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Analysis of non-homogeneous Timoshenko beams with generalized damping distributions

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Abstract

This paper presents a study on the effects of generalized damping distributions on non-homogeneous Timoshenko beams.

On the basis of some fundamentals of modal analysis for damped continuous systems applied to the particular case of the Timoshenko beam model, the eigenproblem is solved by applying a method combining a state-space representation with a transfer matrix technique, yielding closed-form expressions for the eigenfunctions.

After validation by means of numerical examples using the finite element method, response functions in both the time and the frequency domain are discussed and compared according to different damping distributions.

1. Introduction

The effects of generalized damping distributions on vibrating linear continuous systems have not been exhaustively studied in terms of modal analysis. In fact, continuous systems are usually modelled as undamped systems or according to the so-called 'proportional' damping assumption, i.e., considering damping distributions such that the equations of motion can be decoupled by using the undamped system eigenfunctions [1]. Clearly, this way of modelling is so often adopted since it carries little analytical and computational further effort in addition to the undamped case analysis. But in many real situations the 'proportional' damping assumption is not valid and such a simplified approach does not describe the dynamics of the system with sufficient accuracy. So, in this paper generalized damping distributions are considered.

Vibrating Timoshenko beams have been studied by a few authors in recent years, dealing with problems such as free vibrations of beams with intermediate flexible constraints [2], free and forced vibrations of non-homogeneous beams with varying cross-section [3] and carrying a concentrated mass [4]. Even smaller is the number of papers regarding continuous vibrating systems with generalized damping distributions. A study on externally damped Timoshenko beams subjected to distributed loads can be found in Ref. [5], a variational formulation is proposed in Ref. [6] to compute the eigenvalues of cantilevered Timoshenko beams with a tip

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body and boundary damping, whilst free and forced analyses are performed in Ref. [7] for multispan Timoshenko beams supported by resilient joints with damping, by considering a dynamic matrix approach.

Since the existing literature about modal analysis for distributed parameter systems with generalized damping concerns particular cases [8,9], a complete theoretical statement valid in the general case is herein included. After reducing the differential boundary problem to an eigenvalue problem through the separation of variables, the existence of orthogonality relations valid for the case of generalized damping is demonstrated. Using such relations, the solution can then be expressed as a superposition of modes through the expansion theorem [10].

A method for the solution of the differential eigenproblem is then described [11], suitable for a class of vibrating continuous systems with generalized damping distributions, according to different models. This technique is based on the partition of a system in homogeneous substructures or segments, and it matches a reduction of the differential equation order with a transfer matrix technique. So it can be easily applied, provided that closed-form solutions of the undamped case for each substructure are known. Moreover, it leads to an easy computer implementation and presents a high computational efficiency, due to the invariance of the matrix dimensions with respect to the number of substructures considered.

At this stage the developed analytical tools are applied to non-homogeneous Timoshenko beams with different generalized internal or/and external damping distributions, yielding both the modal parameters and mode shapes and thus the expressions of the frequency response functions (FRFs) and impulse response functions (IRFs).

Finally, numerical examples are presented, the results being validated by means of a finite element model (FEM), showing both the accuracy and efficiency of the proposed methodology.

2. Modal analysis of continuous systems with viscous generalized damping

The dynamic behaviour of a continuous system with viscous generalized damping can be described by the following system of equilibrium equations:

$$\mathbf{M}\left[\frac{\partial^2}{\partial t^2}\mathbf{u}(x,t)\right] + \mathbf{C}\left[\frac{\partial}{\partial t}\mathbf{u}(x,t)\right] + \mathbf{K}[\mathbf{u}(x,t)] = \mathbf{f}(x,t), \quad x \in D, \tag{1}$$

where $\mathbf{M}[\cdot]$, $\mathbf{C}[\cdot]$ and $\mathbf{K}[\cdot]$ are linear homogeneous differential operators (which in general can be written in matrix form) and are referred to as mass operator, damping operator and stiffness operator, $\mathbf{f}$ is the external forces density, t is time, $\mathbf{u}$ is the vector of displacements and rotations and x is the spatial coordinate in a domain of extension D [1].

