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# Free vibration of an extensible rotating inclined Timoshenko beam

Sen Yung Lee[a,*], Jer Jia Sheu[b]

[a]Mechanical Engineering Department, National Cheng Kung University, Tainan, Taiwan 701, ROC
[b]Mechanical Engineering Department, Southern Taiwan University of Technology, Tainan, Taiwan 710, ROC

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**Abstract**

An exact power-series solution for free vibration of a rotating inclined Timoshenko beam is developed. It is shown that both the extensional deformation and the Coriolis force will have significant influence on the natural frequencies of the rotating beam when the dimensionless rotating extension parameter is large. Without numerical analysis, several general qualitative relations among the inclination angle, the hub radius and the natural frequencies of the beam are revealed. In addition, the influence of extensional deformation in the centrifugal stiffening force term on the natural frequencies evaluated by the Timoshenko and the Euler beam theories is also studied and compared.


## 1. Introduction

Rotating beams are of importance in engineering applications such as turbine blades, helicopter rotors, airplane propellers, robot manipulators and various cooling fans. Interesting reviews about the subject can be found in the papers given by, amongst others, Leissa [1], Ramamurti et al. [2] and Rao [3].

In the vibration analysis of rotating blades, the structures with large aspect ratio are often modeled as beams vibrating in flexural motion. The influences of various parameters, such as hub radius, tip mass, rotating speed, shear deformation and rotary inertia, Coriolis force, setting angle, taper ratio, pretwisted angle and elastic root restraints on the natural frequencies of flexural vibration of a rotating beam have been studied by many investigators [4–14]. Simo and Quoc [10] showed that appropriate account of the influence of centrifugal force on bending stiffness requires the use of a geometrically nonlinear beam theory. Lin and Hsiao [11] studied the coupling effects of extensional deformation and Coriolis force on the natural frequencies of a Timoshenko beam. In all these studies, the steady-state normal force (or centrifugal force) was used to examine the centrifugally stiffened effect. However, the extensional deformation was not considered in the centrifugal stiffening force term, even though it might have been considered in the governing differential equations.

*Corresponding author.
*E-mail address:* sylee@mail.ncku.edu.tw (S.Y. Lee).

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Besides, although the vibration of a rotating beam has been extensively studied, little study on the vibration of a rotating beam with inclination angle is found. Lee [6] studied the vibration on an inclined rotating cantilevered Euler beam with tip mass using a numerical method. In his study, the extensional deformation and the Coriolis force were not considered.

In this paper, based on the Timoshenko beam theory, the free vibration of a rotating uniform beam with inclination angle is investigated. The extensional deformation and the Coriolis force effect are both taken into account. The beam considered is doubly symmetric so that the centroidal axis and the neutral axis are coincident. By utilizing the Hamilton's principle and the consistent linearization of a geometrically nonlinear beam theory [10], three coupled governing differential equations with variable coefficients are derived. In general, the exact solutions of the system are not available. This type of problem is mainly solved by approximated methods such as the Rayleigh–Ritz method [7], the finite element method [8], the Galerkin method [12], and the finite difference method [13,14].

In this study, two explicit relations are developed. These two relations explicitly express the centroidal axial displacement parameter and the rotating angle in terms of the transverse displacements parameter, respectively. With these two relations, the coupled characteristic differential equations can be decoupled and reduced to a sixth-order ordinary differential equation with variable coefficients. In this way, by Lee and Kuo's work [15], the exact series solution of the system can be developed.

It is known that numerical results can only provide partial qualitative conclusions. In addition, it requires a wide range of data to achieve this. In this paper, several general qualitative relations among the inclination angle, the hub radius and the natural frequencies of the beam are revealed without numerical analysis. The general qualitative relations are also verified using some numerical illustrations. With regard to the numerical analysis, four different approaches are presented and a dimensionless rotating extension parameter is introduced to illustrate the influence of the Coriolis force and the extensional deformation on the natural frequencies of the beam system. In addition, the influence of extensional deformation in the centrifugal stiffening force term on the natural frequencies evaluated by both the Timoshenko and the Euler beam theories is also studied.

## 2. Dynamic system

Consider the free in-plane vibration of a rotating inclined Timoshenko beam, as shown in Fig. 1. The beam is mounted with an inclination angle $\theta$ on a hub with radius $r_h$. It rotates with a constant angular velocity $\Omega$. There are two coordinate systems, $O{-}X_0Y_0$ and $A{-}X_1Y_1$, used for the configuration. The beam deflection is confined in the $X_1$–$Y_1$ plane only. The centroidal axis of the beam is coincident with $X_1$ axis. Let $P_1$ (see Fig. 1) be an arbitrary point in the beam element, and $P$ be the point corresponding to $P_1$ on the centroidal axis. The position vector of a point $P_1$, after deformation, can be expressed as

$$\overrightarrow{OP_1} = [r_h + (x + u - r\phi)\cos\theta - (v + r)\sin\theta]\vec{i} + [(x + u - r\phi)\sin\theta + (v + r)\cos\theta]\vec{j}, \tag{1}$$

where $u$ and $v$ are the centroidal axial and transverse displacements of point $P$, respectively, $x$ is the distance from the origin point A to the position of point $P$ and $r$ is the distance between point $P$ and $P_1$ in the undeformed configuration, $\phi$ is the angle of rotation due to bending, and $\vec{i},\vec{j}$ are unit vectors in the $O{-}X_0Y_0$ coordinate system. The velocity of point $P_1$ is

$$\begin{aligned}\overrightarrow{v_p} = &\left\{\left(\frac{\mathrm{d}u}{\mathrm{d}t} - r\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)\cos\theta - \frac{\mathrm{d}v}{\mathrm{d}t}\sin\theta - \Omega[(x + u - r\phi)\sin\theta + (v + r)\cos\theta]\right\}\vec{i} \\ &+ \left\{\left(\frac{\mathrm{d}u}{\mathrm{d}t} - r\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)\sin\theta + \frac{\mathrm{d}v}{\mathrm{d}t}\cos\theta + \Omega[r_h + (x + u - r\phi)\cos\theta - (v + r)\sin\theta]\right\}\vec{j}.\end{aligned} \tag{2}$$

The kinetic energy $T$ and the potential energy $U$ of the rotating beam are, respectively,

$$T = \frac{1}{2}\int_0^L \rho A\left(\overrightarrow{v_p}\cdot\overrightarrow{v_p}\right)\mathrm{d}x \tag{3}$$

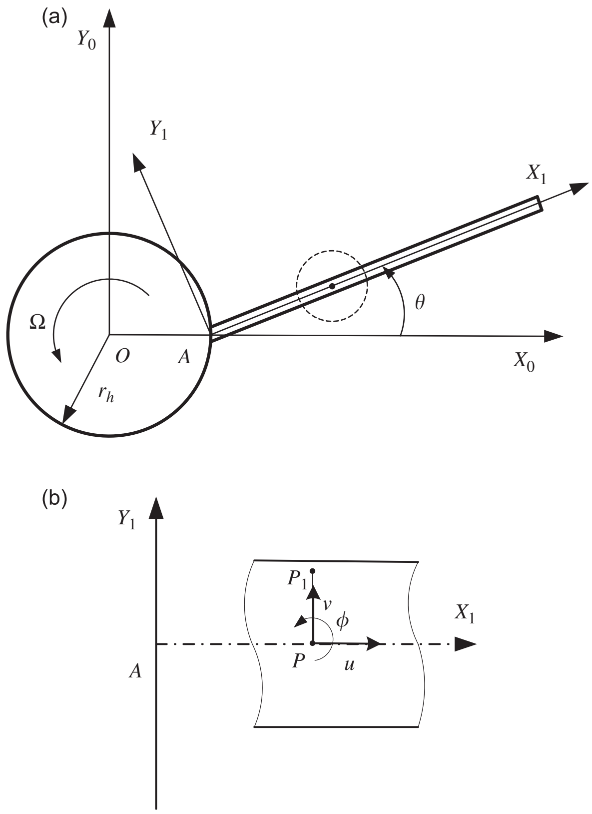


Fig. 1. Geometry and coordinate system of a rotating inclined beam.

and

$$U = \frac{1}{2}\iint E\varepsilon^2 \, \mathrm{d}A \, \mathrm{d}x + \frac{1}{2}\iint kG\gamma^2 \, \mathrm{d}A \, \mathrm{d}x, \tag{4}$$

where $\rho$, $A$, $L$, $E$ and $G$ are the mass per unit volume, the cross-sectional area, the length, the Young's modulus and the shear modulus of the beam, respectively. $k$ is the shear correction factor. $\gamma$ is the shear strain, which is the angle of distortion due to shear. The nonlinear normal strain $\varepsilon$ and the shear strain $\gamma$ may be approximated [11] by

$$\varepsilon = \frac{\partial u}{\partial x} - r\,\frac{\partial \phi}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2, \quad \gamma = \frac{\partial v}{\partial x} - \phi. \tag{5}$$

It should be noted that in order to explain the influence of the centrifugal stiffening force on the bending stiffness in an appropriate manner, the nonlinear term in the normal strain, Eq. (5), is required [10].

