r)$ *is the state transition matrix with respect to the matrix G. Then the sequence* $\{x^{(i)}(t)\}$ *uniformly converges to the solution of system* (8).

### 3.2. SAA designing process

In order to solve TPBV problem (5) using the SAA, one constructs the following TPBV problem sequence

$$\begin{aligned}
&\lambda^{(0)}(t) = 0,\\
&-\dot{\lambda}^{(i)}(t) = Qx_1^{(i)}(t) + A_{11}^{\mathrm{T}}\lambda^{(i)}(t) + A_{12}^{\mathrm{T}}\lambda^{(i-1)}(t+\tau),\\
&\lambda^{(i)}(\infty) = 0;\ \ i = 1,2,\ldots,\\
&x_1^{(0)}(t) = 0,\\
&\dot{x}_1^{(i)}(t) = A_{11}x_1^{(i)}(t) + A_{12}x_1^{(i-1)}(t-\tau) - S\lambda^{(i)}(t) + Dv(t),\\
&x_1^{(i)}(t) = \varphi_1(t),\ \ -\tau \leqslant t \leqslant 0;\ \ i = 1,2,\ldots
\end{aligned} \tag{10}$$

and the corresponding $i$th sliding surface

$$s^{(i)}(t) = Rx_2^{(i)}(t) + A_{13}^{\mathrm{T}}\lambda^{(i)}(t) = 0. \tag{11}$$

For the $i$th iteration, the state trajectory and sliding surface are $x_1^{(i)}(t)$ and $s^{(i)}(t) = 0$, respectively. The following theorem is given out.

**Theorem 1.** *Assume that* $\{x_1^{(i)}(t)\}$ *and* $\{s^{(i)}(t)\}$ *are the solution sequences of Eqs.* (10) *and* (11), *respectively. Then* $\{s^{(i)}(t)\}$ *uniformly converges to the optimal sliding surface* $s^*(t) = 0$ *defined in Eq.* (7) *for system* (1) *with quadratic performance index* (4).

**Proof.** Let

$$\lambda^{(i)}(t) = P_1x_1^{(i)}(t) + P_2v(t) + P_3v_\omega(t) + g^{(i)}(t),\quad i = 1,2,\ldots, \tag{12}$$

where

$$v_\omega(t) = \dot{v}(t) = -\Omega\left[v_1\left(t - \frac{\pi}{2\omega_1}\right),\ v_2\left(t - \frac{\pi}{2\omega_2}\right),\ \ldots,\ v_m\left(t - \frac{\pi}{2\omega_m}\right)\right]^{\mathrm{T}}, \tag{13}$$

$$\Omega = \mathrm{Diag}(\omega_1,\ \omega_2,\ \ldots,\ \omega_m), \tag{14}$$

where $P_1 \in R^{n_1 \times n_1}$ is a positive-definite matrix and $P_2$, $P_3$ are matrices of appropriate dimensions, $g^{(i)}(t) \in R^{n_1}$ is the $i$th adjoint vector. $\square$

Deriving the two sides of (12) produces

$$\dot{\lambda}^{(i)}(t) = P_1\dot{x}_1^{(i)}(t) + P_2v_\omega(t) - P_3\Omega^2v(t) + \dot{g}^{(i)}(t). \tag{15}$$

Substituting the second and fifth equations in Eq. (10) into (15) and comparing the coefficients, one can get the matrix equations

$$\begin{aligned}
&P_1A_{11} + A_{11}^{\mathrm{T}}P_1 - P_1SP_1 + Q = 0,\\
&A_{11}^{\mathrm{T}}P_2 + P_1D - P_3\Omega^2 - P_1SP_2 = 0,\\
&A_{11}^{\mathrm{T}}P_3 + P_2 - P_1SP_3 = 0
\end{aligned} \tag{16}$$

and adjoint vector differential equation as follows:

$$\begin{aligned}
&\dot{g}^{(i)}(t) = (P_1S - A_{11}^{\mathrm{T}})g^{(i)}(t) - P_1A_{12}x_1^{(i-1)}(t-\tau) - A_{12}^{\mathrm{T}}\lambda^{(i-1)}(t+\tau),\\
&g^{(i)}(\infty) = 0,\ \ i = 1,2,\ldots.
\end{aligned} \tag{17}$$

It is obvious that the first equation of (16) is the *Riccati* matrix equation, so the unique positive-definite matrix solution $P_1$ can be easily gotten. Substituting $P_1$ into the second and the third equations of (16), $P_2$ and $P_3$ can be calculated.

