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Eigenvalue problems of linear Hamiltonian systems arising from H_∞ filtering ☆

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Abstract

From the viewpoint of differential eigenvalue problem of Hamiltonian system, a linear finite-time H_∞ filtering problem can be addressed by computing eigenvalue and solving Riccati differential equation of an associated linear Hamiltonian system. This paper shows how a problem of determining the minimum induced norm of the H_∞ filter is formulated as a Hamiltonian differential eigenvalue problem. The H_∞ filters concerned here include central filter, perturbed filter and decentralised filter. The methods presented in the paper are based on characteristics of eigensolution of the corresponding Hamiltonian system and Riccati differential equation. With eigensolution of the Hamiltonian system arising from the central H_∞ filtering problem, variational methods are proposed to compute eigenvalues of perturbed Hamiltonian systems and large-scale Hamiltonian systems derived from perturbed H_∞ filters and decentralised H_∞ filters, respectively. Then, eigenvalues can be obtained by calculating stationary values of corresponding extended Rayleigh's quotient with dual argument functions, which is essentially different from the well-known Rayleigh's quotient of Lagrange systems with only one independent argument function. Numerical examples are also presented to illustrate the variational approaches presented in this paper.

1. Introduction

Design of finite-time H_∞ filters depends on computing the minimum H_∞ induced norm and solving Riccati differential equations[1], which can be approached by solving eigenvalue problems of associated linear Hamiltonian systems [2,3]. The aim of H_∞ filtering is to ensure that energy gain from disturbances to state estimation error is less than a pre-specified level γ^2, and the infimum of γ is defined as the minimum (optimal) H_∞ induced norm of the filter, denoted by γ_{cr} in this paper, it gives the bound for achievable performance of disturbance rejection [1]. Therefore, computation of γ_{cr} is a key step in designing H_∞ filters. Many approaches have been proposed to compute this minimum H_∞ norm as shown in Refs. [4,5] and the references therein. As pointed out in Ref. [2], γ_{cr}^{-2} is the fundamental eigenvalue of an extended Rayleigh's quotient with two

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independent argument functions, i.e. the first order eigenvalue of a linear Hamiltonian system, where a numerical method was also presented to compute γ_{cr}^{-2} and solve the Riccati equations. This paper will demonstrate that analysis and synthesis of parameter perturbed H_∞ filters and decentralised H_∞ filters lead to eigenvalue problems of Hamiltonian systems too.

Full exposition of eigenvalue problems of linear Hamiltonian systems can be found in Refs. [6,7]. However, thanks to their applications in applied mechanics and robust control [8–10], it is still necessary to develop numerical algorithms to compute eigenvalues and eigenfunctions of the Hamiltonian systems with dual argument functions, which are different to the well-established computational methods for Lagrange systems with only one independent argument function [11]. In terms of structural vibration, an eigenvalue extraction method for dual systems with mixed variables was presented in Ref. [8], which was also the basis of the algorithm proposed in Ref. [2] for H_∞ filtering problems. In Ref. [3], a method for the critical value computation of Riccati difference equations was presented, which relates linear discrete Hamiltonian systems with discrete H_∞ control problems also. Recently, to evaluate the disturbance attenuation performance of decentralised H_∞ control systems, a subsystem modal synthesis method for Hamiltonian systems was developed in Ref. [12].

Eigenvalue problems of the Hamiltonian systems arising from finite-time H_∞ filtering are investigated in the remaining sections, including the Hamiltonian systems associated with central H_∞ filter, perturbed H_∞ filter and decentralised H_∞ filter. Section 2 briefly describes the structure of a central H_∞ filter, whose existence depends on property of the associated Riccati differential equation and Hamiltonian system. In Section 3, orthonormal properties of eigenfunctions of the Hamiltonian system are presented. This section also proposes a new argumentation relating the minimum H_∞ norm and the first order eigenvalue of the Hamiltonian system. It is based on the property of conjugate points of the Hamiltonian system. In addition, an equivalent Sturm–Liouville eigenvalue problem is also presented for the purpose of comparison. In Section 4, a variational method is formulated for computing eigenvalues of a perturbed Hamiltonian system belongs to the perturbed H_∞ filtering system, which is a Rayleigh–Ritz-type method adapted to accommodate to the eigenvalue problems of the Hamiltonian systems with dual independent argument functions. In Section 5, this variational method is extended to form a subsystem modal synthesis method to compute eigenvalues of large-scale Hamiltonian systems consisting of subsystems, which are essential in designing decentralised H_∞ filters.

2. H_∞ filtering and associated linear Hamiltonian system

The H_∞ filter plays a similar role in the H_∞ control as the Kalman filter in the linear quadratic Gaussian (LQG) control, i.e. that the H_∞ control is generated by an H_∞ state feedback control law operating on estimates of the system states provided by an H_∞ filter. Suppose the signal to be estimated is generated by the linear time-invariant system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{w}, \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{2.1}$$

$$\mathbf{y} = \mathbf{C}_y\mathbf{x} + \mathbf{v}, \tag{2.2}$$

$$\mathbf{z} = \mathbf{C}_z\mathbf{x}, \tag{2.3}$$

where $\mathbf{x} \in R^n$ is the system state, $\mathbf{y} \in R^q$ the measured output, and $\mathbf{z} \in R^p$ the signal to be estimated. The process disturbance $\mathbf{w} \in R^m$ and the measurement disturbance $\mathbf{v} \in R^q$ are signals belong to the square integrabel set $\mathscr{L}_2[0, t_f]$. It is assumed that $(\mathbf{A}, \mathbf{B})$ is completely controllable and $(\mathbf{A}, \mathbf{C})$ is completely observable. The aim is to find an estimate of the linear combination of the system state $\mathbf{z} = \mathbf{C}_z\mathbf{x}$ of the form

$$\hat{\mathbf{z}} = \mathscr{F}\boldsymbol{y}, \tag{2.4}$$

such that the ratio of the estimation error energy to the disturbance energy is less than a pre-specified performance level γ^2. This objective can be expressed as the requirement that

$$\int_0^{t_f} (\hat{\mathbf{z}} - \mathbf{C}_z\mathbf{x})^{\mathrm{T}}(\hat{\mathbf{z}} - \mathbf{C}_z\mathbf{x})\,\mathrm{d}t < \gamma^2 \int_0^{t_f} (\mathbf{w}^{\mathrm{T}}\mathbf{w} + \mathbf{v}^{\mathrm{T}}\mathbf{v})\,\mathrm{d}t + \frac{1}{2}\mathbf{x}_0^{\mathrm{T}}\mathbf{Q}_0^{-1}\mathbf{x}_0 \tag{2.5}$$

for all $\mathbf{w} \in \mathscr{L}_2[0, t_f]$ and $\mathbf{v} \in \mathscr{L}_2[0, t_f]$. The filter $\mathscr{F}$ is also required to be causal and linear.

According to Ref. [13], there exits a causal linear finite-time H_∞ filter (2.4) on $[0, t_f]$ such that the system satisfies the requirement of (2.5) if and only if the Riccati differential equation

$$\dot{\mathbf{Q}} = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A}^{\mathrm{T}} - \mathbf{Q}(\mathbf{C}_y^{\mathrm{T}}\mathbf{C}_y - \gamma^{-2}\mathbf{C}_z^{\mathrm{T}}\mathbf{C}_z)\mathbf{Q}, \quad \mathbf{Q}(0) = \mathbf{Q}_0 \tag{2.6}$$

has a solution on $[0, t_f]$. One filter that satisfies this objective is given by

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{Q}\mathbf{C}_y^{\mathrm{T}}(\mathbf{y} - \mathbf{C}_y\hat{\mathbf{x}}), \quad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0, \tag{2.7}$$

$$\hat{\mathbf{z}} = \mathbf{C}_z\hat{\mathbf{x}}. \tag{2.8}$$

This filter is a central H_∞ filter and has an observer structure like the Kalman filter. Since Riccati equation (2.6) has no bounded solution if γ^{-2} is larger than the critical value $\gamma_{\mathrm{cr}}^{-2}$, it is necessary to find $\gamma_{\mathrm{cr}}^{-2}$ first, then to select a suitable value of γ^{-2} according to $\gamma_{\mathrm{cr}}^{-2}$ and pre-specified performance index for the H_∞ filter. With this value, Riccati differential Eq. (2.6) can be solved numerically to construct the time-varying filter gain.