By utilizing the Hamilton's principle and the consistent linearization of the nonlinear beam theory [10], the governing differential equations for the rotating inclined Timoshenko beam are

$$EA\,\frac{\partial^2 u}{\partial x^2} + \rho A\Omega^2 u - \rho A\,\frac{\partial^2 u}{\partial t^2} + 2\rho A\Omega\frac{\partial v}{\partial t} = -\rho A\partial^2(x + r_h\,\cos\,\theta), \tag{6a}$$

$$kGA\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial x}\left(N_p\frac{\partial v}{\partial x}\right) + \rho A\Omega^2 v - \rho A\,\frac{\partial^2 v}{\partial t^2} - 2\rho A\Omega\frac{\partial u}{\partial t} = 0, \tag{6b}$$

$$EI\,\frac{\partial^2 \phi}{\partial x^2} + \rho I\Omega^2 \phi - \rho I\,\frac{\partial^2 \phi}{\partial t^2} + kGA\left(\frac{\partial v}{\partial x} - \phi\right) = 0, \tag{6c}$$

where $I$ is the second moment of area of the beam cross-section. The last terms in the left-hand side of Eqs. (6a)–(6b) are the Coriolis forces in the axial and the transverse directions, respectively.

The associated boundary conditions are

$$\text{at} \quad x = 0: \quad v = 0, \tag{7a}$$

$$\phi = \tan\theta, \tag{7b}$$

$$u = 0 \tag{7c}$$

and

$$\text{at} \quad x = L: \quad kGA\left(\frac{\partial v}{\partial x} - \phi\right) = 0, \tag{8a}$$

$$EI\,\frac{\partial \phi}{\partial x} = 0, \tag{8b}$$

$$EA\,\frac{\partial u}{\partial x} = 0. \tag{8c}$$

Here, $N_p = EA(\mathrm{d}u/\mathrm{d}x)$ is the centrifugal stiffening force term and is used to be considered as the steady-state normal force or the centrifugal force [4–16]. It is determined in the following.

For steady-state deformations, the differential equation (6a) is reduced to

$$EA\,\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \rho A \Omega^2 u = -\rho A \Omega^2 (x + r_h \cos\theta). \tag{9}$$

The solution that satisfies the associated boundary conditions (7c) and (8c) is

$$u(x) = r_h \cos\theta \cos\frac{\lambda}{L}x + \left(\frac{L}{\lambda\cos\lambda} + r_h\cos\theta\tan\lambda\right)\sin\frac{\lambda}{L}x - (x + r_h\cos\theta), \tag{10}$$

where $\lambda = \sqrt{(\rho/E)}\Omega L$ is defined as a dimensionless rotating extension parameter. As a result, the axial steady-state normal force is

$$N_p = EA\,\frac{\mathrm{d}u}{\mathrm{d}x}\,EA\left[-\frac{\lambda}{L}r_h\cos\theta\sin\frac{\lambda}{L}x + \left(\frac{1}{\cos\lambda} + \frac{\lambda r_h\cos\theta\tan\lambda}{L}\right)\cos\frac{\lambda}{L}x - 1\right]. \tag{11}$$

If $\lambda$ is relatively small and one just retains two terms in the power-series approximation of the above trigonometric functions, it is reduced to

$$N_{pc} = \rho A\Omega^2[r_h(L - x)\cos\theta + \tfrac{1}{2}(L^2 - x^2)]. \tag{12}$$

It should be mentioned that this $N_{pc}$ is equal to the axial centrifugal force of the rotating beam when the extensional deformation is ignored. The difference between $N_p$ and $N_{pc}$ will increase as the dimensionless rotating extension parameter $\lambda$ is increased.

## 3. Reduced governing differential equations

The displacements of the beam can be considered as the superposition of the static displacements reacted by the centrifugal force and the dynamic displacements generated from the beam vibration. They are

$$u = u_s(x) + u_d(x,t), \quad v = v_s(x) + v_d(x,t) \ \text{and} \ \phi = \phi_s(x) + \phi_d(x,t). \tag{13}$$

Here $u_s$, $v_s$ and $\phi_s$ are the static displacements. They satisfy the following three differential equations:

$$EA\,\frac{\partial^2 u_s}{\partial x^2} + \rho A\Omega^2 u_s = -\rho A\Omega^2(x + r_h\cos\theta), \tag{14a}$$

$$kGA\left(\frac{\partial^2 v_s}{\partial x^2}-\frac{\partial \phi_s}{\partial x}\right)+\frac{\partial}{\partial x}\left(N_p\frac{\partial v_s}{\partial x}\right)+\rho A\Omega^2 v_s=0, \tag{14b}$$

$$EI\,\frac{\partial^2 \phi_s}{\partial x^2}+\rho I\Omega^2\phi_s+kGA\left(\frac{\partial v_s}{\partial x}-\phi_s\right)=0. \tag{14c}$$

The associated boundary conditions are

$$\text{at}\quad x=0:\quad v_s=0,\quad \phi_s=\tan\theta,\quad u_s=0, \tag{15a}$$

$$\text{at}\quad x=L:\quad kGA\left(\frac{\partial v_s}{\partial x}-\phi_s\right)=0,\quad EI\,\frac{\partial \phi_s}{\partial x}=0,\quad EA\,\frac{\partial u_s}{\partial x}=0. \tag{15b}$$

Subsequently, $u_d$, $v_d$ and $\phi_d$ are the dynamic displacements. They satisfy the following three differential equations:

$$EA\,\frac{\partial^2 u_d}{\partial x^2}+\rho A\Omega^2 u_d-\rho A\,\frac{\partial^2 u_d}{\partial t^2}+2\rho A\Omega\,\frac{\partial v_d}{\partial t}=0, \tag{16a}$$

$$kGA\left(\frac{\partial^2 v_d}{\partial x^2}-\frac{\partial \phi_d}{\partial x}\right)+\frac{\partial}{\partial x}\left(N_p\frac{\partial v_d}{\partial x}\right)+\rho A\Omega^2 v_d-\rho A\,\frac{\partial^2 v_d}{\partial t^2}-2\rho A\Omega\,\frac{\partial u_d}{\partial t}=0, \tag{16b}$$

$$EI\,\frac{\partial^2 \phi_d}{\partial x^2}+\rho I\Omega^2\phi_d-\rho I\,\frac{\partial^2 \phi_d}{\partial t^2}+kGA\left(\frac{\partial v_d}{\partial x}-\phi_d\right)=0. \tag{16c}$$

The associated boundary conditions are

$$\text{at}\quad x=0:\quad v_d=0,\quad \phi_d=0,\quad u_d=0, \tag{17a}$$

$$\text{at}\quad x=L:\quad kGA\left(\frac{\partial v_d}{\partial x}-\phi_d\right)=0,\quad EI\,\frac{\partial \phi_d}{\partial x}=0,\quad EA\,\frac{\partial u_d}{\partial x}=0. \tag{17b}$$

In the free vibration analysis, the static displacements will have no influence on the natural frequencies of the beam. Therefore, the governing differential equations of the system are reduced to Eqs. (16a)–(16c). It can be observed that if the Coriolis force is not considered, the centroidal axial displacement will be independent with the other two displacements.