As $x_1^{(0)}(t)=0$ and $\lambda^{(0)}(t)=0$, $g^{(0)}(t)=0$ can be known from Eq. (12). Substituting (12) into (17), one can obtain

$$\begin{aligned}\dot{g}^{(i)}(t) &= (P_1S - A_{11}^{\mathrm{T}})g^{(i)}(t) - P_1A_{12}x_1^{(i-1)}(t-\tau) - A_{12}^{\mathrm{T}}P_1x_1^{(i-1)}(t+\tau)\\ &\quad - A_{12}^{\mathrm{T}}g^{(i-1)}(t+\tau) - A_{12}^{\mathrm{T}}P_2v(t+\tau) - A_{12}^{\mathrm{T}}P_3v_\omega(t+\tau),\\ g^{(i)}(\infty) &= 0, \quad i=1,2,\ldots .\end{aligned} \tag{18}$$

Substituting (12) into the fifth equation of (10), it produces

$$\begin{aligned}\dot{x}_1^{(i)}(t) &= (A_{11}-SP_1)x_1^{(i)}(t) + (D-SP_2)v(t) - SP_3v_\omega(t) - Sg^{(i)}(t) + A_{12}x_1^{(i-1)}(t-\tau),\\ x_1^{(i)}(t) &= \varphi_1(t), \quad -\tau \leqslant t \leqslant 0.\end{aligned} \tag{19}$$

Substituting (12) into (11), it gives the $i$th sliding surface

$$s^{(i)}(t) = Rx_2^{(i)}(t) + A_{13}^{\mathrm{T}}P_1x_1^{(i)}(t) + A_{13}^{\mathrm{T}}P_2v(t) + A_{13}^{\mathrm{T}}P_3v_\omega(t) + A_{13}^{\mathrm{T}}g^{(i)}(t) = 0. \tag{20}$$

According to Lemma 1, sequences $\{g^{(i)}(t)\}$ and $\{x_1^{(i)}(t)\}$ in Eqs. (18) and (19) are uniformly convergent, respectively. From Eq. (20), $\{x_2^{(i)}(t)\}$ is only related to $\{x_1^{(i)}(t)\}$ and $\{g^{(i)}(t)\}$, so it is also uniformly convergent. Then the optimal sliding surface is obtained

$$s^*(t) = \lim_{i\to\infty} s^{(i)}(t) = Rx_2(t) + A_{13}^{\mathrm{T}}P_1x_1(t) + A_{13}^{\mathrm{T}}P_2v(t) + A_{13}^{\mathrm{T}}P_3v_\omega(t) + A_{13}^{\mathrm{T}}g^{(\infty)}(t) = 0. \tag{21}$$

The proof is complete. □

Actually, the optimal sliding surface in Eq. (21) cannot be calculated. A suboptimal sliding surface can be found in practical applications by replacing $\infty$ with $M$ in Eq. (21)

$$s_M(t) = Rx_2(t) + A_{13}^{\mathrm{T}}P_1x_1(t) + A_{13}^{\mathrm{T}}P_2v(t) + A_{13}^{\mathrm{T}}P_3v_\omega(t) + A_{13}^{\mathrm{T}}g^{(M)}(t) = 0. \tag{22}$$

**Remark 1.** $x_1(t)$ and $x_2(t)$ in Eq. (22) are the accurate solutions. Only $g^{(M)}(t)$ is the $M$th iterative result in place of $g^{(\infty)}(t)$, so suboptimal sliding surface (22) is closer to the optimal sliding surface shown in Eq. (21) than the $M$th iterative sliding surface, which can be obtained by replacing $i$ with $M$ in Eq. (20).

We give a design *algorithm* of suboptimal sliding surface (22) as follows.

**Algorithm 1.** Suboptimal sliding surface design

*Step* 1: Solve the unique positive-definite matrix $P_1$ from the first *Riccati* matrix equation of (16) and $P_2$, $P_3$ from the other two matrix equations of (16). Let $x_1^{(0)}(t)=g^{(0)}(t)=0$, $J_0=0$ and $i=1$. Give some positive constant $\varepsilon>0$.

*Step* 2: Obtain the $i$th adjoint vector $g^{(i)}(t)$ from Eq. (18).