The following linear Hamiltonian system:

$$\begin{Bmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\lambda}} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{B}^{\mathrm{T}} \\ \mathbf{C}_y^{\mathrm{T}}\mathbf{C}_y - \gamma^{-2}\mathbf{C}_z^{\mathrm{T}}\mathbf{C}_z & -\mathbf{A}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{Bmatrix} \tag{2.9}$$

with boundary conditions

$$\begin{Bmatrix} \mathbf{x}(0) \\ \boldsymbol{\lambda}(t_f) \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q}_0\boldsymbol{\lambda}(0) \\ 0 \end{Bmatrix} \tag{2.10}$$

is the Hamiltonian system associated with aforementioned finite-time H_∞ filtering problem. Boundary condition (2.10) can also be written in a general form as

$$\mathbf{R}_1 \begin{Bmatrix} -\mathbf{x}(0) \\ \mathbf{x}(t_f) \end{Bmatrix} + \mathbf{R}_2 \begin{Bmatrix} \boldsymbol{\lambda}(0) \\ \boldsymbol{\lambda}(t_f) \end{Bmatrix} = 0, \tag{2.11}$$

where

$$\mathbf{R}_1 = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{R}_2 = \begin{bmatrix} \mathbf{Q}_0 & 0 \\ 0 & \mathbf{I} \end{bmatrix}. \tag{2.12}$$

Let $\boldsymbol{\Phi}(0, t)$ be the transition matrix of Hamiltonian system (2.9):

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Phi}(0, t) = \mathbf{H}\boldsymbol{\Phi}(0, t), \quad \boldsymbol{\Phi}(0, 0) = \mathbf{I} \tag{2.13}$$

in which

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{B}^{\mathrm{T}} \\ \mathbf{C}_y^{\mathrm{T}}\mathbf{C}_y - \gamma^{-2}\mathbf{C}_z^{\mathrm{T}}\mathbf{C}_z & -\mathbf{A}^{\mathrm{T}} \end{bmatrix}. \tag{2.14}$$

Then, solution of Eq. (2.9) is given by

$$\begin{Bmatrix} \mathbf{x}(t) \\ \boldsymbol{\lambda}(t) \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_{11}(0, t) & \boldsymbol{\Phi}_{12}(0, t) \\ \boldsymbol{\Phi}_{21}(0, t) & \boldsymbol{\Phi}_{22}(0, t) \end{bmatrix} \begin{Bmatrix} \mathbf{x}(0) \\ \boldsymbol{\lambda}(0) \end{Bmatrix}. \tag{2.15}$$

It can be verified that

$$\mathbf{Q}(t) = [\boldsymbol{\Phi}_{11}(0, t)\mathbf{Q}_0 + \boldsymbol{\Phi}_{12}(0, t)][\boldsymbol{\Phi}_{21}(0, t)\mathbf{Q}_0 + \boldsymbol{\Phi}_{22}(0, t)]^{-1} \tag{2.16}$$

is solution to the Riccati differential equation (2.6).

The minimum H_∞ induced norm of the concerned filtering problem can be obtained by eigenvalue algorithms, e.g. the extended Williams–Wittrick algorithm for dual systems [2,8], since $\gamma_{\mathrm{cr}}^{-2}$ is the first-order eigenvalue of the Hamiltonian system (2.9). In this avenue, other H_∞ filters, such as perturbed filters and decentralised filters, which have their roots in the aforementioned central H_∞ filtering problem can also be

approached by investigating eigenvalue problems of corresponding Hamiltonian systems. And the derived eigenvalue problems can be solved by variational methods with less computational efforts provided that eigensolutions of Eqs. (2.9) and (2.10) have been known.

3. Eigensolutions of linear Hamiltonian system

The eigenvalue problem of linear Hamiltonian system (2.9) and (2.10) provides a new measure to approach finite-time H_∞ filtering problems. This section will give a brief introduction on properties of the Hamiltonian systems eigensolutions and Riccati differential equations, which are basis of the variational methods proposed in Sections 4 and 5. To relate the eigenvalue problem here with differential eigenvalue problem in structural vibration, an equivalent Sturm–Liouville eigenvalue problem is also presented in this section.

3.1. Eigenvalues of linear Hamiltonian system

In terms of Ref. [7], γ^{-2} is an eigenvalue of the linear Hamiltonian system (2.9) and (2.10) if and only if the matrix

$$\mathbf{\Omega} = \mathbf{R}_1 \begin{bmatrix} -\mathbf{\Phi}_{11}(0,0) & -\mathbf{\Phi}_{12}(0,0) \\ \mathbf{\Phi}_{11}(0,t_f) & \mathbf{\Phi}_{12}(0,t_f) \end{bmatrix} + \mathbf{R}_2 \begin{bmatrix} \mathbf{\Phi}_{21}(0,0) & \mathbf{\Phi}_{22}(0,0) \\ \mathbf{\Phi}_{21}(0,t_f) & \mathbf{\Phi}_{22}(0,t_f) \end{bmatrix} \tag{3.1}$$

is invertable, where $\mathbf{R}_1$ and $\mathbf{R}_2$ are given by (2.12). Substituting Eq. (2.12) into (3.1) leads to

$$\mathbf{\Omega} = \begin{bmatrix} -\mathbf{I} & \mathbf{Q}_0 \\ \mathbf{\Phi}_{21}(0,t_f) & \mathbf{\Phi}_{22}(0,t_f) \end{bmatrix}. \tag{3.2}$$

Because the determinant of $\mathbf{\Omega}$ is given by

$$\det \mathbf{\Omega} = \det \begin{bmatrix} -\mathbf{I} & \mathbf{Q}_0 \\ \mathbf{\Phi}_{21}(0,t_f) & \mathbf{\Phi}_{22}(0,t_f) \end{bmatrix} = \det(-\mathbf{I}) \cdot \det[\mathbf{\Phi}_{21}(0,t_f)\mathbf{Q}_0 + \mathbf{\Phi}_{22}(0,t_f)], \tag{3.3}$$

the singularity of $\mathbf{\Omega}$ depends on matrix $\mathbf{\Phi}_{21}(0, t_f)\mathbf{Q}_0 + \mathbf{\Phi}_{22}(0, t_f)$, so that γ^{-2} is a eigenvalue of (2.9) and (2.10) if and only if $\mathbf{\Phi}_{21}(0, t_f)\mathbf{Q}_0 + \mathbf{\Phi}_{22}(0, t_f)$ is a singular matrix.

The H_∞ filtering problem has a solution whenever Riccati differential equation (2.6) has a solution. However, in the light of the existence of conjugate points, the matrix $\mathbf{\Phi}_{21}(0, t_f)\mathbf{Q}_0 + \mathbf{\Phi}_{22}(0, t_f)$ is also a criterion for the existence of solutions to the Riccati differential equation (2.6). Conjugate points are two time points t_a and $t_b (t_a \leqslant t_b)$ of the Hamiltonian system (2.9) and (2.10), for which (2.9) has a nontrivial solution such that $\mathbf{x}(t_a) = \mathbf{Q}_0\boldsymbol{\lambda}(t_a)$ and $\boldsymbol{\lambda}(t_b) = 0$ for a given fixed $\mathbf{Q}_0$ [1,13]. The matrix $\mathbf{\Phi}_{21}(0, t)\mathbf{Q}_0 + \mathbf{\Phi}_{22}(0, t)$ is singular if and only if 0 and t are conjugate points. The argumentation of this conclusion is given as follows.

Suppose t_0 and t_1 are conjugate points, then there exists a nontrivial solution to (2.9) such that $\mathbf{x}(t_0) = \mathbf{Q}_0\boldsymbol{\lambda}(t_0)$ and $\boldsymbol{\lambda}(t_1) = 0$. Hence,

$$\begin{Bmatrix} \mathbf{x}(t) \\ \boldsymbol{\lambda}(t) \end{Bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{11}(t_0,t) & \mathbf{\Phi}_{12}(t_0,t) \\ \mathbf{\Phi}_{21}(t_0,t) & \mathbf{\Phi}_{22}(t_0,t) \end{bmatrix} \begin{Bmatrix} \mathbf{Q}_0 \\ \mathbf{I} \end{Bmatrix} \boldsymbol{\lambda}(t_0)$$

and one notes that $\boldsymbol{\lambda}(t_0) \neq 0$, because $\mathbf{x}(t)$, $\boldsymbol{\lambda}(t)$ are not identically zero and $\mathbf{\Phi}(0, t)$ is non-singular for all t, t_0. Since $\boldsymbol{\lambda}(t_1) = 0$, one has $\boldsymbol{\lambda}(t_1) = [\mathbf{\Phi}_{21}(t_0, t_1)\mathbf{Q}_0 + \mathbf{\Phi}_{22}(t_0, t_1)]\boldsymbol{\lambda}(\mathbf{t_0}) = 0$, which means that $\mathbf{\Phi}_{21}(t_0, t_1)\mathbf{Q}_0 + \mathbf{\Phi}_{22}(t_0, t_1)$ is singular.

Now suppose that $\mathbf{\Phi}_{21}(t_0, t_1)\mathbf{Q}_0 + \mathbf{\Phi}_{22}(t_0, t_1)$ is singular. Then there exists a vector $\mathbf{g} \neq 0$ such that $0 = [\mathbf{\Phi}_{21}(t_0, t_1)\mathbf{Q}_0 + \mathbf{\Phi}_{22}(t_0, t_1)]\mathbf{g}$. Let $\boldsymbol{\lambda}(t_0) = \mathbf{g}$, then $\mathbf{x}(t_0) = \mathbf{Q}_0\boldsymbol{\lambda}(t_0)$, $\boldsymbol{\lambda}(t_1) = 0$, and $\boldsymbol{\lambda}(t)$ is not identically zero. Hence t_0 and t_1 are conjugates points. Hence, the required proof is complete.

Note that

$$\mathbf{Q}(t) = [\mathbf{\Phi}_{11}(0,t)\mathbf{Q}_0 + \mathbf{\Phi}_{12}(0,t)][\mathbf{\Phi}_{21}(0,t)\mathbf{Q}_0 + \mathbf{\Phi}_{22}(0,t)]^{-1}$$

the Riccati differential equation (2.6) has a solution on $[0, t_f]$ provided that $\mathbf{\Phi}_{21}(0, t)\mathbf{Q}_0 + \mathbf{\Phi}_{22}(0, t)$ is non-singular for all $t \in [0, t_f]$. Therefore, it can be concluded that the Riccati equation (2.6) does not have a solution on $[0, t_f]$ if there is a $t \in [0, t_f]$ for which 0 and t are conjugate points.