## 4. Governing characteristic differential equations

For time-harmonic vibration of a rotating inclined beam with angular frequency $\omega$, one assumes

$$v_d(x,t)=\tilde{V}(x)\mathrm{e}^{\mathrm{i}\omega t},\quad u_d(x,t)=\tilde{U}(x)\mathrm{e}^{\mathrm{i}\omega t}\ \text{and}\ \phi_d(x,t)=\widetilde{\phi_d}(x)\mathrm{e}^{\mathrm{i}\omega t}. \tag{18}$$

In terms of the following dimensionless parameters

$$\begin{aligned}&\bar{V}=\frac{\tilde{V}}{L},\ \ \bar{U}=\frac{\tilde{U}}{L},\ \ \bar{\phi}(\xi)=\widetilde{\phi_d}(x),\ \ \xi=\frac{x}{L},\ \ \Lambda=\sqrt{\frac{\rho A}{EI}}\omega L^2,\\ &L_z=L\sqrt{\frac{A}{I}},\ \ \mu=\frac{r_h}{L},\ \ \alpha=L_z\lambda=\sqrt{\frac{\rho A}{EI}}\Omega L^2,\ \ \overline{N_p}=\frac{N_p}{\rho A\Omega^2 L^2},\\ &\overline{N_{pc}}=\frac{N_{pc}}{\rho A\Omega^2 L^2},\ \ k_s=k\,\frac{G}{E},\end{aligned} \tag{19}$$

the governing characteristic differential equations (16a)–(16c) can be expressed as

$$\bar{U}''+\frac{\Lambda^2+\alpha^2}{L_z^2}\,\bar{U}+\frac{2i\Lambda\alpha}{L_z^2}\,\bar{V}=0, \tag{20a}$$

$$k_s L_z^2\left(\bar{V}'' - \bar{\phi}'\right) + \alpha^2\left(\overline{N_p}\bar{V}'\right)' + (\Lambda^2 + \alpha^2)\bar{V} - 2i\Lambda\alpha\bar{U} = 0, \tag{20b}$$

$$\bar{\phi}'' + \frac{\Lambda^2 + \alpha^2}{L_z^2}\bar{\phi} + k_s L_z^2(\bar{V}' - \bar{\phi}) = 0, \tag{20c}$$

where the prime denotes the derivative with respect to the dimensionless variable $\xi$. The dimensionless axial steady-state normal force is

$$\overline{N_p} = C_a \sin\lambda\xi + C_b \cos\lambda\xi - \frac{1}{\lambda^2}, \tag{21}$$

where $C_a$ and $C_b$ are, respectively,

$$C_a = -\frac{\mu}{\lambda}\cos\theta \text{ and } C_b = \frac{1}{\lambda^2 \cos\lambda} + \frac{\tan\lambda}{\lambda}\mu\cos\theta. \tag{22}$$

The other dimensionless axial steady-state normal force neglecting extensional deformation is

$$\overline{N_{pc}} = \mu(1-\xi)\cos\theta + \frac{1}{2}(1-\xi^2). \tag{23}$$

The associated dimensionless boundary conditions are

$$\text{at} \quad \xi = 0: \quad \bar{V} = 0, \tag{24a}$$

$$\bar{\phi} = 0, \tag{24b}$$

$$\bar{U} = 0 \tag{24c}$$

and

$$\text{at} \quad \xi = 1: \quad \bar{V}' - \bar{\phi} = 0, \tag{25a}$$

$$\bar{\phi}' = 0, \tag{25b}$$

$$\bar{U}' = 0. \tag{25c}$$

### *4.1. Explicit relations*

Equations (20a)–(20c) are three coupled differential equations. The coefficient of the second term in Eq. (20b) is a variable function. After taking a series of differentiation and variable cancellation operations on these equations, one can explicitly expressed variables $\overline{\phi}$ and $\overline{U}$ in terms of variable $\bar{V}$

$$\bar{U} = \zeta_{11}\bar{V} + \zeta_{12}\bar{V}' + \zeta_{13}\bar{V}'' + \zeta_{14}\bar{V}''' + \zeta_{15}\bar{V}^{(4)} + \zeta_{16}\bar{V}^{(5)}, \tag{26}$$

$$\bar{\phi} = \zeta_{21}\bar{V} + \zeta_{22}\bar{V}' + \zeta_{23}\bar{V}'' + \zeta_{24}\bar{V}''' + \zeta_{25}\bar{V}^{(4)} + \zeta_{26}\bar{V}^{(5)}, \tag{27}$$

where $\zeta_{ij}$ are listed in Appendix A.

### *4.2. Uncoupled characteristic differential equations*

After differentiating relation (27) once and substituting it and relation (26) back to Eq. (20b), the coupled governing equations (20a)–(20c) are uncoupled and reduced to a sixth-order ordinary differential equation with variable coefficients, in terms of the variable $\bar{V}$

$$a_6\bar{V}^{(6)} + a_5\bar{V}^{(5)} + a_4\bar{V}^{(4)} + a_3\bar{V}''' + a_2\bar{V}'' + a_1\bar{V}' + a_0\bar{V} = 0, \tag{28}$$

where the corresponding coefficients $a_i$ are listed in Appendix B.

The associated boundary conditions are

$$\text{at} \quad \xi = 0: \quad b_{16}\bar{V}^{(5)} + b_{15}\bar{V}^{(4)} + b_{14}\bar{V}''' + b_{13}\bar{V}'' + b_{12}\bar{V}' + b_{11}\bar{V} = 0, \tag{29a}$$

$$b_{26}\bar{V}^{(5)} + b_{25}\bar{V}^{(4)} + b_{24}\bar{V}''' + b_{23}\bar{V}'' + b_{22}\bar{V}' + b_{21}\bar{V} = 0, \tag{29b}$$

$$b_{36}\bar{V}^{(5)} + b_{35}\bar{V}^{(4)} + b_{34}\bar{V}''' + b_{33}\bar{V}'' + b_{32}\bar{V}' + b_{31}\bar{V} = 0, \tag{29c}$$

$$\text{at} \quad \xi = 1: \quad b_{46}\bar{V}^{(5)} + b_{45}\bar{V}^{(4)} + b_{44}\bar{V}''' + b_{43}\bar{V}'' + b_{42}\bar{V}' + b_{41}\bar{V} = 0, \tag{30a}$$

$$b_{56}\bar{V}^{(5)} + b_{55}\bar{V}^{(4)} + b_{54}\bar{V}''' + b_{53}\bar{V}'' + b_{52}\bar{V}' + b_{51}\bar{V} = 0, \tag{30b}$$

$$b_{66}\bar{V}^{(5)} + b_{65}\bar{V}^{(4)} + b_{64}\bar{V}''' + b_{63}\bar{V}'' + b_{62}\bar{V}' + b_{61}\bar{V} = 0, \tag{30c}$$

where the coefficients $b_{ij}$ are listed in Appendix C.

### 4.3. Without Coriolis force effect

If one neglects the Coriolis force and retains the extensional deformation effect, the longitudinal motion, Eq. (20a), is uncoupled to the motions in $\bar{V}$ and $\bar{\phi}$ directions, Eqs. (20b)–(20c). The governing characteristic differential equations for the flexural motion become

$$k_s L_z^2\left(\bar{V}'' - \bar{\phi}'\right) + \alpha^2\left(\overline{N_p}\bar{V}'\right)' + (\Lambda^2 + \alpha^2)\bar{V} = 0, \tag{31a}$$

$$\bar{\phi}'' + \frac{\alpha^2 + \Lambda^2}{L_z^2}\bar{\phi} + k_s L_z^2(\overline{V}' - \bar{\phi}) = 0 \tag{31b}$$

and the associated boundary conditions are Eqs. (24a)–(24b), (25a)–(25b)

The explicit relation between $\bar{\phi}$ and $\bar{V}$ is reduced to

$$\bar{\phi} = \frac{1}{c_{n3}}\left[\left(k_s L_z^2 + \alpha^2\overline{N_p}\right)\bar{V}''' + 2\alpha^2\overline{N_p}'\bar{V}'' + \left(\Lambda^2 + \alpha^2 + k_s^2 L_z^4 + \alpha^2\overline{N_p}''\right)\bar{V}'\right], \tag{32}$$

where

$$c_{n3} = \frac{1}{k_s^2 L_z^4 - k_s(\Lambda^2 + \alpha^2)}.$$

As a result, the coupled equations can be reduced into a fourth-order ordinary differential equation in terms of the variable $\bar{V}$

$$s_4\bar{V}^{(4)} + s_3\bar{V}''' + s_2\bar{V}'' + s_1\bar{V}' + s_0\bar{V} = 0, \tag{33}$$

where

$$\begin{aligned} &s_0 = c_{n4}(\Lambda^2 + \alpha^2), && s_1 = c_{n4}\alpha^2\overline{N_p}' + \alpha^2\overline{N_p}''', \\ &s_2 = (\Lambda^2 + \alpha^2)(k_s + 1) + c_{n4}\alpha^2\overline{N_p}, && s_3 = 3\alpha^2\overline{N_p}', \\ &s_4 = k_s L_z^2 + \alpha^2\overline{N_p}, \end{aligned} \tag{34}$$

where

$$c_{n4} = \frac{\Lambda^2 + \alpha^2}{L_z^2} - k_s L_z^2.$$

The associated boundary conditions can be rearranged accordingly. When the extensional deformation is ignored, the term $\overline{N_p}$ is replaced by$\overline{N_{pc}}$.