Recalling that self-adjointness of differential operators corresponds to symmetric matrices in the case of discrete systems, in the following the operators $\mathbf{M}[\cdot]$, $\mathbf{C}[\cdot]$ and $\mathbf{K}[\cdot]$ will be supposed to be self-adjoint [1,12].

The differential boundary problem can be reduced to a differential eigenvalue problem by separating the variables. Eq. (1) can then be rewritten in the state-space form as follows:

$$\mathbf{A}[\dot{\mathbf{v}}] + \mathbf{B}[\mathbf{v}] = \mathbf{F}, \tag{2}$$

where the dot denotes derivation with respect to time, $\mathbf{A}[\cdot]$ and $\mathbf{B}[\cdot]$ are linear homogeneous differential operators, $\mathbf{v}$ and $\mathbf{F}$ are the state vector and the external forces density vector, respectively. They can be expressed as

$$\mathbf{A}[\cdot] = \begin{bmatrix} \mathbf{C}[\cdot] & \mathbf{M}[\cdot] \\ \mathbf{M}[\cdot] & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}[\cdot] = \begin{bmatrix} \mathbf{K}[\cdot] & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}[\cdot] \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}, \tag{3}$$

so that also $\mathbf{A}[\cdot]$ and $\mathbf{B}[\cdot]$ are self-adjoint.

The state-space Eq. (2) leads to the differential eigenproblem:

$$s\mathbf{A}[\mathbf{z}] + \mathbf{B}[\mathbf{z}] = \mathbf{0} \text{ with eigenvectors } \mathbf{z} = \begin{bmatrix} \boldsymbol{\phi}(x) \\ s\boldsymbol{\phi}(x) \end{bmatrix}. \tag{4}$$

The solution of this eigenproblem forms an infinite set of pairs of discrete values, each pair characterizing a mode and being related to a pair of eigenvectors (i.e. to a pair of eigenfunctions). The orthogonality properties holding for the eigenvectors can be written in inner-product form [12]:

$$(\mathbf{z}_m, \mathbf{A}[\mathbf{z}_n]) = \delta_{mn} a_n,$$
$$(\mathbf{z}_m, \mathbf{B}[\mathbf{z}_n]) = \delta_{mn} b_n, \tag{5}$$

where δ_{mn} denotes the Kronecker symbol and $b_n/a_n = -s_n$. Due to the orthogonality properties of the eigenvectors $\mathbf{z}_n$, any other vector in the same space of functions can be expressed as their linear combination, so that the response can be written in the form:

$$\mathbf{u}(x,t) = \sum_{i=1}^{\infty} \boldsymbol{\phi}_i(x) q_i(t). \tag{6}$$

The factors $q_i(t)$ can then be evaluated according to the Laplace transform technique. The IRF due to a concentrated unit impulse applied in the jth degree of freedom at a coordinate x_f and evaluated along the jth degree of freedom takes the following form:

$$h_{x,x_f}(t) = \sum_{i=1}^{\infty} \frac{\phi_{ij}(x_f)\phi_{ij}(x)}{a_i} \exp(s_i t), \tag{7}$$

whilst the receptance due to a concentrated harmonic force applied in the jth degree of freedom at a coordinate x_f and evaluated along the jth degree of freedom can be expressed as follows [12]:

$$H_{x,x_f}(\omega) = \sum_{i=1}^{\infty} \frac{\phi_{ij}(x_f)\phi_{ij}(x)}{a_i[(i\omega) - s_i]} \tag{8}$$

the expressions of other FRFs following immediately from Eq. (8).

3. Further investigations for a class of continuous systems

In this section, the eigenproblem related to non-homogeneous Timoshenko beams with generalized damping is solved by applying a method based on a partition in homogeneous substructures, combining a state-space representation with a transfer matrix technique, yielding closed-form expressions for the eigenfunctions.