It should be noted that eigenvalues of the Hamiltonian system (2.9) and (2.10) are also stationary values of an extended Rayleigh's quotient with dual independent argument functions [2]

$$\rho = \frac{\Pi_1(\mathbf{x}, \boldsymbol{\lambda})}{\Pi_2(\mathbf{x})} \tag{3.4}$$

in which

$$\Pi_1(\mathbf{x}, \boldsymbol{\lambda}) = \int_0^{t_f} \left(\boldsymbol{\lambda}^{\mathrm{T}} \dot{\mathbf{x}} - \boldsymbol{\lambda}^{\mathrm{T}} \mathbf{A}\mathbf{x} + \frac{1}{2}\mathbf{x}^{\mathrm{T}} \mathbf{C}_y^{\mathrm{T}} \mathbf{C}_y \mathbf{x} - \frac{1}{2}\boldsymbol{\lambda}^{\mathrm{T}} \mathbf{B}\mathbf{B}^{\mathrm{T}} \boldsymbol{\lambda} \right) \mathrm{d}t + \frac{1}{2}\mathbf{x}_0^{\mathrm{T}} \mathbf{Q}_0^{-1} \mathbf{x}_0, \tag{3.5}$$

$$\Pi_2(\mathbf{x}) = \frac{1}{2} \int_0^{t_f} \mathbf{x}^{\mathrm{T}} \mathbf{C}_z^{\mathrm{T}} \mathbf{C}_z \mathbf{x} \, \mathrm{d}t. \tag{3.6}$$

Such expression facilitates direct variational methods for eigenvalues computation as shown in the following sections.

By setting $\gamma^{-2} = 0$, the Hamiltonian system (2.9) and (2.10) is changed into

$$\begin{Bmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\lambda}} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{B}^{\mathrm{T}} \\ \mathbf{C}_y^{\mathrm{T}} \mathbf{C}_y & -\mathbf{A}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{Bmatrix}, \quad \begin{Bmatrix} \mathbf{x}(0) \\ \boldsymbol{\lambda}(t_f) \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q}_0 \boldsymbol{\lambda}(0) \\ 0 \end{Bmatrix}, \tag{3.7}$$

which is the Hamiltonian system associated with the Kalman filtering problem and does not have conjugate points on $[0, t_f]$ at all time, so that the associated Riccati equation

$$\dot{\mathbf{Q}} = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A}^{\mathrm{T}} - \mathbf{Q}\mathbf{C}_y^{\mathrm{T}} \mathbf{C}_y \mathbf{Q}, \quad \mathbf{Q}(0) = \mathbf{Q}_0 \tag{3.8}$$

always has a solution, that is that the Kalman filtering problem always has a solution. This also means that the Kalman filter should emerge from the H_∞ filter theory in the limit as $r \to \infty$ [13].

3.2. Orthogonality of eigenfunctions

To simplify following expressions, define differential operators

$$\mathscr{H} = \mathbf{J}_s \frac{\mathrm{d}}{\mathrm{d}t} - \begin{bmatrix} -\mathbf{C}_y^{\mathrm{T}} \mathbf{C}_y & \mathbf{A}^{\mathrm{T}} \\ \mathbf{A} & \mathbf{B}\mathbf{B}^{\mathrm{T}} \end{bmatrix}, \tag{3.9}$$

$$\mathrm{M} = \begin{bmatrix} \mathbf{C}_z^{\mathrm{T}} \mathbf{C}_z & 0 \\ 0 & 0 \end{bmatrix} \tag{3.10}$$

and let

$$\mathbf{v} = \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix}, \tag{3.11}$$

$$\rho = \gamma^{-2}, \tag{3.12}$$

where the symplectic matrix

$$\mathbf{J}_s = \begin{bmatrix} 0 & -\mathbf{I} \\ \mathbf{I} & 0 \end{bmatrix} \tag{3.13}$$

and $\boldsymbol{I}$ represents an identity matrix with proper dimensions. Then Eq. (2.9) is given by

$$\mathscr{H}\mathbf{v} = \rho \mathscr{M}\mathbf{v}. \tag{3.14}$$

According to Ref. [6], the definiteness assumptions that $\mathscr{M} \geqslant 0$ and $\int_0^{t_f} \mathbf{v}^{\mathrm{T}} \mathscr{M} \mathbf{v} \, \mathrm{d}t > 0$ are made for (3.14) to guarantee the concerned eigenvalue problem have a real and discrete spectrum.

Assume that $\boldsymbol{\mu} = [\mathbf{x}_\mu^{\mathrm{T}}, \boldsymbol{\lambda}_\mu^{\mathrm{T}}]^{\mathrm{T}}$ and $\boldsymbol{\theta} = [\mathbf{x}_\theta^{\mathrm{T}}, \boldsymbol{\lambda}_\theta^{\mathrm{T}}]^{\mathrm{T}}$ are differentiable functions, and satisfy boundary condition (2.10) of the eigenvalue problem, then

$$\int_0^{t_f} \boldsymbol{\mu}^{\mathrm{T}} \mathscr{H} \boldsymbol{\theta} \, \mathrm{d}t = \int_0^{t_f} (\boldsymbol{\lambda}_\mu^{\mathrm{T}} \dot{\mathbf{x}}_\theta + \boldsymbol{\lambda}_\theta^{\mathrm{T}} \dot{\mathbf{x}}_\mu - \boldsymbol{\lambda}_\mu^{\mathrm{T}} \mathbf{A} \mathbf{x}_\theta - \boldsymbol{\lambda}_\theta^{\mathrm{T}} \mathbf{A} \mathbf{x}_\mu - \boldsymbol{\lambda}_\mu^{\mathrm{T}} \mathbf{B} \mathbf{B}^{\mathrm{T}} \boldsymbol{\lambda}_\theta + \mathbf{x}_\mu^{\mathrm{T}} \mathbf{C}_y^{\mathrm{T}} \mathbf{C}_y \mathbf{x}_\theta) \, \mathrm{d}t + \mathbf{x}_\mu^{\mathrm{T}}(0) \mathbf{Q}_0^{-1} \mathbf{x}_\theta(0). \tag{3.15}$$

Hence,

$$\int_0^{t_f} (\boldsymbol{\theta}^{\mathrm{T}} \mathscr{H} \boldsymbol{\mu} - \boldsymbol{\mu}^{\mathrm{T}} \mathscr{H} \boldsymbol{\theta}) \, \mathrm{d}t = 0, \tag{3.16}$$

which means that operator (3.9) is a self-adjoint differential operator and the system (2.9) and (2.10) is a self-adjoint system.

Let ρ_i, $\mathbf{v}_i = [\mathbf{x}_i^{\mathrm{T}}, \boldsymbol{\lambda}_i^{\mathrm{T}}]^{\mathrm{T}}$ and ρ_j, $\mathbf{v}_j = [\mathbf{x}_j^{\mathrm{T}}, \boldsymbol{\lambda}_j^{\mathrm{T}}]^{\mathrm{T}}$ denote the ith and jth order distinct eigenvalue and eigenfunction, respectively ($i \neq j$, $\rho_i \neq \rho_j$), orthogonality relations of the eigenfunctions are given by

$$\int_0^{t_f} \mathbf{v}_j^{\mathrm{T}} \mathscr{M} \mathbf{v}_i \, \mathrm{d}t = \int_0^{t_f} \mathbf{x}_j^{\mathrm{T}} \mathbf{C}_z^{\mathrm{T}} \mathbf{C}_z \mathbf{x}_i \, \mathrm{d}t = 0 \quad (i \neq j, \;\; i,j = 1,2,\ldots), \tag{3.17}$$

$$\int_0^{t_f} \mathbf{v}_j^{\mathrm{T}} \mathscr{H} \mathbf{v}_i \, \mathrm{d}t = 0 \quad (i \neq j, \;\; i,j = 1,2,\ldots). \tag{3.18}$$

The orthogonality of the eigenfunctions is a direct consequence of the system being self-adjoint. The normalization condition is given by

$$\frac{1}{2} \int_0^{t_f} \mathbf{v}_i^{\mathrm{T}} \mathscr{M} \mathbf{v}_i \, \mathrm{d}t = \frac{1}{2} \int_0^{t_f} \mathbf{x}_i^{\mathrm{T}} \mathbf{C}_z^{\mathrm{T}} \mathbf{C}_z \mathbf{x}_i \, \mathrm{d}t = 1, \tag{3.19}$$

which implies that

$$\int_0^{t_f} \mathbf{v}_i^{\mathrm{T}} \mathscr{H} \mathbf{v}_i \, \mathrm{d}t = \rho_i \int_0^{t_f} \mathbf{v}_i^{\mathrm{T}} \mathscr{M} \mathbf{v}_i \, \mathrm{d}t = 2\rho_i. \tag{3.20}$$

With the assumptions that $\mathscr{M} \geqslant 0$ and $\int_0^{t_f} \mathbf{v}^{\mathrm{T}} \mathscr{M} \mathbf{v} \, \mathrm{d}t > 0$, the eigenfunctions $\mathbf{v}_i$ ($i = 1, 2, \ldots$) constitute a complete orthonormal set of infinite dimension, and can be used as a basis for a function space [6]. Therefore, every differential function $\mathbf{v}$ satisfying the boundary condition (2.10) can be expanded in an absolutely and uniformly convergent series in the eigenfunctions in the form

$$\mathbf{v}(t) = \sum_{i=1}^{\infty} \beta_i \mathbf{v}_i(t) = \sum_{i=1}^{\infty} \beta_i \begin{Bmatrix} \boldsymbol{\lambda}_i(t) \\ \mathbf{x}_i(t) \end{Bmatrix}, \tag{3.21}$$

where the coefficients β_i are such that

$$\beta_i = \frac{1}{2} \int_0^{t_f} \mathbf{v}_i^{\mathrm{T}} \mathscr{M} \mathbf{v} \, \mathrm{d}t \quad (j = 1,2,\ldots). \tag{3.22}$$

3.3. An equivalent Sturm–Liouville eigenvalue problem

Because the Hamiltonian system is a self-adjoint differential system, it can be transformed into an equivalent Sturm–Liouville form. This transformation connects the eigenvalue problem discussed in this paper with the well-known differential eigenvalue problems in structural vibration [11].