*4.4. Euler beam theory*

Without consideration of the shear deformation and the rotary inertia effects, the associated governing differential equations (20a)–(20c) are reduced to

$$\bar{U}'' + \frac{\Lambda^2 + \alpha^2}{L_z^2}\bar{U} + \frac{2i\Lambda\alpha}{L_z^2}\bar{V} = 0, \tag{35a}$$

$$\bar{V}^{(4)} - \alpha^2\left(\overline{N_p}\bar{V}'\right)' - (\Lambda^2 + \alpha^2)\bar{V} + 2i\Lambda\alpha\bar{U} = 0. \tag{35b}$$

When the extensional deformation is neglected and the inclination angle $\theta = 0$, Eqs. (35a)–(35b) are the same as those given in the literature [9,13].

## 5. Fundamental solutions and frequency equation

The decoupled governing differential equations (28) is a sixth-order ordinary differential equation with variable coefficients. In general, the exact fundamental solutions are not available. However, if the coefficients of the differential equation can be expressed in the following polynomial form:

$$a_0 = \sum_{i=0}^{n0} c_i\xi^i, \quad a_1 = \sum_{i=0}^{n1} e_i\xi^i, \quad a_2 = \sum_{i=0}^{n2} f_i\xi^i, \quad a_3 = \sum_{i=0}^{n3} g_i\xi^i, \quad a_4 = \sum_{i=0}^{n4} h_i\xi^i,$$
$$a_5 = \sum_{i=0}^{n5} p_i\xi^i, \quad a_6 = \sum_{i=0}^{n6} q_i\xi^i, \tag{36}$$

then the six linearly independent fundamental solutions, $w_i(\xi)$, $i = 1$–$6$, of the differential equation (28), which satisfy the following normalization condition at the origin of the coordinate system

$$\begin{bmatrix} w_1(0) & w_2(0) & w_3(0) & w_4(0) & w_5(0) & w_6(0) \\ w_1'(0) & w_2'(0) & w_3'(0) & w_4'(0) & w_5'(0) & w_6'(0) \\ w_1''(0) & w_2''(0) & w_3''(0) & w_4''(0) & w_5''(0) & w_6''(0) \\ w_1'''(0) & w_2'''(0) & w_3'''(0) & w_4'''(0) & w_5'''(0) & w_6'''(0) \\ w_1^{(4)}(0) & w_2^{(4)}(0) & w_3^{(4)}(0) & w_4^{(4)}(0) & w_5^{(4)}(0) & w_6^{(4)}(0) \\ w_1^{(5)}(0) & w_2^{(5)}(0) & w_3^{(5)}(0) & w_4^{(5)}(0) & w_5^{(5)}(0) & w_6^{(5)}(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{37}$$

can be obtained by extending Lee and Kuo's work [15] and using the method of Frobenius [16,17]. Otherwise, the approximated solutions can be obtained by following the algorithm developed by Lee and Kuo [18].

The six independent solutions are assumed to be in the form of

$$w_i(\xi) = \sum_{n=0}^{\infty} k_{i,n}\xi^n, \quad i = 1, 2, \ldots, 6, \tag{38a}$$

where

$$\begin{aligned} &\text{for} \quad w_1(\xi): \; k_{1,0} = 1, \; k_{1,1} = k_{1,2} = k_{1,3} = k_{1,4} = k_{1,5} = 0, \\ &\text{for} \quad w_2(\xi): \; k_{2,1} = 1, \; k_{2,0} = k_{2,2} = k_{2,3} = k_{2,4} = k_{2,5} = 0, \\ &\text{for} \quad w_3(\xi): \; k_{3,2} = 1/2, \; k_{3,0} = k_{3,1} = k_{3,3} = k_{3,4} = k_{3,5} = 0, \\ &\text{for} \quad w_4(\xi): \; k_{4,3} = 1/6, \; k_{4,0} = k_{4,1} = k_{4,2} = k_{4,4} = k_{4,5} = 0, \\ &\text{for} \quad w_5(\xi): \; k_{5,4} = 1/24, \; k_{5,0} = k_{5,1} = k_{5,2} = k_{5,3} = k_{5,5} = 0, \\ &\text{for} \quad w_6(\xi): \; k_{6,5} = 1/120, \; k_{6,0} = k_{6,1} = k_{6,2} = k_{6,3} = k_{6,4} = 0. \end{aligned} \tag{38b}$$

It is noted that Eq. (38b) is required for establishing the six normalized fundamental solutions $w_i(\xi)$. Upon substituting Eqs. (36), (38) into Eq. (28) and collecting the coefficients of like powers of $\xi$, the following recurrence formula can be obtained:

$$k_{i,m+6} = \frac{-1}{(m+6)(m+5)\cdots(m+1)q_0}\left\{\left[\sum_{j=1}^{m}(m-j+6)\cdots(m-j+1)q_jk_{i,m-j+6}\right]\right.$$
$$+\left[\sum_{j=0}^{m}(m-j+5)\cdots(m-j+1)p_jk_{i,m-i+5}\right]+\left[\sum_{j=0}^{m}(m-j+4)\cdots(m-j+1)h_jk_{i,m-i+4}\right]$$
$$+\left[\sum_{j=0}^{m}(m-j+3)\cdots(m-j+1)g_jk_{i,m-j+3}\right]+\left[\sum_{j=0}^{m}(m-j+2)(m-j+1)f_jk_{i,m-j+2}\right]$$
$$\left.+\left[\sum_{j=0}^{m}(m-j+1)e_jk_{i,m-j+1}\right]+\left[\sum_{j=0}^{m}c_jk_{i,m-j}\right]\right\}. \qquad m=0,1,\ldots,\infty. \tag{39}$$

With this recurrence formula, the six exact normalized fundamental solutions (28) can be generated. Therefore, the general solution of the system is

$$\bar{V}(\xi)=\sum_{=1}^{6}\overline{c_i}w_i(\xi), \tag{40}$$

where $\{\overline{c_i}\}$are the constants to be determined.

Substituting the general solution into the associated boundary conditions yields a set of equations

$$[B_{ij}]\{\overline{c_i}\}=0, \quad i,j=1{-}6, \tag{41}$$

where

$$B_{ij}=b_{ij}, \quad i=1,2,3, \tag{42a}$$

$$B_{ij}=\sum_{m=1}^{6}b_{im}w_j^{(m-1)}(1), \quad i=4,5,6; \quad j=1,2,\ldots,6. \tag{42b}$$

The natural frequencies of the rotating inclined beam can now be obtained from the associated frequency equation.

## 6. Frequency relations for the systems with different inclination angle and hub radius

The natural frequencies of the system can be numerically determined by the method described in the previous section. However, most of the numerical results can only provide partial qualitative conclusions. In addition, it requires a wide range of data to achieve this. In this section, qualitative relations are explored without numerical analysis. To specify two different systems, subscripts "a" and "b" are added to the associated physical parameters.

Consider two dynamic systems with the same physical parameters except the inclination angle $\theta$ and the hub radius $r_h$. In terms of dimensionless quantities, $\theta$ and $\mu$, one can observe and obtain the following conclusions:

(1) If $\mu_a\cos\theta_a=\mu_b\cos\theta_b$ the governing characteristic differential equations and the associated boundary conditions of two dynamic systems will be the same. Therefore, the fundamental solutions and the natural frequencies of two systems will be the same.
(2) It is well known that the natural frequencies of a rotating beam will increase as the hub radius is increased [4]. Combining the fact with the first conclusion, one can also conclude that if the hub radius is not zero and the inclination angle is increased, the natural frequencies of the dynamic system decrease.
(3) If the hub radius is zero, the coefficients in the governing characteristic differential equations and the associated boundary conditions of the dynamic system will be independent with the inclination angle. Therefore, the natural frequencies of the dynamic system will be independent with the inclination angle.