*Step* 3: Letting $M=i$, calculate $x_2^{(M)}(t)$ from Eq. (20).

*Step* 4: Calculate $J_M$ from

$$J_M = \lim_{T\to\infty}\frac{1}{T}\int_0^T [x_1^{\mathrm{T}}(t)Qx_1(t) + (x_2^{(M)}(t))^{\mathrm{T}}Rx_2^{(M)}(t)]\,\mathrm{d}t. \tag{23}$$

*Step* 5: If $|(J_M - J_{M-1})|/J_M<\varepsilon$, then stop and output the suboptimal sliding surface $s_M(t)$ in Eq. (22), else calculate $x_1^{(i)}(t)$ from Eq. (19).

*Step* 6: Letting $i=i+1$, go to step 2.

## 4. Sliding mode control design

A sliding reachability condition [19] is chosen as

$$\dot{s} = \gamma(s,t) = -Ks - E\,\mathrm{sign}(s), \tag{24}$$

where $K = \text{diag}\{k_1, k_2, \ldots, k_{n_2}\}$, $E = \text{diag}[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n_2}]$, with $k_i$, $\varepsilon_i > 0$,

$$\text{sign}(s) = [\text{sign}(s_1),\ \text{sign}(s_2), \ldots,\ \text{sign}(s_{n_2})]^{\mathrm{T}}. \tag{25}$$

For linear systems with external disturbances, we develop the method of the sliding reachability condition and propose the following sliding reachability condition

$$\begin{aligned} \dot{s}_i &\geqslant -k_i s_i - \varepsilon_i \,\text{sign}(s_i), \quad s_i < 0, \\ \dot{s}_i &\leqslant -k_i s_i - \varepsilon_i \,\text{sign}(s_i), \quad s_i > 0, \quad i = 1, 2, \ldots, n_2. \end{aligned} \tag{26}$$

Choosing the switching function as (22) and using the following control:

$$\begin{aligned} u(t) = &-(RB)^{-1}\{(RA_{21} + A_{13}^{\mathrm{T}}P_1A_{11} + KA_{13}^{\mathrm{T}}P_1)x_1(t) + (RA_{22} + A_{13}^{\mathrm{T}}P_1A_{12})x_1(t-\tau) \\ &+ (KR + RA_{23} + A_{13}^{\mathrm{T}}P_1A_{13})x_2(t) + RA_{24}x_2(t-\tau) + (A_{13}^{\mathrm{T}}P_1D - A_{13}^{\mathrm{T}}P_3\Omega^2 + KA_{13}^{\mathrm{T}}P_2)v(t) \\ &+ (KA_{13}^{\mathrm{T}}P_3 + A_{13}^{\mathrm{T}}P_2)v_\omega(t) + KA_{13}^{\mathrm{T}}g^{(M)}(t) + A_{13}^{\mathrm{T}}\dot{g}^{(M)}(t) + E\,\text{sign}(s) + \rho(x)R\,\text{sign}(s)\}. \end{aligned} \tag{27}$$

One can ensure that the trajectories reach the ideal sliding surface in finite time and maintain on it thereafter.

If $v(t) \equiv 0$, the following theorem is given out.

**Theorem 2.** *Consider system* (1) *with control law* (27). *Assume that the sliding surface is chosen as* (22). *Then the state trajectories of system* (1) *can reach the sliding surface in finite time and the closed-loop system is asymptotically stable.*

**Proof.** It is known that the state trajectories can be driven to suboptimal sliding surface (22) by control law (27). From Eqs. (1), (22), (24), on the sliding surface (22), one has

$$\dot{x}_1(t) = [A_{11} - A_{13}R^{-1}A_{13}^{\mathrm{T}}P_1]x_1(t) + A_{12}x_1(t-\tau) - A_{13}R^{-1}A_{13}^{\mathrm{T}}g^{(M)}(t). \tag{28}$$

And

$$x_2(t) = -R^{-1}A_{13}^{\mathrm{T}}[P_1x_1(t) + g^{(M)}(t)]. \tag{29}$$

According to the optimal control theory, system (28) is asymptotically stable. Moreover, noting that $g^{(M)}(\infty) = 0$, we obtain $\lim_{t\to\infty} x_2(t) = 0$. The proof is complete. □