The Hamiltonian eigenvalue problem (2.9) and (2.10) is a canonical form with two independent argument functions, which corresponds to a variational problem of the form

$$\delta\left\{\int_0^{t_f}\left[\boldsymbol{\lambda}^{\mathrm{T}}\dot{\mathbf{x}}-H(\mathbf{x},\boldsymbol{\lambda})\right]\mathrm{d}t+\frac{1}{2}\mathbf{x}^{\mathrm{T}}(0)\mathbf{Q}_0^{-1}\mathbf{x}(0)\right\}=0, \tag{3.23}$$

where

$$H(\mathbf{x},\boldsymbol{\lambda})=\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{A}\mathbf{x}-\tfrac{1}{2}\mathbf{x}^{\mathrm{T}}(\mathbf{C}_y^{\mathrm{T}}\mathbf{C}_y-\gamma^{-2}\mathbf{C}_z^{\mathrm{T}}\mathbf{C}_z)\mathbf{x}+\tfrac{1}{2}\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}. \tag{3.24}$$

As shown in Ref. [14], a variational problem with only one independent argument function can be transformed into canonical form with two independent argument functions, whereas the inverse transformation can also be carried out, as shown in the following.

Consider the variational problem

$$\delta\left\{\int_0^{t_f}L(\mathbf{x},\dot{\mathbf{x}})\,\mathrm{d}t+\frac{1}{2}\mathbf{x}^{\mathrm{T}}(0)\mathbf{Q}_0^{-1}\mathbf{x}(0)\right\}=0, \tag{3.25}$$

where

$$L(\mathbf{x},\dot{\mathbf{x}})=\tfrac{1}{2}\dot{\mathbf{x}}^{\mathrm{T}}\mathbf{K}_{22}\dot{\mathbf{x}}+\dot{\mathbf{x}}^{\mathrm{T}}\mathbf{K}_{21}\mathbf{x}+\tfrac{1}{2}\mathbf{x}^{\mathrm{T}}(\mathbf{K}_{11}-\rho\mathbf{M})\mathbf{x}, \tag{3.26}$$

which corresponds to a Sturm–Liouville eigenvalue problem

$$\mathbf{K}_{22}\ddot{\mathbf{x}}+(\mathbf{K}_{21}-\mathbf{K}_{21}^{\mathrm{T}})\dot{\mathbf{x}}-\mathbf{K}_{11}\mathbf{x}+\rho\mathbf{M}\mathbf{x}=0 \tag{3.27}$$

defined on $[0,\ t_{\mathrm{f}}]$ with boundary conditions

$$\mathbf{x}_0-\mathbf{Q}_0(\mathbf{K}_{22}\dot{\mathbf{x}}_0+\mathbf{K}_{21}\mathbf{x}_0)=0,\quad \mathbf{K}_{22}\dot{\mathbf{x}}_f+\mathbf{K}_{21}\mathbf{x}_f=0, \tag{3.28}$$

where $\mathbf{K}_{22}$, $\mathbf{K}_{11}$, and $(\mathbf{K}_{21}-\mathbf{K}_{21}^{\mathrm{T}})$ are symmetric matrices, the matrix $\mathbf{M}>0$. Introduce new argument functions $\boldsymbol{\lambda}=\partial L(\mathbf{x},\dot{\mathbf{x}})/\partial\dot{\mathbf{x}}$ and $\dot{\boldsymbol{\lambda}}=-\partial[\boldsymbol{\lambda}^{\mathrm{T}}\dot{\mathbf{x}}-L(\mathbf{x},\dot{\mathbf{x}})]/\partial\mathbf{x}$, gives

$$\begin{Bmatrix}\dot{\mathbf{x}}\\ \dot{\boldsymbol{\lambda}}\end{Bmatrix}=\begin{bmatrix}-\mathbf{K}_{22}^{-1}\mathbf{K}_{21} & \mathbf{K}_{22}^{-1}\\ \mathbf{K}_{11}-\mathbf{K}_{21}^{\mathrm{T}}\mathbf{K}_{22}^{-1}\mathbf{K}_{21}-\rho\mathbf{M} & \mathbf{K}_{21}^{\mathrm{T}}\mathbf{K}_{22}^{-1}\end{bmatrix}\begin{Bmatrix}\mathbf{x}\\ \boldsymbol{\lambda}\end{Bmatrix}. \tag{3.29}$$

Assume that $\mathbf{B}\mathbf{B}^{\mathrm{T}}$ is invertible, let

$$\mathbf{K}_{22}=(\mathbf{B}\mathbf{B}^{\mathrm{T}})^{-1}, \tag{3.30}$$

$$\mathbf{K}_{21}=-(\mathbf{B}\mathbf{B}^{\mathrm{T}})^{-1}\mathbf{A}, \tag{3.31}$$

$$\mathbf{K}_{11}=\mathbf{C}_y^{\mathrm{T}}\mathbf{C}_y+\mathbf{A}^{\mathrm{T}}(\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{A}, \tag{3.32}$$

$$\mathbf{M}=\mathbf{C}_z^{\mathrm{T}}\mathbf{C}_z, \tag{3.33}$$

then Eqs. (3.28) and (3.29) can be expressed in a form identical to equations (2.9) and (2.10), respectively. Hence, the Sturm–Liouville eigenvalue problem (3.29) and (3.28) is an equivalent form of the Hamiltonian eigenvalue problem (2.9) and (2.10).

4. Perturbed H_∞ filter and corresponding eigenvalue problem

Parameters of an H_∞ filtering system may be changed by design modifications or improved knowledge of the system, etc. [15]. Correspondingly, disturbance attenuation performance of the filter changes also. From the viewpoint of eigenvalue perturbation, the minimum H_∞ norm γ_{cr} of the perturbed filtering system can be calculated by eigenvalue perturbation technique, since $\gamma_{\mathrm{cr}}^{-2}$ is the first-order eigenvalue of the Hamiltonian system (2.9) and (2.10). However, in terms of the extended Rayleigh's quotient (3.4), eigenvalues of the perturbed Hamiltonian system may also be approximated by a Rayleigh–Ritz method adapted for two independent argument functions[12], which replaces the considered differential eigenvalue problem by algebraic eigenvalue problems. Such variational method reduces computational effort if eigenfunctions of the

unperturbed system are known. Because all the eigensolutions can be obtained in the procedure of computing the minimum H_∞ norm of the original system [2,12], the trial functions for this problem may consist of eigenfunctions of the unperturbed Hamiltonian system.

4.1. Variational method for eigenvalue problem of perturbed Hamiltonian systems

Consider only perturbations of the plant matrix $\mathbf{A}$, let $\mathbf{A}_0$ and $\varepsilon\mathbf{A}_0$ denote the unperturbed plant matrix and the perturbation, respectively, then Eq. (2.1) is given as

$$\dot{\mathbf{x}} = (\mathbf{A}_0 + \varepsilon\mathbf{A}_1)\mathbf{x} + \mathbf{B}\mathbf{w}, \quad \mathbf{x}(0) = \mathbf{x}_0. \tag{4.1}$$

Let ρ_{0j}, $\mathbf{v}_{0j} = [\mathbf{x}_{0j}^{\mathrm{T}}, \boldsymbol{\lambda}_{0j}^{\mathrm{T}}]^{\mathrm{T}}$ $(j = 1, 2, \ldots)$ denote eigensolutions of the unperturbed Hamiltonian system

$$\begin{Bmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\lambda}} \end{Bmatrix} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{B}\mathbf{B}^{\mathrm{T}} \\ \mathbf{C}_y^{\mathrm{T}}\mathbf{C}_y - \rho\mathbf{C}_z^{\mathrm{T}}\mathbf{C}_z & -\mathbf{A}_0^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{Bmatrix}, \quad \begin{Bmatrix} \mathbf{x}(0) \\ \boldsymbol{\lambda}(t_f) \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q}_0\boldsymbol{\lambda}(0) \\ 0 \end{Bmatrix}. \tag{4.2}$$

Eigenvalues of the perturbed Hamiltonian system are still stationary values of an extended Rayleigh's quotient, so that the beginning eigenvalues may be computed by variational methods directly. Argument functions of the extended Rayleigh's quotient (3.4) are two independent functions $\mathbf{x}$ and $\boldsymbol{\lambda}$. The state vector function $\mathbf{x}$ should be expanded by using the eigenfunctions $\mathbf{x}_{0j}$ and its dual vector function $\boldsymbol{\lambda}$ should be expanded by using $\boldsymbol{\lambda}_{0j}$. In terms of variational principles, coefficients of two expansions are independent of each other, so the state vector and its dual vector should be expanded independently.