(4) If the inclination angle is $\pm 90^\circ$, the coefficients in the governing characteristic differential equations and the associated boundary conditions of the dynamic system will be independent with the hub radius. Therefore, the natural frequencies of the dynamic system will be independent with the hub radius.

It should be mentioned that the conclusions are valid for both systems with and without considering the Coriolis force effect. In addition, the conclusions are also valid for both Timoshenko and Euler beam systems.

## 7. Numerical analysis

To illustrate the previous analysis, the accuracy of the numerical analysis and the influence of extensional deformation and Coriolis effects on the natural frequencies of a rotating inclined Timoshenko beam, numerical results are presented and discussed. Here, the case considering the extensional deformation in the centrifugal stiffening force term ($\overline{N_p}$) and the Coriolis force is referred to as case A. The case considering the extensional deformation in the centrifugal stiffening force term ($\overline{N_p}$) and without considering the Coriolis force is referred to as case B. The case without considering both the extensional deformation ($\overline{N_{pc}}$) and the Coriolis force is referred to as case C. Case C is the one most commonly considered in the existing literature. The case considering the Coriolis force and without considering the extensional deformation ($\overline{N_{pc}}$) is referred to as case D. Data without special indication are obtained through case A approach.

To illustrate the accuracy of the numerical results in the present analysis, the first six natural frequencies of the rotating beam, evaluated for case A, are compared with those given by Lin and Hsiao [11] and given in Table 1. It shows that the results are very consistent as the dimensionless rotating extension parameter $\lambda = 0$, 0.05 and 0.1, and the natural frequencies increase as $\lambda$ is increased.

In Figs. 2(a) and (b), the influences of the inclination angle on the first five natural frequencies of rotating inclined beams with different hub radius are shown. One can observe that the frequencies increase as the hub radius is increased. When the hub radius is zero, the natural frequencies are independent with the inclination angle. When the hub radius is not zero, the natural frequencies decrease as the inclination angle of the dynamic system is increased. These observations from the numerical results are consistent with those qualitative frequency relations revealed in Section 6.

Fig. 3 shows the mode shapes of the first five natural frequencies in Fig. 2 with $\theta = 30^\circ$, $\mu = 2$ and $\alpha = 5$. One can observe that the third mode is mainly dominated by the axial deformation. Therefore, the influences of the inclination angle and the hub radius on the third natural frequencies of beams are almost negligible. This observation is consistent with that given in Fig. 2.

Table 1
First six natural frequencies of a rotating Timoshenko beam ($\theta = 0$, $\mu = 0$, $L_z = 20$, $k_s = 0.32693$)

| $\lambda$ | | $\Lambda_1$ | $\Lambda_2$ | $\Lambda_3$ | $\Lambda_4$ | $\Lambda_5$ | $\Lambda_6$ |
|---|---|---|---|---|---|---|---|
| 0 | A | 3.436 | 19.139 | 31.416 | 46.751 | 79.240 | 94.248 |
| | # | 3.436 | 19.140 | 31.416 | 46.752 | 79.240 | 94.248 |
| 0.05 | A | 3.454 | 19.259 | 31.466 | 46.906 | 79.431 | 94.264 |
| | # | 3.456 | 19.260 | 31.466 | 46.906 | 79.432 | 94.264 |
| 0.1 | A | 3.503 | 19.613 | 31.618 | 47.367 | 80.003 | 94.313 |
| | # | 3.506 | 19.616 | 31.618 | 47.368 | 80.002 | 94.312 |
| 0.2 | A | 3.675 | 20.982 | 32.221 | 49.177 | 82.266 | 94.514 |
| | # | — | — | — | — | — | — |
| 0.5 | A | 4.680 | 29.050 | 36.397 | 60.773 | 96.121 | 97.310 |
| | # | — | — | — | — | — | — |
| 1.0 | A | 8.524 | 47.202 | 56.206 | 92.840 | 111.570 | 143.778 |
| | # | — | — | — | — | — | — |

A: case A; #: Ref. [11].

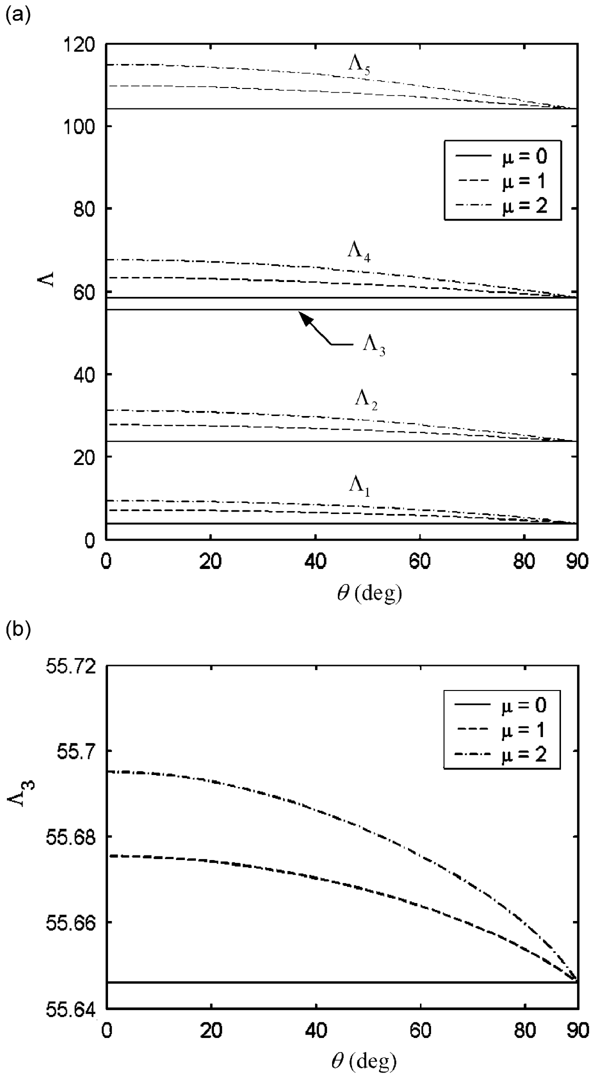


Fig. 2. Influence of inclination angle $\theta$ and hub radius on the first five natural frequencies of a rotating beam ($\alpha = 5$, $L_z = 35$, $k_s = 0.32693$).

In Table 2, the first five natural frequencies in Fig. 2 with $\theta = 30^\circ$, $\mu = 2$ and $\alpha = 5$ are also evaluated via cases A and C approaches. Since the extensional deformation is not considered in case C approach, the third natural frequency, which is mainly dominated by the extensional deformation is missed in case C approach. This analysis shows that the vibration modes mainly dominated by the extensional deformation will be lost in the pure bending vibration analysis [6–8].

Table 3 shows the first two natural frequencies of rotating inclined beams with the same physical parameters except the inclination angle and the hub radius. The inclination angle and the hub radius satisfy the relation $\mu_a \cos\theta_a = \mu_b \cos\theta_b$. It can be found that for two dynamic systems with the same physical parameters except the inclination angle and the hub radius, if the relation $\mu_a \cos\theta_a = \mu_b \cos\theta_b$ exists, the natural frequencies of two systems will be the same. The results in this table are consistent with the frequency relation revealed in the Section 6.

Fig. 4 shows the influence of the inclination angle on the centrifugal stiffening force at the top, the middle and the root of the beam, respectively. It is obvious that there will be no centrifugal stiffening force at the top of the beam. It can be observed that the centrifugal stiffening force at the other positions of the beam will decrease as the inclination angle is increased. The results are consistent with those revealed in Fig. 2.