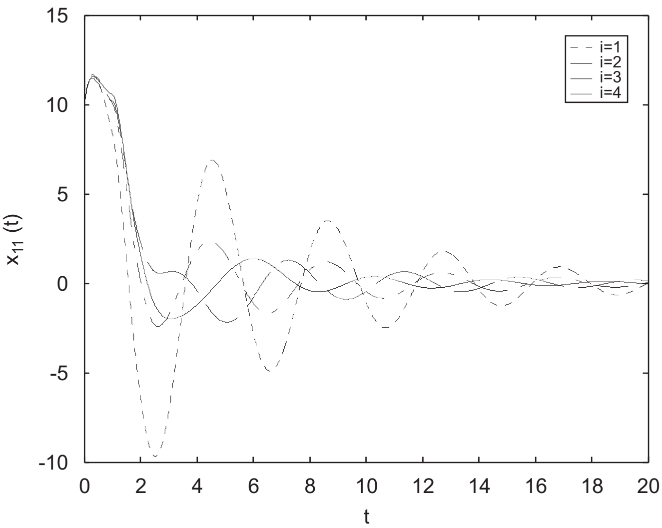


Fig. 1. State variable $x_{11}(t)$ in the $i$th iteration.

## 5. A simulation example

Consider the linear time-delay system with sinusoidal disturbances described by (1), where

$$A_{11} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},\ A_{12} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},\ A_{13} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},\ D = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}^{\mathrm{T}},\ A_{22} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}^{\mathrm{T}},\ \varphi_1 = \begin{bmatrix} 10 \\ 6 \end{bmatrix},\ x_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix},\ \varphi_2 = -4,$$

$$A_{23} = 1,\ A_{24} = 1,\ B = 1,\ \tau = 1,\ \bar{d} = a\cos x_2,\ a = 0.04$$

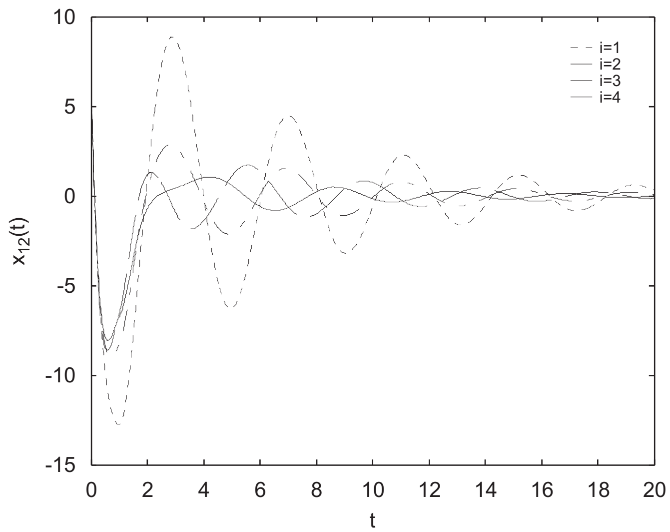


Fig. 2. State variable $x_{12}(t)$ in the $i$th iteration.

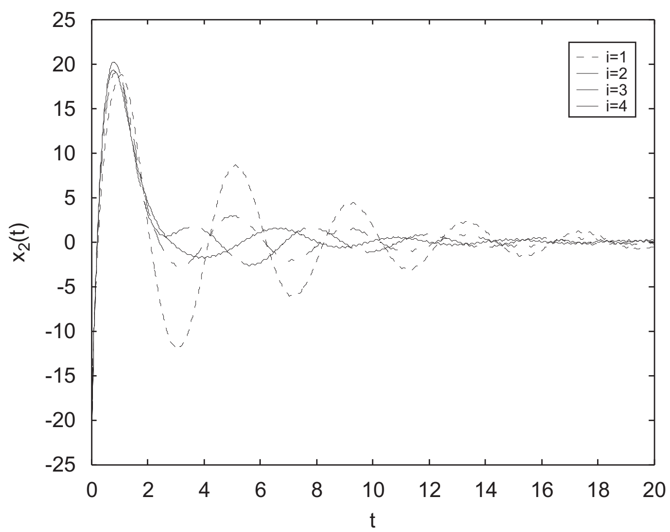


Fig. 3. State variable $x_2(t)$ in the $i$th iteration.