The eigenfunctions of the unperturbed Hamiltonian system (4.2) constitute a complete orthonormal set, so that $\mathbf{x}$ and $\boldsymbol{\lambda}$ are expanded as

$$\mathbf{x} = \sum_{j=1}^{\infty} b_j\mathbf{x}_{0j}, \tag{4.3}$$

$$\boldsymbol{\lambda} = \sum_{k=1}^{\infty} a_k\boldsymbol{\lambda}_{0k}. \tag{4.4}$$

According to the linear combination principle, the trial functions for the coefficients b_k can be selected as the complete eigenfunctions of the subsystem, so that

$$\mathbf{x} = \sum_{j=1}^{\infty} b_j\mathbf{x}_{0j}, \tag{4.5}$$

$$\boldsymbol{\lambda} = \sum_{k=1}^{\infty} (a_k + b_k)\boldsymbol{\lambda}_{0k}. \tag{4.6}$$

Theoretically, an infinite number of eigenfunctions should be used in computation, but in practice, only a finite number of eigenfunctions can be provided. Suppose that the numbers of eigenfunctions available are n_e, then

$$\mathbf{x} \cong \sum_{j=1}^{n_e} b_j\mathbf{x}_{0j}, \tag{4.7}$$

$$\boldsymbol{\lambda} \cong \sum_{k=1}^{n_e} (a_k + b_k)\boldsymbol{\lambda}_{0k}, \tag{4.8}$$

where coefficients a_k and b_j are independent parameters to be determined by finding stationary values of the extended Rayleigh's quotient (3.4). Substituting Eqs. (4.7), (4.8) into Eqs. (3.5), (3.6), respectively, and using the orthonormal properties of eigenfunctions and

$$\dot{\mathbf{x}}_{0i} = \mathbf{A}_0\mathbf{x}_{0i} + \mathbf{B}\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}_{0i}, \tag{4.9}$$

$$\dot{\boldsymbol{\lambda}}_{0i} = (\mathbf{C}_y^{\mathrm{T}}\mathbf{C}_y - \rho_{0i}\mathbf{C}_z^{\mathrm{T}}\mathbf{C}_z)\mathbf{x}_{0i} + \mathbf{A}_0^{\mathrm{T}}\boldsymbol{\lambda}_{0i}, \tag{4.10}$$

one obtains

$$\begin{aligned}
\Pi_1(\mathbf{x},\boldsymbol{\lambda}) &\cong \tilde{\Pi}_1(\mathbf{a},\mathbf{b}) \\
&= \sum_{j=1}^{n_e}\rho_{0j}b_j^2 - \sum_{j=1}^{n_e}\sum_{k=1}^{n_e}\int_0^{t_f}\frac{1}{2}(a_j a_k)\cdot(\boldsymbol{\lambda}_{0j}^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}_{0k})\mathrm{d}t \\
&\quad - \sum_{j=1}^{n_e}\sum_{k=1}^{n_e}\int_0^{t_f}\frac{1}{2}[\varepsilon b_j(a_k+b_k)\cdot(\boldsymbol{\lambda}_{0j}^{\mathrm{T}}\mathbf{A}_1\mathbf{x}_{0k}) + \varepsilon b_j(a_k+b_k)\cdot(\mathbf{x}_{0k}^{\mathrm{T}}\mathbf{A}_1^{\mathrm{T}}\boldsymbol{\lambda}_{0j})]\mathrm{d}t \\
&= \mathbf{b}^{\mathrm{T}}\boldsymbol{\Gamma}\mathbf{b} - \frac{1}{2}\mathbf{a}^{\mathrm{T}}\boldsymbol{\Psi}\mathbf{a} - \frac{\varepsilon}{2}[(\mathbf{a}+\mathbf{b})^{\mathrm{T}}\boldsymbol{\Psi}_1\mathbf{b} + \mathbf{b}^{\mathrm{T}}\boldsymbol{\Psi}_1^{\mathrm{T}}(\mathbf{a}+\mathbf{b})],
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\Pi_2(\mathbf{x}) &\cong \tilde{\Pi}_2(\mathbf{b}) \\
&= \sum_{j=1}^{n_e}\sum_{k=1}^{n_e}\int_0^{t_f}\frac{1}{2}[(b_j b_k)\cdot(\mathbf{x}_{0j}^{\mathrm{T}}\mathbf{C}_z^{\mathrm{T}}\mathbf{C}_z\mathbf{x}_{0k})]\mathrm{d}t \\
&= \sum_{j=1}^{n_e} b_j^2 = \mathbf{b}^{\mathrm{T}}\mathbf{b},
\end{aligned} \tag{4.12}$$

in which

$$\boldsymbol{\Psi} = \int_0^{t_f}\boldsymbol{\Lambda}^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\boldsymbol{\Lambda}\,\mathrm{d}t, \tag{4.13}$$

$$\boldsymbol{\Psi}_1 = \int_0^{t_f}\boldsymbol{\Lambda}^{\mathrm{T}}\mathbf{A}_1\mathbf{X}\,\mathrm{d}t, \tag{4.14}$$

$$\boldsymbol{\Gamma} = \mathrm{diag}[\rho_{01},\rho_{02},\ldots,\rho_{0n_e}], \tag{4.15}$$

$$\boldsymbol{\Lambda} = [\boldsymbol{\lambda}_{01},\boldsymbol{\lambda}_{02},\ldots,\boldsymbol{\lambda}_{0n_e}], \tag{4.16}$$

$$\mathbf{X} = [\mathbf{x}_{01},\mathbf{x}_{02},\ldots,\mathbf{x}_{0n_e}]. \tag{4.17}$$

In addition, coefficients a_i and b_i constitute vectors $\mathbf{a} = \{a_1,a_2,\ldots,a_{n_e}\}^{\mathrm{T}}$ and $\mathbf{b} = \{b_1,b_2,\ldots,b_{n_e}\}^{\mathrm{T}}$ of Eqs. (4.11), (4.12). Maximisation with respect to the vector $\mathbf{a}$ can be carried out first for $\tilde{\Pi}_1(\mathbf{a},\mathbf{b})$, which gives

$$\mathbf{a} = -\varepsilon\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_1\mathbf{b} \tag{4.18}$$

and the following quadratic form for vector $\mathbf{b}$

$$\tilde{\Pi}_1(\mathbf{b}) = \mathbf{b}^{\mathrm{T}}\mathbf{K}\mathbf{b}, \tag{4.19}$$

$$\tilde{\Pi}_2(\mathbf{b}) = \mathbf{b}^{\mathrm{T}}\mathbf{b}, \tag{4.20}$$

where

$$\mathbf{K} = \boldsymbol{\Gamma} - \frac{\varepsilon}{2}(\boldsymbol{\Psi}_1 + \boldsymbol{\Psi}_1^{\mathrm{T}}) + \frac{1}{2}\varepsilon^2\boldsymbol{\Psi}_1^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_1. \tag{4.21}$$

Then finding eigenvalues of the perturbed Hamiltonian system becomes computing the stationary values of the following Rayleigh's quotient

$$\tilde{\rho} = \frac{\tilde{\Pi}_1(\mathbf{b})}{\tilde{\Pi}_2(\mathbf{b})} = \frac{\mathbf{b}^{\mathrm{T}}\mathbf{K}\mathbf{b}}{\mathbf{b}^{\mathrm{T}}\mathbf{b}}, \tag{4.22}$$

which provides approximation of the beginning eigenvalues of the perturbed Hamiltonian system.

4.2. Examples

Example 4.1: A system of order 4 with $\varepsilon\mathbf{A}_1 = \varepsilon\mathbf{A}$ and

$$\mathbf{A} = \begin{bmatrix} 3.0 & 1.0 & -0.3 & -0.3 \\ -0.934 & 2.0 & -0.1 & -0.2 \\ -0.1 & -0.3 & 2.0 & 1.0 \\ -0.2 & -0.4 & -0.934 & 0.9 \end{bmatrix},$$

$$\mathbf{B} = \mathrm{diag}(1.0, 0.09, 1.0, 0.09),$$

$$\mathbf{C}_z = \mathbf{I}_4, \quad \mathbf{C}_y = \mathbf{I}_4, \quad \mathbf{Q}_0 = 0.1 \times \mathbf{I}_4, \quad t_f = 0.8,$$

where $\mathbf{I}_n$ denotes an identity matrix of order n. Table 1 shows the numerical results of the eigenvalue problem. Using the method of [2] directly, the first seven eigenvalues of the perturbed and unperturbed Hamiltonian systems were computed precisely. Then, after calculating twenty eigensolutions of the original system, the first seven eigenvalues of the perturbed system were also found by the aforementioned variational method.