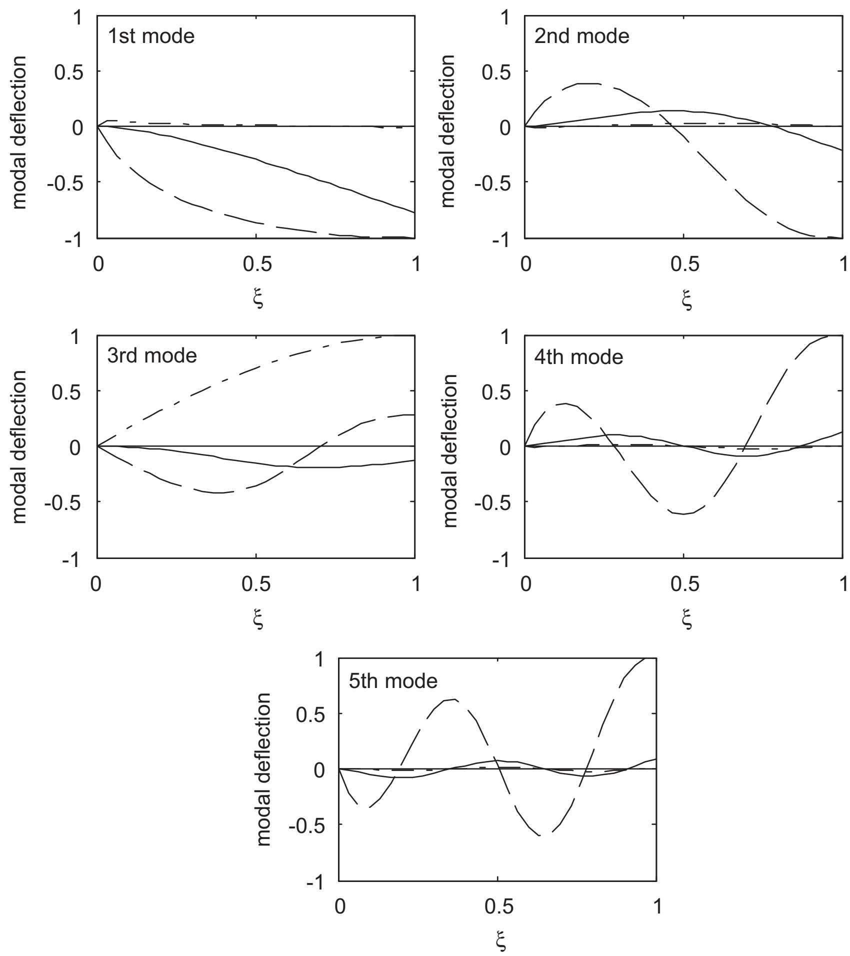


Fig. 3. Mode shapes of the first five natural frequencies of a rotating beam ($\alpha = 5$, $L_z = 35$, $\theta = 30°$, $\mu = 2$, $\mu_s = 0.32693$; $\bar{V} = (\xi)$ : "–"; $\bar{\phi} = (\xi)$ : "- - -"; $\bar{U}(\xi)$ : "—-—").

Table 2
First five natural frequencies evaluated via two different approaches ($\theta = 30°$, $\mu = 2$, $\alpha = 5$, $L_z = 35$, $k_s = 0.32693$)

| Approach | Natural frequency | | | | |
|---|---|---|---|---|---|
| | $\Lambda_1$ | $\Lambda_2$ | $\Lambda_3$ | $\Lambda_4$ | $\Lambda_5$ |
| Case A | 8.9224 | 30.3892 | 55.6901 | 66.6180 | 113.6236 |
| Case C | 9.0207 | 30.4013 | # | 66.5798 | 113.5631 |

Fig. 5 shows that when the extensional deformation is ignored, the centrifugal stiffening force $\overline{N_{pc}}$ will be independent with the dimensionless rotating extension parameter $\lambda$. Otherwise, as the dimensionless rotating extension parameter is increased, the centrifugal stiffening force $\overline{N_p}$ increases.

Fig. 6 shows the influence of the dimensionless rotating extension parameter $\lambda$, the extensional deformation and the Coriolis force on the first two natural frequencies of a rotating beam. The following conclusions can be observed:

1. The natural frequencies of the beam considering the Coriolis force effect will be smaller than those of the beam without considering the Coriolis force effect. This conclusion is consistent with those in the existing literature [11].

Table 3
First two natural frequencies of rotating inclined beams with different $\mu$ and $\theta$ ($L_z = 30$)

| $\alpha$ | $\Lambda_1$ | $\mu = 1$, $\cos\theta = 1$ | | $\mu = 2$, $\cos\theta = 0.5$ | |
|---|---|---|---|---|---|
| | | Case A | Case B | Case A | Case B |
| 0 | $\Lambda_1$ | 3.4798 | 3.4798 | 3.4798 | 3.4798 |
| | $\Lambda_2$ | 20.5892 | 20.5892 | 20.5892 | 20.5892 |
| 1 | $\Lambda_1$ | 3.7173 | 3.7204 | 3.7173 | 3.7204 |
| | $\Lambda_2$ | 20.9100 | 20.9134 | 20.9100 | 20.9134 |
| 2 | $\Lambda_1$ | 4.3435 | 4.3592 | 4.3435 | 4.3592 |
| | $\Lambda_2$ | 21.8431 | 21.8573 | 21.8431 | 21.8573 |
| 3 | $\Lambda_1$ | 5.2054 | 5.2453 | 5.2054 | 5.2453 |
| | $\Lambda_2$ | 23.3125 | 23.3460 | 23.3125 | 23.3460 |
| 4 | $\Lambda_1$ | 6.1841 | 6.2693 | 6.1841 | 6.2693 |
| | $\Lambda_2$ | 25.2207 | 25.2843 | 25.2207 | 25.2843 |
| 5 | $\Lambda_1$ | 7.2153 | 7.3724 | 7.2153 | 7.3724 |
| | $\Lambda_2$ | 27.4718 | 27.5794 | 27.4718 | 27.5794 |

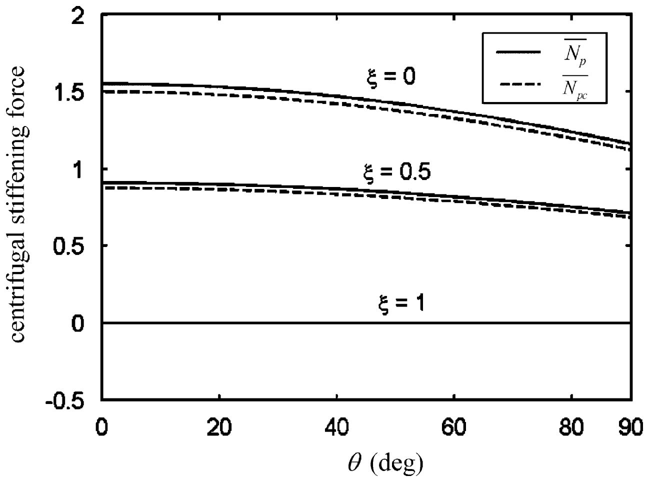


Fig. 4. Influence of the inclination angle on the centrifugal stiffening force ($\mu = 1$, $\lambda = 0.3$).

2. The natural frequencies of a rotating beam generally increase as the dimensionless rotating extension parameter $\lambda$ is increased. It is due to the fact that as the dimensionless rotating extension parameter $\lambda$ is increased, the centrifugal stiffening force increases. However, if $\lambda$ is large enough (for instance, see the curve near $\lambda = 1$ obtained for case D), the effect of the Coriolis force and the centrifugal force term $\rho A \Omega^2 v_d$ in Eq. (13b) will be probably larger than that of the centrifugal stiffening force, and consequently the natural frequencies will decrease.
3. The natural frequencies of the beam considering the extensional deformation in the centrifugal stiffening force term are greater than those of the beam without considering the extensional deformation in the centrifugal stiffening force term. It is due to the fact that the centrifugal stiffening force of a rotating beam with extensional deformation will be greater than that without extensional deformation, i.e. $\overline{N_p} \geqslant \overline{N_{pc}}$.
4. Both the extensional deformation and the Coriolis force will have very significant influence on the natural frequencies of the rotating beam as the dimensionless rotating extension parameter $\lambda$ is large. The error for

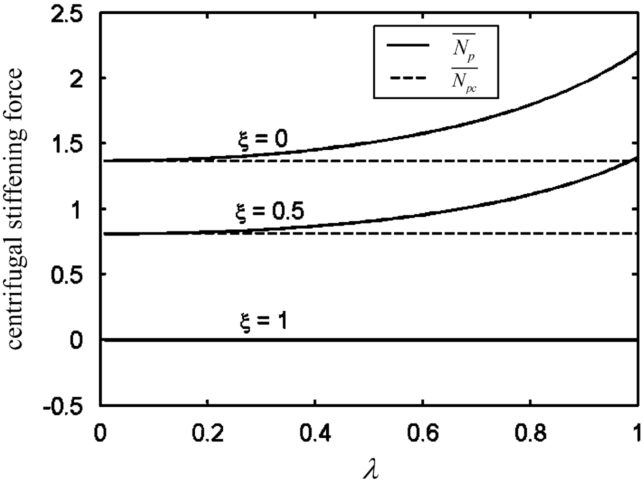


Fig. 5. Influence of the extension parameter $\lambda$ on the centrifugal stiffening force ($\mu = 1$, $\theta = 30°$).