3.1. Solution of the eigenproblem

For a Timoshenko beam in bending vibration, $\mathbf{u}$, $\mathbf{M}[\cdot]$ and $\mathbf{K}[\cdot]$ can be written in the form:

$$\mathbf{u}(x,t) = [\,\vartheta(x,t) \quad w(x,t)\,]^{\mathrm{T}},$$
$$\mathbf{M}[\cdot] = \begin{bmatrix} \rho I & 0 \\ 0 & \rho A \end{bmatrix}, \quad \mathbf{K}[\cdot] = \begin{bmatrix} \kappa GA - \dfrac{\partial}{\partial x} EI \dfrac{\partial}{\partial x}(\cdot) & -\kappa GA \dfrac{\partial}{\partial x}(\cdot) \\ \dfrac{\partial}{\partial x}\kappa GA(\cdot) & -\dfrac{\partial}{\partial x}\kappa GA \dfrac{\partial}{\partial x}(\cdot) \end{bmatrix}, \tag{9}$$

where ϑ is the section rotation, w the transverse displacement, ρ is the mass density of the beam, A is the cross-section area, EI is the bending stiffness or flexural rigidity, in which E is the Young's modulus of the material and I is the area moment of inertia, G and κ are the shear modulus of elasticity and the shear coefficient, respectively [1]. All these parameters in general can vary with respect to x.

Moment and shear are related to rotation and displacement as follows:

$$M(x,t) = EI\,\frac{\partial}{\partial x}\,\vartheta(x,t),$$
$$T(x,t) = \kappa GA\left[\vartheta(x,t) - \frac{\partial}{\partial x}\,w(x,t)\right]. \tag{10}$$

As a consequence, the simplest boundary conditions can be expressed in the following form:

$$\text{Clamped}:\begin{cases}\vartheta = 0\\ w = 0\end{cases},\quad \text{Pinned}:\begin{cases}M = 0\\ w = 0\end{cases},\quad \text{Free}:\begin{cases}M = 0\\ T = 0\end{cases},\quad \text{Sliding}:\begin{cases}\vartheta = 0\\ T = 0\end{cases}. \tag{11}$$

Assuming that:

$$G = \frac{E}{2(1+v)} \tag{12}$$

which holds for isotropic materials, where v is the Poisson's ratio, then the dissipative phenomena can be taken into account according to the so-called elastic–viscoelastic correspondence principle [13], stating that it is possible to replace in the Laplace domain (or in the frequency domain) the real-valued Young's moduli with their complex representations, i.e.:

$$E = k + cs. \tag{13}$$

Eq. (13) derives from the Kelvin–Voigt model, which is adopted in the following, noticing that it is suitable only in the case of small amounts of damping. More complicated damping laws, even involving fractional derivatives, could be easily taken into account by modifying the definition of Eq. (13) as a function of s. For example:

$$E = \frac{k + cs^{\alpha}}{1 + ds^{\alpha}} \quad \text{with} \quad 0<\alpha\leqslant 1 \tag{14}$$

derives from the Fractional Standard Linear Solid model [13].

In such cases, the developments presented in this subsection still hold, but attention must be paid to the consequent modifications in the equation of motion and in its state-space representation, affecting the expressions of the modal parameters, the IRFs and the FRFs. Eq. (13) represents a model for the internal damping of the beam, depending on the material properties, but it is worth noting that distributions of external damping can be taken into account as well. For example, in the case of external viscous damping the operator $\mathbf{C}$ in Eq. (1) takes the form:

$$\mathbf{C}[\cdot] = \begin{bmatrix} C_{\vartheta} & 0 \\ 0 & C_w \end{bmatrix}, \tag{15}$$

where the parameters in general can vary with respect to x.

In order to highlight the effects of generalized damping distributions, the differential eigenvalue problem is solved in the special case in which all the parameters of the model can be considered piecewise constant on D.

Dividing the beam into P segments of length $\Delta x_p = x_p - x_{p-1}$ (where $x_0 = 0$, $x_P = l$, total length of the beam), and assuming every parameter constant on each segment, the differential eigenvalue problem can be reduced to two decoupled sets of P fourth-order ordinary differential equations with constant coefficients:

$$\varphi_p^{\mathrm{IV}}(x) = \alpha_p\varphi_p^{\mathrm{II}}(x) + \beta_p\varphi_p(x),$$
$$\phi_p^{\mathrm{IV}}(x) = \alpha_p\phi_p^{\mathrm{II}}(x) + \beta_p\phi_p(x), \tag{16}$$

with appropriate boundary conditions, where:

$$\alpha = \rho s^2\left(\frac{1}{\kappa G} + \frac{1}{E}\right), \quad \beta = -\frac{\rho A s^2}{EI}\left(\frac{\rho I s^2}{\kappa GA} + 1\right) \tag{17}$$

and φ are the eigenfunctions related to the rotation and ϕ those related to the transverse displacement.