Example 4.2: A system of order 5 with $\varepsilon\mathbf{A}_1 = \varepsilon\mathrm{diag}[-1, -2, -5, -2, -4]$ and

$$\mathbf{A} = \begin{bmatrix} -1.0 & 0.1 & 0.2 & 0.1 & 0.4 \\ 0.5 & -2.0 & 0.0 & -0.2 & 0.1 \\ 0.1 & -0.5 & -5.0 & 0.0 & -0.5 \\ 0.5 & 0.2 & 0.5 & -2.0 & 0.0 \\ 0.0 & -0.1 & -1.0 & 0.0 & -4.0 \end{bmatrix},$$

$$\mathbf{C}_y = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 1.0 \end{bmatrix},$$

$$\mathbf{B} = \mathbf{I}_5, \quad \mathbf{C}_z = \mathbf{I}_5, \quad \mathbf{Q}_0 = 0.1 \times \mathbf{I}_5, \quad t_f = 0.8.$$

The first seven eigenvalues of the perturbed and unperturbed Hamiltonian systems were also computed directly by using the method of Ref. [2]. Then, after calculating twenty eigensolutions of the unperturbed system, the first seven eigenvalues of the perturbed system were also found by the variational method. All the results of this eigenvalue problem are displayed in Table 2. It can be found from Tables 1 and 2 that the variational method gives satisfying numerical results with less computational efforts.

5. Decentralised H_∞ filter and corresponding eigenvalue problem

The conventional decentralised filtering strategies are mostly based on the Kalman filter algorithms, which offer powerful methods for solving state estimation problems of large-scale systems [16]. Nevertheless, decentralised H_∞ filtering is a more appropriate choice if the plant models and the statistics of the exogenous

Table 1
Eigenvalues of Example 4.1

Eigenvalue order		1	2	3	4	5	6	7
Unperturbed system		1.1970	1.5847	2.2458	5.5332	30.331	33.592	78.287
$\varepsilon = 0.2$	Method of Ref. [2]	1.1014	1.3917	1.7552	4.7177	31.699	37.059	78.214
	Variational method	1.1104	1.4006	1.7802	4.7286	31.703	37.070	78.348
$\varepsilon = -0.2$	Method of Ref. [2]	1.3713	1.8494	3.0145	6.4481	29.292	30.988	78.535
	Variational method	1.3774	1.8570	3.0555	6.4636	29.294	30.994	78.684

Table 2
Eigenvalues of Example 4.2

Eigenvalue order		1	2	3	4	5	6	7
Unperturbed system		6.2230	10.041	11.390	21.078	30.675	34.880	35.339
$\varepsilon = 0.1$	Method of Ref. [2]	6.5998	11.070	12.415	24.368	31.253	36.035	36.956
	Variational method	6.5999	11.070	12.415	24.370	31.252	36.033	36.955
$\varepsilon = 0.2$	Method of Ref. [2]	6.9873	12.159	13.499	27.916	31.786	37.190	38.276
	Variational method	6.9876	12.160	13.500	27.924	31.782	37.181	38.269

signals are not known exactly [17], where information obtained from the local filters is combined to generate the global estimation of the system state. To obtain the minimum achievable performance of a decentralised H_∞ filter, the eigenvalue problem of large-scale Hamiltonian systems is investigated in this section. Eigenvalues of this Hamiltonian system are computed by a subsystem synthesis method based on eigensolutions of all the local Hamiltonian subsystems constituting the global Hamiltonian system. It should be noted that, in some sense, this method is similar to the substructure synthesis method for complex structures [11].

5.1. Subsystem modal synthesis method for eigenvalue problem of large-scale Hamiltonian systems

Without loss of generality, subdivide system (2.1)–(2.3) into two subsystems and let

$$\mathbf{x} = \begin{Bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{Bmatrix}, \quad \mathbf{y} = \begin{Bmatrix} \mathbf{y}^{(1)} \\ \mathbf{y}^{(2)} \end{Bmatrix}, \quad \mathbf{z} = \begin{Bmatrix} \mathbf{z}^{(1)} \\ \mathbf{z}^{(2)} \end{Bmatrix}, \quad \mathbf{w} = \begin{Bmatrix} \mathbf{w}^{(1)} \\ \mathbf{w}^{(2)} \end{Bmatrix}, \quad \mathbf{v} = \begin{Bmatrix} \mathbf{v}^{(1)} \\ \mathbf{v}^{(2)} \end{Bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \varepsilon\mathbf{A}_{12} \\ \varepsilon\mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & 0 \\ 0 & \mathbf{B}_{22} \end{bmatrix},$$

$$\mathbf{C}_y = \begin{bmatrix} \mathbf{C}_{y11} & 0 \\ 0 & \mathbf{C}_{y22} \end{bmatrix}, \quad \mathbf{C}_z = \begin{bmatrix} \mathbf{C}_{z11} & 0 \\ 0 & \mathbf{C}_{z22} \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{011} & 0 \\ 0 & \mathbf{Q}_{022} \end{bmatrix},$$

so that the original system can be described by the following two subsystems combined by the ε terms:

$$\dot{\mathbf{x}}^{(1)} = \mathbf{A}_{11}\mathbf{x}^{(1)} + \varepsilon\mathbf{A}_{12}\mathbf{x}^{(2)} + \mathbf{B}_{11}\mathbf{w}^{(1)}, \tag{5.1}$$

$$\mathbf{y}^{(1)} = \mathbf{C}_{y11}\mathbf{x}^{(1)} + \mathbf{v}^{(1)}, \tag{5.2}$$

$$\mathbf{z}^{(1)} = \mathbf{C}_{z11}\mathbf{x}^{(1)}, \tag{5.3}$$

$$\dot{\mathbf{x}}^{(2)} = \mathbf{A}_{22}\mathbf{x}^{(2)} + \varepsilon\mathbf{A}_{21}\mathbf{x}^{(1)} + \mathbf{B}_{22}\mathbf{w}^{(2)}, \tag{5.4}$$

$$\mathbf{y}^{(2)} = \mathbf{C}_{y22}\mathbf{x}^{(2)} + \mathbf{v}^{(2)}, \tag{5.5}$$

$$\mathbf{z}^{(2)} = \mathbf{C}_{z22}\mathbf{x}^{(2)}. \tag{5.6}$$

Then the associated global Hamiltonian system for the decentralised H_∞ filter is given as

$$\begin{Bmatrix} \dot{\mathbf{x}}^{(1)} \\ \dot{\mathbf{x}}^{(2)} \\ \dot{\boldsymbol{\lambda}}^{(1)} \\ \dot{\boldsymbol{\lambda}}^{(2)} \end{Bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \varepsilon\mathbf{A}_{12} & \mathbf{B}_{11}\mathbf{B}_{11}^{\mathrm{T}} & 0 \\ \varepsilon\mathbf{A}_{21} & \mathbf{A}_{22} & 0 & \mathbf{B}_{22}\mathbf{B}_{22}^{\mathrm{T}} \\ \mathbf{C}_{y11}^{\mathrm{T}}\mathbf{C}_{y11} - \rho\mathbf{C}_{z11}^{\mathrm{T}}\mathbf{C}_{z11} & 0 & -\mathbf{A}_{11}^{\mathrm{T}} & -\varepsilon\mathbf{A}_{12}^{\mathrm{T}} \\ 0 & \mathbf{C}_{y22}^{\mathrm{T}}\mathbf{C}_{y22} - \rho\mathbf{C}_{z22}^{\mathrm{T}}\mathbf{C}_{z22} & -\varepsilon\mathbf{A}_{21}^{\mathrm{T}} & -\mathbf{A}_{22}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \\ \boldsymbol{\lambda}^{(1)} \\ \boldsymbol{\lambda}^{(2)} \end{Bmatrix}. \tag{5.7}$$

If $\varepsilon = 0$, these two subsystems are completely independent, i.e. the ε terms alone combine the two subsystems together. Hence, in the spirit of the subsystem modal synthesis technique [12], it can be advantageous to compute eigensolutions of the local Hamiltonian systems obtained by setting $\varepsilon = 0$ independently and then to combine them afterwards by re-introducing the ε terms, even though it is not necessarily a small number.

The local Hamiltonian system for a subsystem H_∞ filter is given as

$$\begin{Bmatrix} \dot{\mathbf{x}}^{(i)} \\ \dot{\boldsymbol{\lambda}}^{(i)} \end{Bmatrix} = \begin{bmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}\mathbf{B}_{ii}^{\mathrm{T}} \\ \mathbf{C}_{yii}^{\mathrm{T}}\mathbf{C}_{yii} - \rho^{(i)}\mathbf{C}_{zii}^{\mathrm{T}}\mathbf{C}_{zii} & -\mathbf{A}_{ii}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{x}^{(i)} \\ \boldsymbol{\lambda}^{(i)} \end{Bmatrix}, \quad \begin{Bmatrix} \mathbf{x}^{(i)}(0) \\ \boldsymbol{\lambda}^{(i)}(t_f) \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q}_{0ii}\boldsymbol{\lambda}^{(i)}(0) \\ 0 \end{Bmatrix}, \tag{5.8}$$

where i ($i = 1, 2$) stands for the subsystem number. In terms of Eq. (3.4), the extended Rayleigh's quotient for this local Hamiltonian system is

$$\rho^{(i)} = \frac{\Pi_1^{(i)}(\mathbf{x}, \boldsymbol{\lambda})}{\Pi_2^{(i)}(\mathbf{x})}, \tag{5.9}$$

where

$$\Pi_1^{(i)}(\mathbf{x}, \boldsymbol{\lambda}) = \int_0^{t_f} \left[\boldsymbol{\lambda}^{(i)\mathrm{T}}\dot{\mathbf{x}}^{(i)} - \boldsymbol{\lambda}^{(i)\mathrm{T}}\mathbf{A}_{ii}\mathbf{x}^{(i)} + \frac{1}{2}\mathbf{x}^{(i)\mathrm{T}}\mathbf{C}_{yii}^{\mathrm{T}}\mathbf{C}_{yii}\mathbf{x}^{(i)} - \frac{1}{2}\boldsymbol{\lambda}^{(i)\mathrm{T}}\mathbf{B}_{ii}\mathbf{B}_{ii}^{\mathrm{T}}\boldsymbol{\lambda}^{(i)} \right] \mathrm{d}t + \frac{1}{2}\mathbf{x}_0^{(i)\mathrm{T}}\mathbf{Q}_{0ii}^{-1}\mathbf{x}_0^{(i)}, \tag{5.10}$$

$$\Pi_2^{(i)}(\mathbf{x}) = \frac{1}{2}\int_0^{t_f} \mathbf{x}^{(i)\mathrm{T}}\mathbf{C}_{zii}^{\mathrm{T}}\mathbf{C}_{zii}\mathbf{x}^{(i)}\,\mathrm{d}t. \tag{5.11}$$