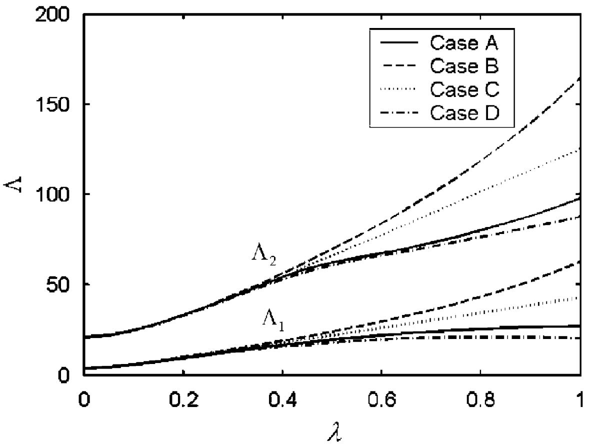


Fig. 6. Influence of dimensionless rotating extension parameter $\lambda$ and Coriolis force on the first two natural frequencies of a rotating beam ($\mu = 1$, $L_z = 35$, $\theta = 0$, $k_s = 0.32693$).

Table 4
The fundamental natural frequencies evaluated via four different approaches ($\mu = 1$, $\theta = 0$, $L_z = 35$)

| Parameter $\lambda$ | Approach | | | | | | |
|---|---|---|---|---|---|---|---|
| | Case A | Case B | | Case C | | Case D | |
| | $\Lambda$ | $\Lambda$ | Error | $\Lambda$ | Error | $\Lambda$ | Error |
| 0.2 | 9.4128 | 9.7122 | 3.18 | 9.6002 | 2.00 | 9.3045 | 1.15 |
| 0.6 | 21.9090 | 29.5040 | 34.67 | 26.0357 | 18.84 | 19.5493 | 10.77 |
| 1.0 | 26.8976 | 62.7494 | 133.3 | 42.7629 | 58.98 | 20.3084 | 24.50 |

$\text{Error} = \frac{(\text{case X-case A})}{\text{case A}} \times 100\%, \quad \text{X} = \text{B, C, D}.$

the natural frequencies of the beam without considering any one of the both factors will turn to be considerable as the dimensionless rotating extension parameter $\lambda$ is large.

In Table 4, the discrepancy between the fundamental natural frequencies evaluated for different cases and those evaluated via case A approach is illustrated. It can be found that when the dimensionless rotating

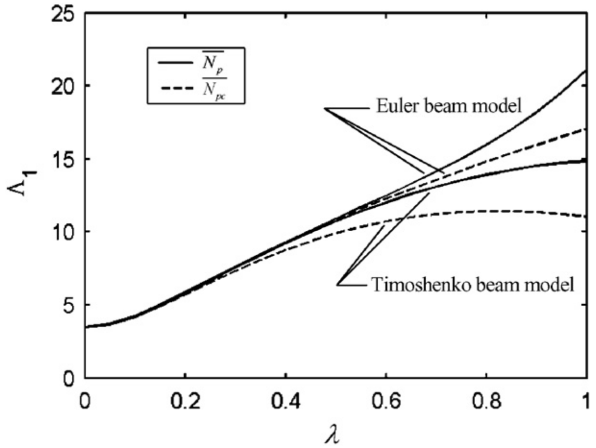


Fig. 7. The first natural frequencies of a rotating beam evaluated by two different beam theories, using two different approaches cases A and D ($L_z = 20$, $\theta = 30°$, $\mu = 1$, $k_s = 0.32693$).

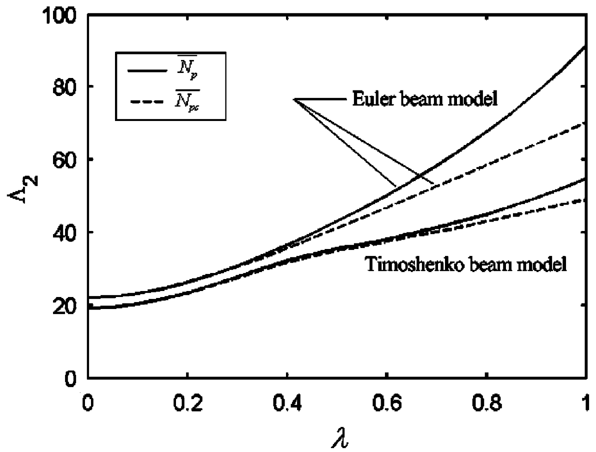


Fig. 8. The second natural frequencies of a rotating beam evaluated by two different beam theories, using two different approaches cases A and D ($L_z = 20$, $\theta = 30°$, $\mu = 1$, $k_s = 0.32693$).

extension parameter$\lambda$ is large, the error is considerable. The natural frequencies evaluated for case D have the smallest error, as expected.

In Figs. 7 and 8, using two different approaches, cases A and D, the first two natural frequencies of a rotating beam evaluated by two different beam theories are shown. In the analysis, the Coriolis force effect is taken into account. It can be found that when the dimensionless rotating extension parameter $\lambda$ is increased, the difference between the first natural frequencies of a rotating Timonshenko beam considering two different centrifugal stiffening forces is greater than that of a rotating beam when $\lambda$ is less than approximately 0.95. The difference between the second natural frequencies of a rotating Timonshenko beam considering two different centrifugal stiffening forces is less than that of a rotating beam.

## 8. Conclusions

By utilizing the Hamilton's principle, three coupled governing differential equations for a rotating inclined Timoshenko beam are derived. Both the extensional deformation and the Coriolis force effect are considered.

An exact series solution of the system is developed. The frequency relations and analytical numerical results have led to the following conclusions:

1. For two dynamic systems with the same physical parameters except the inclination angle and the hub radius:
   (a) If $\mu_a \cos\theta_a = \mu_b \cos\theta_b$, the natural frequencies of two systems will be the same.
   (b) If the hub radius is not zero and the inclination angle of the dynamic system is increased, the natural frequencies decrease.
   (c) If the hub radius is zero, the inclination angle will have no influence on the natural frequencies of the dynamic system.
   (d) If the inclination angle is $\pm 90^\circ$, the natural frequencies of the dynamic system will be independent with the hub radius.
2. Both the extensional deformation and the Coriolis force will have very significant influence on the natural frequencies of the rotating beam as the dimensionless rotating extension parameter $\lambda$ is large. The error for the natural frequencies of the beam without considering any one of the both factors will increase as the dimensionless rotating extension parameter $\lambda$ is increased. The error will turn to be considerable as the dimensionless rotating extension parameter $\lambda$ is large.
3. The difference between the first natural frequencies of a rotating Timonshenko beam considering two different centrifugal stiffening forces is greater than that of a rotating beam when $\lambda$ is less than approximately 0.95. The difference between the second natural frequencies of a rotating Timonshenko beam considering two different centrifugal stiffening forces is less than that of a rotating beam.

**Acknowledgments**

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## Appendix A. The coefficients of Eqs. (26)–(27)

$$\zeta_{11} = 0, \quad \zeta_{12} = \frac{1}{\zeta_{10}}\left[k_s^3 L_z^6 + \frac{(\Lambda^2-\alpha^2)^2}{L_z^2} + \frac{\alpha^2(\Lambda^2+\alpha^2)}{L_z^2}\overline{N_p}'' + \alpha^2 \overline{N_p}^{(4)}\right],$$

$$\zeta_{13} = \frac{1}{\zeta_{10}}\left[\frac{2\alpha^2(\Lambda^2+\alpha^2)}{L_z^2}\overline{N_p}' + 4\alpha^2\overline{N_p}'''\right],$$

$$\zeta_{14} = \frac{1}{\zeta_{10}}\left[k_s^2 L_z^4 + (\Lambda^2+\alpha^2)(k_s+1) + \frac{\alpha^2(\Lambda^2+\alpha^2)}{L_z^2}\overline{N_p} + 6\alpha^2\overline{N_p}''\right],$$

$$\zeta_{15} = \frac{1}{\zeta_{10}}\left(4\alpha^2\overline{N_p}'\right), \quad \zeta_{16} = \frac{1}{\zeta_{10}}\left(k_s L_z^2 + \alpha^2\overline{N_p}\right),$$