and the sinusoidal disturbances $v(t)$ can be expressed as

$$v(t) = \begin{bmatrix} 0.5\ \sin\left((\pi/10)t + \pi\right) \\ \sin\left((\pi/18)t\right) \end{bmatrix}.$$

We only know $|a| \leqslant 0.05$, but the exact value of $a$ cannot be known. So we get $\rho(x) = 0.05$. The quadratic performance index is chosen as (4), where $Q = I_2$, $R = 1$. The matrices $K$ and $E$ in Eq. (24) are chosen as $K = 1$, $E = 0.01$. Choosing $\varepsilon = 0.1$ and using Algorithm 1, we can get the simulation results of the state variables and the performance index values, as shown in Figs. 1–3. It clearly shows that the state variables of the closed-loop system are asymptotically stable. The relative error of performance index values satisfies $|(J_4 - J_3)/J_4| < \varepsilon$. Furthermore, the errors of the curves are smaller and smaller with the increase of the iteration

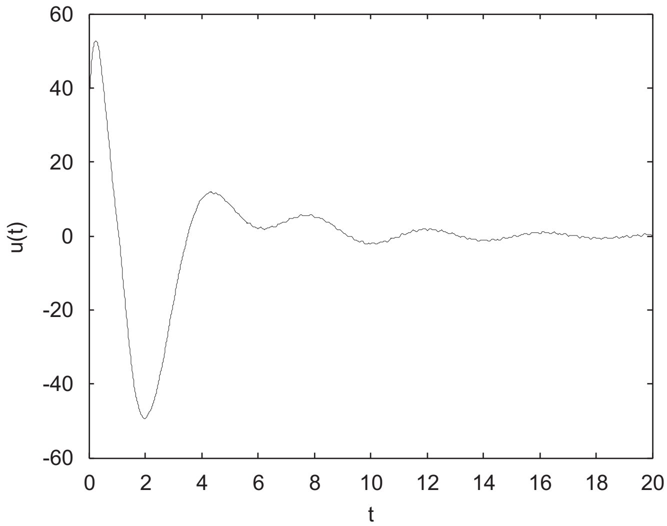


Fig. 4. Control input $u(t)$ of the closed-loop system.

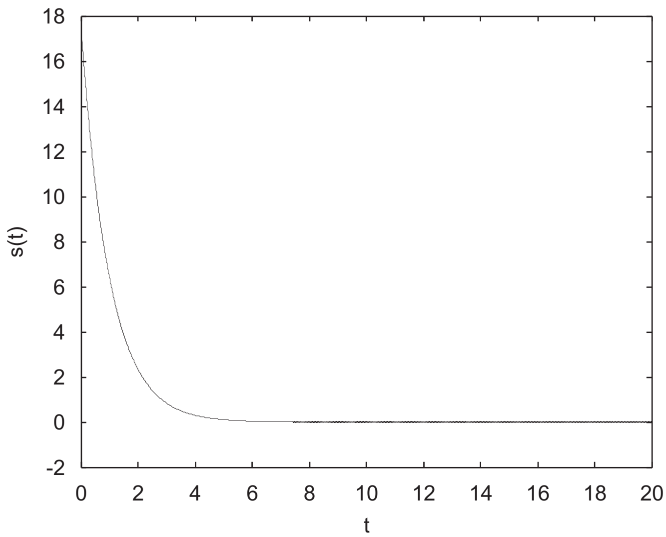


Fig. 5. Switching function $s(t)$ of the closed-loop system.

times. It shows that the fourth suboptimal sliding surface $s_4(t)$ is sufficiently close to the optimal sliding surface $s^*(t)$, and therefore switching function defined as (22) is obtained.

The new SMC can also be gotten according to Theorem 2. The simulation results of the control vector and the sliding function are shown in Figs. 4 and 5. It is obvious that the state variables of the closed-loop system can reach the sliding mode surface in finite time and remain on it. Furthermore, the controller can be implemented easily as shown in Fig. 5.

## 6. Conclusions

The paper has designed a suboptimal sliding surface for linear time-delay systems with sinusoidal disturbances based on the SAA. Moreover, a new SMC has been designed to force the state trajectories to reach the sliding surface in finite time. The stability of the sliding mode has been proved.

## Acknowledgments

This research was supported by the National Natural Science Foundation of China (60574023), the Key Natural Science Foundation of Shandong Province (Z2005G01), and the Natural Science Foundation of Qingdao City (05-1-JC-94).

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