The trial functions $\mathbf{x}$ and $\boldsymbol{\lambda}$ of the extended Rayleigh's quotient (3.4) of the global Hamiltonian system can be expanded as

$$\mathbf{x}(t) = \begin{Bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} \sum_{j=1}^{\infty} b_j^{(1)}\mathbf{x}_j^{(1)} \\ \sum_{k=1}^{\infty} b_k^{(2)}\mathbf{x}_k^{(2)} \end{Bmatrix}, \tag{5.12}$$

$$\boldsymbol{\lambda}(t) = \begin{Bmatrix} \boldsymbol{\lambda}^{(1)} \\ \boldsymbol{\lambda}^{(2)} \end{Bmatrix} = \begin{Bmatrix} \sum_{j=1}^{\infty} (a_j^{(1)} + b_j^{(1)})\boldsymbol{\lambda}_j^{(1)} \\ \sum_{k=1}^{\infty} (a_k^{(2)} + b_k^{(2)})\boldsymbol{\lambda}_k^{(2)} \end{Bmatrix}, \tag{5.13}$$

where $a_j^{(1)}$, $a_k^{(2)}$, $b_j^{(1)}$, and $b_k{}^{(1)}$ are independent parameters to be determined by (3.4), and the superscript stands for the subsystem number, whereas the subscript denotes the order of the eigensolutions.

Now consider the extended Rayleigh's quotient of the global system, its composition is still Π_1 and Π_2. Because all the matrices of the global system are block diagonal except for the plant matrix $\mathbf{A}$, the composition of Π_1 and Π_2 are the sums of all the $\Pi_1^{(i)}$ and $\Pi_2^{(i)}$ of subsystems, and also the inter-subsystem terms from the off-diagonal sub-matrices of $\mathbf{A}$ with multiplier ε. Therefore, the numerator and denominator of the extended Rayleigh's quotient of the global system consists of

$$\Pi_1 = \Pi_1^{(1)} + \Pi_1^{(2)} + \Pi_m, \tag{5.14}$$

$$\Pi_2 = \Pi_2^{(1)} + \Pi_2^{(2)}, \tag{5.15}$$

where Π_m represents the inter-subsystem term. Suppose that the numbers of eigensolutions used for subsystems 1 and 2 are, respectively, n_{e1} and n_{e2}. Note, when $\mathbf{x}^{(i)}$ and $\boldsymbol{\lambda}^{(i)}$ are substituted with the expansion

form ($i = 1, 2$), because of the ortho-normalization condition, $\Pi_1^{(i)}$ and $\Pi_2^{(i)}$ can be computed by

$$\begin{aligned}\Pi_1^{(i)}(\mathbf{x},\boldsymbol{\lambda}) &\cong \tilde{\Pi}_1^{(i)}(\mathbf{a}^{(i)},\mathbf{b}^{(i)})\\ &= \sum_{j=1}^{n_{ei}}\left[\rho_j^{(i)}b_j^{(i)2} - \int_0^{t_f}\sum_{j=1}^{n_{ei}}\sum_{k=1}^{n_{ei}}(\boldsymbol{\lambda}_j^{(i)\mathrm{T}}\mathbf{B}_{ii}\mathbf{B}_{ii}^{\mathrm{T}}\boldsymbol{\lambda}_k^{(i)}/2)\cdot(a_j^{(i)}a_k^{(i)})\mathrm{d}t\right]\\ &= \sum_{j=1}^{n_{ei}}\rho_j^{(i)}b_j^{(i)2} - \frac{1}{2}\mathbf{a}^{(i)\mathrm{T}}\boldsymbol{\Psi}^{(i)}\mathbf{a}^{(i)},\end{aligned} \tag{5.16}$$

where

$$\boldsymbol{\Psi}^{(i)} = \int_0^{t_f}\boldsymbol{\Lambda}^{(i)\mathrm{T}}\mathbf{B}_{ii}\mathbf{B}_{ii}^{\mathrm{T}}\boldsymbol{\Lambda}^{(i)}\,\mathrm{d}t, \tag{5.17}$$

$$\boldsymbol{\Lambda}^{(i)} = [\boldsymbol{\lambda}_1^{(i)},\boldsymbol{\lambda}_2^{(i)},\ldots,\boldsymbol{\lambda}_{n_{ei}}^{(i)}] \tag{5.18}$$

and $\mathbf{a}^{(i)} = [a_1^{(i)}, a_2^{(i)},\ldots,a_{n_{ei}}^{(i)}]^{\mathrm{T}}$ is the vector of the expansion parameters to be determined, $\rho_j{}^{(i)}$ represents the jth eigenvalue of the ith subsystem. Since

$$\int_0^{t_f}[\mathbf{x}_j^{(i)\mathrm{T}}\mathbf{C}_{zii}^{\mathrm{T}}\mathbf{C}_{zii}\mathbf{x}_j^{(i)}/2]\,\mathrm{d}t = 1,$$

one has

$$\Pi_2^{(i)}(\mathbf{x}) \cong \tilde{\Pi}_2^{(i)}(\mathbf{b}^{(i)}) = \sum_{j=1}^{n_{ei}}b_j^{(i)2}\cdot\int_0^{t_f}[\mathbf{x}_j^{(i)\mathrm{T}}\mathbf{C}_{zii}^{\mathrm{T}}\mathbf{C}_{zii}\mathbf{x}_j^{(i)}/2]\mathrm{d}t = \sum_{j=1}^{n_{ei}}b_j^{(i)2} = \mathbf{b}^{(i)\mathrm{T}}\mathbf{b}^{(i)}, \tag{5.19}$$

The inter-subsystem term Π_m can be derived as

$$\begin{aligned}\Pi_m(\mathbf{x},\boldsymbol{\lambda}) &\cong \tilde{\Pi}_m(\mathbf{a}^{(1)},\mathbf{b}^{(1)},\mathbf{a}^{(2)},\mathbf{b}^{(2)})\\ &= -\varepsilon\int_0^{t_f}[\boldsymbol{\lambda}^{(1)\mathrm{T}}\mathbf{A}_{12}\mathbf{x}^{(2)} + \boldsymbol{\lambda}^{(2)\mathrm{T}}\mathbf{A}_{21}\mathbf{x}^{(1)}]\,\mathrm{d}t\\ &= -\varepsilon\sum_{j=1}^{n_{e1}}\sum_{k=1}^{n_{e2}}[\psi_{jk}^{(12)}(a_j^{(1)}+b_j^{(1)})b_k^{(2)} + \psi_{kj}^{(21)}(a_k^{(2)}+b_k^{(2)})b_j^{(1)}]\\ &= -\varepsilon[(\mathbf{a}^{(1)}+\mathbf{b}^{(1)})^{\mathrm{T}}\boldsymbol{\Psi}^{(12)}\mathbf{b}^{(2)} + (\mathbf{a}^{(2)}+\mathbf{b}^{(2)})^{\mathrm{T}}\boldsymbol{\Psi}^{(21)}\mathbf{b}^{(1)}],\end{aligned} \tag{5.20}$$

where the matrices $\boldsymbol{\Psi}^{(12)}$ and $\boldsymbol{\Psi}^{(21)}$ can be composed as

$$\psi_{jk}^{(12)} = \int_0^{t_f}[\boldsymbol{\lambda}_j^{(1)\mathrm{T}}\mathbf{A}_{12}\mathbf{x}_k^{(2)}]\,\mathrm{d}t, \tag{5.21}$$

$$\psi_{jk}^{(21)} = \int_0^{t_f}[\boldsymbol{\lambda}_j^{(2)\mathrm{T}}\mathbf{A}_{21}\mathbf{x}_k^{(1)}]\,\mathrm{d}t. \tag{5.22}$$