$$\zeta_{21} = \frac{1}{\zeta_{20}}\left[\frac{(\Lambda^2-\alpha^2)^2}{k_s L_z^4} - (\Lambda^2+\alpha^2)\right], \quad \zeta_{22} = \frac{1}{\zeta_{20}}\left[\alpha^2\left(\frac{\Lambda^2+\alpha^2}{k_s L_z^4} - 1\right)\overline{N_p}' + \frac{\alpha^2}{k_s L_z^2}\overline{N_p}'''\right],$$

$$\zeta_{23} = \frac{1}{\zeta_{20}}\left[\frac{(\Lambda^2+\alpha^2)}{L_z^2}\left(\frac{1}{k_s}+1\right) + \alpha^2\left(\frac{\Lambda^2+\alpha^2}{k_s L_z^4} - 1\right)\overline{N_p} + \frac{3\alpha^2}{k_s L_z^2}\overline{N_p}''\right],$$

$$\zeta_{24}=\frac{1}{\zeta_{20}}\left(\frac{3\alpha^2}{k_sL_z^2}\overline{N_p}'\right),\quad \zeta_{25}=\frac{1}{\zeta_{20}}\left(1+\frac{\alpha^2}{k_sL_z^2}\overline{N_p}\right),\quad \zeta_{26}=0,$$

where $\zeta_{10}=k_s^3L_z^6-k_s^2L_z^2(\alpha^2+\Lambda^2)$ and $\zeta_{20}=-2\alpha\Lambda i$; ($i=\sqrt{-1}$).

## Appendix B. The coefficients $a_i$ of the governing equation (28)

$$a_0=(\Lambda^2-\alpha^2)^2\left(\frac{(\Lambda^2+\alpha^2)}{L_z^4}-k_s\right),\quad a_1=c_{n1}\alpha^2\overline{N_p}'+c_{n2}\alpha^2\overline{N_p}'''+\alpha^2\overline{N_p}^{(5)},$$

$$a_2=\frac{(\Lambda^2-\alpha^2)^2}{L_z^2}+\frac{(\Lambda^2+\alpha^2)^2(k_s+1)}{L_z^2}-(\Lambda^2+\alpha^2)k_sL_z^2+c_{n1}\alpha^2\overline{N_p}+3c_{n2}\alpha^2\overline{N_p}''$$
$$+5\alpha^2\overline{N_p}^{(4)},$$

$$a_3=3c_{n2}\alpha^2\overline{N_p}'+10\alpha^2\overline{N_p}''',\quad a_4=(\Lambda^2+\alpha^2)(2k_s+1)+c_{n2}\alpha^2\overline{N_p}+10\alpha^2\overline{N_p}'',$$

$$a_5=5\alpha^2\overline{N_p}',\quad a_6=k_sL_z^2+\alpha^2\overline{N_p},$$

where

$$c_{n1}=\frac{(\Lambda^2+\alpha^2)^2}{L_z^4}-k_s(\Lambda^2+\alpha^2)\ \text{and}\ c_{n2}=\frac{2(\Lambda^2+\alpha^2)}{L_z^2}-k_sL_z^2.$$

## Appendix C. The coefficients of the boundary conditions (29)–(30)

$$b_{11}=1,\quad b_{12}=b_{13}=b_{14}=b_{15}=b_{16}=0,$$

$$b_{21}=0,\quad b_{22}=k_s^3L_z^6+\frac{(\Lambda^2-\alpha^2)^2}{L_z^2}+\frac{\alpha^2(\Lambda^2+\alpha^2)}{L_z^2}\overline{N_p}''(0)+\alpha^2\overline{N_p}^{(4)}(0),$$

$$b_{23}=\frac{2\alpha^2(\Lambda^2+\alpha^2)}{L_z^2}\overline{N_p}'(0)+4\alpha^2\overline{N_p}'''(0),$$

$$b_{24}=k_s^2L_z^4+(\Lambda^2+\alpha^2)(k_s+1)+\frac{\alpha^2(\Lambda^2+\alpha^2)}{L_z^2}\overline{N_p}(0)+6\alpha^2\overline{N_p}''(0),$$

$$b_{25}=4\alpha^2\overline{N_p}'(0),\quad b_{26}=k_sL_z^2+\alpha^2\overline{N_p}(0),$$

$$b_{31}=\frac{(\Lambda^2-\alpha^2)^2}{k_sL_z^4}-(\Lambda^2+\alpha^2),\quad b_{32}=\alpha^2\left(\frac{\Lambda^2+\alpha^2}{k_sL_z^4}-1\right)\overline{N_p}'(0)+\frac{\alpha^2}{k_sL_z^2}\overline{N_p}'''(0),$$

$$b_{33}=\frac{(\Lambda^2+\alpha^2)}{L_z^2}\left(\frac{1}{k_s}+1\right)+\alpha^2\left(\frac{\Lambda^2+\alpha^2}{k_sL_z^4}-1\right)\overline{N_p}(0)+\frac{3\alpha^2}{k_sL_z^2}\overline{N_p}''(0),$$

$$b_{34}=\frac{3\alpha^2}{k_sL_z^2}\overline{N_p}'(0),\quad b_{35}=1+\frac{\alpha^2}{k_sL_z^2}\overline{N_p}(0),\quad b_{36}=0,$$

$$b_{41}=0,\quad b_{42}=\frac{(\Lambda^2-\alpha^2)^2}{L_z^2}+k_s^2L_z^2(\Lambda^2+\alpha^2)+\frac{\alpha^2(\Lambda^2+\alpha^2)}{L_z^2}\overline{N_p}''(1)+\alpha^2\overline{N_p}^{(4)}(1),$$

$$b_{43} = \frac{2\alpha^2(\Lambda^2 + \alpha^2)}{L_z^2}\overline{N_p}'(1) + 4\alpha^2\overline{N_p}'''(1),$$

$$b_{44} = k_s^2 L_z^4 + (\Lambda^2 + \alpha^2)(k_s + 1) + \frac{\alpha^2(\Lambda^2 + \alpha^2)}{L_z^2}\overline{N_p}(1) + 6\alpha^2\overline{N_p}''(1),$$

$$b_{45} = 4\alpha^2\overline{N_p}'(1), \quad b_{46} = k_s L_z^2 + \alpha^2\overline{N_p}(1),$$

$$b_{51} = \frac{(\Lambda^2 - \alpha^2)^2}{L_z^2}, \quad b_{52} = \frac{\alpha^2(\Lambda^2 + \alpha^2)}{L_z^2}\overline{N_p}'(1) + \alpha^2\overline{N_p}'''(1),$$

$$b_{53} = k_s^2 L_z^4 + (\Lambda^2 + \alpha^2)(k_s + 1) + \frac{\alpha^2(\Lambda^2 + \alpha^2)}{L_z^2}\overline{N_p}(1) + 3\alpha^2\overline{N_p}''(1),$$

$$b_{54} = 3\alpha^2\overline{N_p}'(1), \quad b_{55} = k_s L_z^2 + \alpha^2\overline{N_p}(1), \quad b_{56} = 0,$$

$$b_{61} = 0, \quad b_{62} = \frac{(\Lambda^2 - \alpha^2)^2}{k_s L_z^4} - (\Lambda^2 + \alpha^2) + \alpha^2\left(\frac{\Lambda^2 + \alpha^2}{k_s L_z^4} - 1\right)\overline{N_p}''(1) + \frac{\alpha^2}{k_s L_z^2}\overline{N_p}^{(4)}(1),$$

$$b_{63} = 2\alpha^2\left(\frac{\Lambda^2 + \alpha^2}{k_s L_z^4} - 1\right)\overline{N_p}'(1) + \frac{4\alpha^2}{k_s L_z^2}\overline{N_p}''(1),$$

$$b_{64} = \frac{(\Lambda^2 + \alpha^2)}{L_z^2}\left(\frac{1}{k_s} + 1\right) + \alpha^2\left(\frac{\Lambda^2 + \alpha^2}{k_s L_z^4} - 1\right)\overline{N_p}(1) + \frac{6\alpha^2}{k_s L_z^2}\overline{N_p}''(1),$$

$$b_{65} = \frac{4\alpha^2}{k_s L_z^2}\overline{N_p}'(1), \quad b_{66} = 1 + \frac{\alpha^2}{k_s L_z^2}\overline{N_p}(1).$$

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