In what follows the attention is focused on the second set of equations, but clearly the same analytical technique could also be applied to the first one, and the subscripts ϑ and w refer to vectors and matrices depending on rotation and displacement, respectively.

At this stage, it is convenient converting the fourth-order equation into a set of four first-order equations. According to the state vector definition:

$$\mathbf{y}_w(x) = [\,\phi^{\mathrm{III}}(x) \quad \phi^{\mathrm{II}}(x) \quad \phi^{\mathrm{I}}(x) \quad \phi(x)]^{\mathrm{T}}, \tag{18}$$

the solution for each segment can then be expressed as

$$\mathbf{y}_{w,p}(x) = \mathbf{\Phi}_p e^{\Lambda_p x}\mathbf{c}_{wp}, \tag{19}$$

where $\mathbf{\Phi}_p$ is the pth segment eigenvector matrix, $\mathbf{\Lambda}_p$ is the pth segment eigenvalue matrix and $\mathbf{c}_{wp}$ is the pth segment constant vector. The four eigenvalues are

$$\lambda_{1,2} = \sqrt{\frac{\alpha \pm \sqrt{\alpha^2 + 4\beta}}{2}}, \quad \lambda_{3,4} = -\lambda_{1,2}, \tag{20}$$

Moreover, it is possible to show [11] that the solution at any point x_p can be written as

$$\mathbf{y}_{w,p}(x_p) = \overset{1}{\underset{p}{\mathbf{\Pi}}}\,\mathbf{y}_{w,1}(0) \quad \text{with} \quad \overset{1}{\underset{p}{\mathbf{\Pi}}} = \prod_{i=p}^{1}\left[\mathbf{\Phi}_i e^{\Lambda_i(x_i - x_{i-1})}\mathbf{\Phi}_i^{-1}B_{wi}\right], \tag{21}$$

where the ith segment eigenvector matrix and its inverse, written as functions of $\lambda_{1,2}$, have the form:

$$\mathbf{\Phi}_i = \begin{bmatrix} \lambda_1^3 & \lambda_2^3 & -\lambda_1^3 & -\lambda_2^3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_1^2 & \lambda_2^2 \\ \lambda_1 & \lambda_2 & -\lambda_1 & -\lambda_2 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathbf{\Phi}_i^{-1} = \frac{1}{2\lambda_1\lambda_2\left(\lambda_1^2 - \lambda_2^2\right)}\begin{bmatrix} \lambda_2 & \lambda_1\lambda_2 & -\lambda_2^3 & -\lambda_1\lambda_2^3 \\ -\lambda_1 & -\lambda_1\lambda_2 & \lambda_1^3 & \lambda_1^3\lambda_2 \\ -\lambda_2 & \lambda_1\lambda_2 & \lambda_2^3 & -\lambda_1\lambda_2^3 \\ \lambda_1 & -\lambda_1\lambda_2 & -\lambda_1^3 & \lambda_1^3\lambda_2 \end{bmatrix} \tag{22}$$

the 4×4 $\mathbf{B}_{wi}$ matrices are the result of imposing the continuity of displacement, rotation, moment and shear in $x = x_{i-1}$. Clearly, these constraints represent inner boundary conditions between adjacent beam segments. Using Eqs. (9) and (10), shear, moment and rotation can be immediately written in a function of the displacement w and its derivatives:

$$\left.\begin{aligned} T &= -\frac{\rho A s^2}{\beta}(w^{\mathrm{III}} - \alpha w^{\mathrm{I}}) \\ M &= EI(w^{\mathrm{II}} - \chi w) \\ \vartheta &= -\frac{\chi}{\beta}(w^{\mathrm{III}} - \psi w^{\mathrm{I}}) \end{aligned}\right] \quad \text{with} \quad \left[\begin{aligned} \chi &= \frac{\rho s^2}{\kappa G} \\ \psi &= \chi - \frac{\kappa G A}{EI} \end{aligned}\right. \tag{23}$$

so, in the absence of external constraints in the ith separation point, the non-zero elements of $\mathbf{B}_{wi}$ take the form:

$$B_{11} = \frac{\chi_{i-1}(R_{\kappa GA}\psi_i - \alpha_i)}{\beta_{i-1}}, \quad B_{31} = \frac{\chi_{i-1}(R_{\kappa GA} - 1)}{\beta_{i-1}},$$
$$B_{13} = \frac{\chi_{i-1}\psi_i\alpha_i(R_\psi - R_{\kappa GA}R_\alpha)}{\beta_{i-1}}, \quad B_{33} = \frac{\chi_{i-1}(\psi_i - R_{\kappa GA}\alpha_i)}{\beta_{i-1}},$$

$$
\begin{aligned}
&B_{22} = R_{EI}, \quad B_{44} = 1, \\
&B_{24} = \chi_i(1 - R_{EI} R_\chi),
\end{aligned} \tag{24}
$$

where

$$
R_{\text{parameter}} = \frac{\text{value of the parameter in segment } i-1}{\text{value of the parameter in segment } i}.
$$

The same technique applies also for the eigenproblem related to the rotation. In this case the four eigenvalues are the same, so the matrices $\mathbf{\Lambda}_p$ and $\mathbf{\Phi}_p$ do not change.

The non-zero elements of the matrices $\mathbf{B}_{\vartheta i}$ take the form:

$$
\begin{aligned}
&B_{11} = \frac{R_{EI}}{R_{\rho A}}, \quad B_{24} = \frac{\rho_i s^2}{E_i}(1 - R_{\rho I}) \\
&B_{13} = \frac{\rho_i s^2}{E_i} R_{EI}(1 - R_{EA}^{-1}), \quad B_{33} = R_{EI} \\
&B_{22} = R_{EI}, \quad B_{44} = 1.
\end{aligned} \tag{25}
$$

Note that in both cases $\mathbf{B}_1 = \mathbf{I}$.

The presence of external constraints in the separation points can be taken into account by adding proper terms to the matrices $\mathbf{B}_{wi}$ or $\mathbf{B}_{\vartheta i}$, as described in Ref. [11].

It is now possible to relate the solution $\mathbf{y}(l)$ at one end of the beam to the solution $\mathbf{y}(0)$ at the other end, which enables to express the boundary conditions in the following form:

$$
\begin{aligned}
&\mathbf{B}_w^{(0)} \mathbf{y}_{w,1}(0) = \mathbf{0}, \\
&\mathbf{B}_w^{(l)} \prod_P^1 \mathbf{y}_{w,1}(0) = \mathbf{0},
\end{aligned} \tag{26}
$$

where $\mathbf{y}_{w,1}(0) = \mathbf{\Phi}_1 \mathbf{c}_{w1}$ and $\mathbf{B}_w^{(\cdot)}$ are 2×4 matrices depending on the kind of constraints, which can be obtained from Eqs. (11) and (23) as done in Ref. [11], Eqs. (23).

For example, in the case of free and clamped ends, introducing $T = 0$, $M = 0$ and $\vartheta = 0$ in Eqs. (23) and recalling the state vector definition (18), the 2×4 matrices $\mathrm{B}_w^{(\cdot)}$ can be written in the following form:

$$
\begin{aligned}
&\text{Free} \quad \mathbf{B}_w^{(\cdot)} = \begin{bmatrix} -\dfrac{\chi \kappa G A}{\beta} & 0 & \dfrac{\chi \alpha \kappa G A}{\beta} & 0 \\ 0 & EI & 0 & -\chi EI \end{bmatrix}, \\
&\text{Clamped} \quad \mathbf{B}_w^{(\cdot)} = \begin{bmatrix} -\dfrac{\chi}{\beta} & 0 & \dfrac{\chi \psi}{\beta} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\end{aligned} \tag{27}
$$

where the values of the parameters are those at the ends of the beam.

Eqs. (26) form a linear homogeneous system of four algebraic equations in four unknowns (i.e. the constant $\mathbf{c}_{w1}$). Thus the solution of the eigenproblem follows directly by setting to zero the determinant of the coefficient matrix of the system in Eq. (26), recalling that its elements depend on the (unknown) eigenvalue s.

The same applies for the rotation. In this case the eigenvalues s are the same, and the eigenfunctions can be determined after computing the constants $\mathbf{c}_{\vartheta 1}$.