Combing the independent expansion parameters of the two subsystems forms the unknown vectors as

$$\mathbf{a} = \begin{Bmatrix}\mathbf{a}^{(1)}\\ \mathbf{a}^{(2)}\end{Bmatrix} = \{a_1^{(1)}, a_2^{(1)},\ldots,a_{n_{e1}}^{(1)}; a_1^{(2)}, a_2^{(2)},\ldots,a_{n_{e2}}^{(2)}\}^{\mathrm{T}}, \tag{5.23}$$

$$\mathbf{b} = \begin{Bmatrix}\mathbf{b}^{(1)}\\ \mathbf{b}^{(2)}\end{Bmatrix} = \{b_1^{(1)}, b_2^{(1)},\ldots,b_{n_{e1}}^{(1)}; b_1^{(2)}, b_2^{(2)},\ldots,b_{n_{e2}}^{(2)}\}^{\mathrm{T}}. \tag{5.24}$$

Substituting Eqs. (5.16), (5.19) and (5.20) into (5.14) and (5.15), respectively, gives

$$\begin{aligned}\Pi_1(\mathbf{x},\boldsymbol{\lambda}) &\cong \tilde{\Pi}_1(\mathbf{a},\mathbf{b})\\ &= \mathbf{b}^{\mathrm{T}}\boldsymbol{\Gamma}\mathbf{b} - \frac{1}{2}\mathbf{a}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{\Psi}^{(1)} & 0\\ 0 & \boldsymbol{\Psi}^{(2)}\end{bmatrix}\mathbf{a} - \varepsilon(\mathbf{a}+\mathbf{b})^{\mathrm{T}}\begin{bmatrix}0 & \boldsymbol{\Psi}^{(12)}\\ \boldsymbol{\Psi}^{(21)} & 0\end{bmatrix}\mathbf{b},\end{aligned} \tag{5.25}$$

$$\Pi_2(\mathbf{x}) \cong \tilde{\Pi}_2(\mathbf{b}) = \mathbf{b}^{\mathrm{T}}\mathbf{b}, \tag{5.26}$$

where

$$\boldsymbol{\Gamma} = \mathrm{diag}[\rho_1^{(1)}, \rho_2^{(1)}, \ldots, \rho_{n_{e1}}^{(1)}, \rho_1^{(2)}, \rho_2^{(2)}, \ldots, \rho_{n_{e2}}^{(2)}] \tag{5.27}$$

maximisation with respect to the parameter vector $\mathbf{a}$ can be carried out first for Π_1, which gives

$$\mathbf{a} = -\varepsilon\begin{bmatrix}\boldsymbol{\Psi}^{(1)} & 0\\ 0 & \boldsymbol{\Psi}^{(2)}\end{bmatrix}^{-1}\begin{bmatrix}0 & \boldsymbol{\Psi}^{(12)}\\ \boldsymbol{\Psi}^{(21)} & 0\end{bmatrix}\mathbf{b} \tag{5.28}$$

and the quadratic form for vector $\mathbf{b}$

$$\tilde{\Pi}_1(\mathbf{b}) = \mathbf{b}^{\mathrm{T}}\mathbf{K}\mathbf{b}, \tag{5.29}$$

$$\tilde{\Pi}_2(\mathbf{b}) = \mathbf{b}^{\mathrm{T}}\mathbf{b}, \tag{5.30}$$

where

$$\mathbf{K} = \boldsymbol{\Gamma} - \varepsilon\begin{bmatrix}0 & \boldsymbol{\Psi}^{(12)}\\ \boldsymbol{\Psi}^{(21)} & 0\end{bmatrix} + \frac{1}{2}\varepsilon^2\begin{bmatrix}0 & \boldsymbol{\Psi}^{(12)}\\ \boldsymbol{\Psi}^{(21)} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{\Psi}^{(1)} & 0\\ 0 & \boldsymbol{\Psi}^{(2)}\end{bmatrix}^{-1}\begin{bmatrix}0 & \boldsymbol{\Psi}^{(12)}\\ \boldsymbol{\Psi}^{(21)} & 0\end{bmatrix}^{\mathrm{T}}. \tag{5.31}$$

Therefore, it only remains to minimise with respect to the parameter vector $\mathbf{b}$. Then finding the eigenvalues of the global Hamiltonian system becomes the algebraic eigenvalue problem

$$\tilde{\rho} = \frac{\tilde{\Pi}_1(\mathbf{b})}{\tilde{\Pi}_2(\mathbf{b})} = \frac{\mathbf{b}^{\mathrm{T}}\mathbf{K}\mathbf{b}}{\mathbf{b}^{\mathrm{T}}\mathbf{b}}, \tag{5.32}$$

which has the same formulation as Eq. (4.22) except the structure and elements of the matrix $\mathbf{K}$.

5.2. Examples

Example 5.1: A system of order 4, with $t_f = 0.8$ and

$$\mathbf{A} = \begin{bmatrix}\mathbf{A}_{11} & \varepsilon\mathbf{A}_{12}\\ \varepsilon\mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix} = \left[\begin{array}{cccc:c}0.8 & 1.0 & 1.0 & 1.0 & 1.0\\ 2.0 & 0.9 & 0.0 & 0.0 & 1.0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 1.0 & 0.0 & 0.0 & 0.8 & 1.0\\ 0.0 & 1.0 & 2.0 & 0.4 & \end{array}\right],$$

$$\mathbf{B} = \mathbf{I}_4,\ \ \mathbf{C}_y = \mathbf{I}_4,\ \ \mathbf{C}_z = \mathbf{I}_4,\ \ \mathbf{Q}_0 = \mathbf{I}_4.$$

Table 3 shows the results of the eigenvalue problem. The lowest seven eigenvalues of the global system were computed by the subsystem synthesis modal method described above, after calculating twenty eigensolutions for either of the two subsystems precisely. Solving the global system directly using the method presented in Ref. [2], the first seven eigenvalues were also found.

Table 3
Eigenvalues of Example 5.1

Eigenvalue order	1	2	3	4	5	6	7
Method of Ref. [2]	1.0235	1.3740	2.0959	3.6283	18.544	19.114	21.958
Modal synthesis method	1.0236	1.3784	2.1101	3.6565	18.593	19.135	22.015

Table 4
Eigenvalues of Example 5.2

Eigenvalue order		1	2	3	4	5	6	7
$\varepsilon = 0.5$	Method of Ref. [2]	0.5317	1.5086	3.6965	7.7253	8.8802	12.421	13.381
	Modal synthesis method	0.5329	1.5128	3.8528	7.7416	8.9004	12.437	13.401
$\varepsilon = 1.0$	Method of Ref. [2]	0.4994	1.5909	4.4850	7.1791	9.4027	11.849	13.899
	Modal synthesis method	0.5043	1.5985	4.9717	7.2332	9.4985	11.930	13.981

Example 5.2: A system of order 9, with $t_f = 0.5$ and

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \varepsilon\mathbf{A}_{12} \\ \varepsilon\mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix},$$

$$\mathbf{A}_{11} = \begin{bmatrix} 0.1 & 0.545 & 0.0 & 0.0 & 0.0 \\ -6.0 & -0.05 & 6.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -3.33 & 3.33 & 0.0 \\ 0.0 & -5.21 & 0.0 & -12.5 & 0.0 \\ 1.0 & 0.425 & 0.0 & 0.0 & 0.2 \end{bmatrix}, \quad \mathbf{A}_{12} = \begin{bmatrix} -0.545 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix},$$

$$\mathbf{A}_{21} = \begin{bmatrix} 6.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ -1.0 & 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix}, \quad \mathbf{A}_{22} = \begin{bmatrix} -0.05 & 6.0 & 0.0 & 0.0 \\ 0.0 & -3.33 & 3.33 & 0.0 \\ -5.21 & 0.0 & -12.5 & 0.0 \\ 0.425 & 0.0 & 0.0 & 2.0 \end{bmatrix},$$

$$\mathbf{C}_y = \begin{bmatrix} 0 & 0 & 0 & 12.5 & 0 & \vdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 12.5 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_z = \mathbf{I}_{10}, \quad \mathbf{Q}_0 = \mathbf{I}_{10}, \quad \mathbf{B} = \mathbf{I}_{10}.$$

After calculating twenty eigensolutions for either of the two subsystems, the lowest seven eigenvalues of the global system were computed by the subsystem synthesis method for $\varepsilon = 0.5$ and 1.0, respectively. The lowest seven eigenvalues were also found by solving the global system directly. All the results are displayed in Table 4. It can be found from Tables 3 and 4 that the subsystem synthesis method also gives satisfying numerical results.

6. Concluding remarks

Eigenvalue problems of the linear Hamiltonian systems arising from H_∞ filtering systems are investigated in this paper. Firstly, the Hamiltonian system eigenvalue problem arising from the design of a finite-time central

H_∞ filter is investigated, correspondence between minimum induced norm of the H_∞ filter and fundamental eigenvalue of the Hamiltonian system is demonstrated based on characteristics of the eigensolutions and Riccati equations of the Hamiltonian systems. Secondly, a variational method is proposed to approximate eigenvalues of a perturbed Hamiltonian system derived from a perturbed H_∞ filtering problem. The variational method is a Rayleigh–Ritz-type method adapted to calculate stationary values of an extended Rayleigh's quotient with dual argument functions. Lastly, based on the variational method, a subsystem modal synthesis method is presented to compute eigenvalues of a large-scale Hamiltonian system consisting of subsystems, which comes from a decentralised H_∞ filtering problem. Furthermore, the variational methods and numerical results of this paper also imply that eigenvalue problems of discrete Hamiltonian systems arising from discrete-time H_∞ filtering may be approached by similar variational measures.

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