It should finally be noticed that mathematically the eigenfunctions ϕ result to be classical solutions (i.e. four times continuously differentiable on D) everywhere, except in a finite subset of D (i.e. $x = x_p$, with $p = 1,\ldots,P$–1): here they result to be weak (in this case at least one time continuously differentiable) as a consequence of the discontinuities introduced in the functions α and β, which have been assumed piecewise constants on D (see also Ref. [14]).

3.2. The modal parameters

The modal parameters a_n and b_n can be computed from Eq. (5). For the Voigt internal damping model, according to the definitions in Eqs. (12) and (13), and for any combinations of clamped, pinned, free or sliding boundary conditions, dropping the modal index n they take the form:

$$(\mathbf{z}, \mathbf{A}[\mathbf{z}]) = 2\int_D (\rho A s\phi^2 + \rho I s\varphi^2)\,\mathrm{d}x + \int_D \frac{c}{E}[\kappa GA(\varphi - \phi^{\mathrm{I}})^2 + EI(\varphi^{\mathrm{I}})^2]\,\mathrm{d}x,$$
$$(\mathbf{z}, \mathbf{B}[\mathbf{z}]) = -s(\mathbf{z}, \mathbf{A}[\mathbf{z}]), \tag{28}$$

as shown in Appendix A.

In the case of clamped, pinned, free or sliding boundary conditions, Eqs. (28) are valid for any distribution of the physical parameters of the beam (either distributed or concentrated), so their use is not restricted to the particular case of a stepped beam.

When the last of Eqs. (23) holds true, after introducing the link between rotations and displacements Eqs. (10) and (23) in the expressions of the inner products, the eigenfunctions φ are not necessary any more. Thus Eqs. (28) can be written as a function of ϕ in the form:

$$(\mathbf{z}, \mathbf{A}[\mathbf{z}]) = 2\int_D \left[\rho A s\phi^2 + \rho I s\,\frac{\chi^2}{\beta^2}(\phi^{\mathrm{III}} - \psi\phi^{\mathrm{I}})^2\right]\mathrm{d}x$$
$$+ \int_D \frac{c}{E}\left[\kappa GA\,\frac{\chi^2}{\beta^2}(\phi^{\mathrm{III}} - \alpha\phi^{\mathrm{I}})^2 + EI(\phi^{\mathrm{II}} - \chi\phi)^2\right]\mathrm{d}x. \tag{29}$$

In the case of a stepped beam the integrals in the inner products can be easily computed, since the eigenfunctions are linear combinations of complex exponentials. Eq. (29) reduces to a sum of integrals, which can be written and computed according to

$$\int_{\Delta x_p} \phi_p^{(n)}\phi_p^{(m)}\,\mathrm{d}x = \mathbf{r}_p^{\mathrm{T}}\Lambda_p^n \mathbf{D}_p \Lambda_p^m \mathbf{r}_p, \tag{30}$$

where

$$\mathbf{r}_1 = \mathbf{c}_w,$$
$$\mathbf{r}_2 = \mathbf{\Phi}_2^{-1}\mathbf{B}_{w2}\mathbf{\Phi}_1 e^{\Lambda_1 x_1}\mathbf{c}_w,$$
$$\mathbf{r}_p = \prod_{i=p}^{1} \mathbf{\Phi}_i^{-1}\mathbf{B}_{wi}\mathbf{y}_{w,i-1}(x_{i-1}), \tag{31}$$

and $\mathbf{D}$ is a 4×4 matrix whose ij element is

$$[\mathbf{D}_p]_{ij} = \frac{\exp[(\lambda_i + \lambda_j)\,\Delta x_p] - 1}{\lambda_i + \lambda_j}, \tag{32}$$

the eigenvalues λ depending on the pth beam segment.

4. Numerical examples

Two numerical examples are presented as applications of the proposed method, in order both to seek for a validation by comparisons to other less general methods and to show its reliability in problems involving non-proportional damping. The results are also compared with those obtained by means of a finite element (FE) model, which also includes rotary inertia and shear deformation effects. Moreover, it should be noticed that the proposed method has also been successfully applied by the authors to solve some particular cases, separately proposed in two papers concerning non-homogeneous continuous systems [3,4].

As already shown in Refs. [11,12], the proposed method is characterized by a high computational efficiency, due to the reduced dimensions (4×4) of the matrices involved in the numerical procedure.

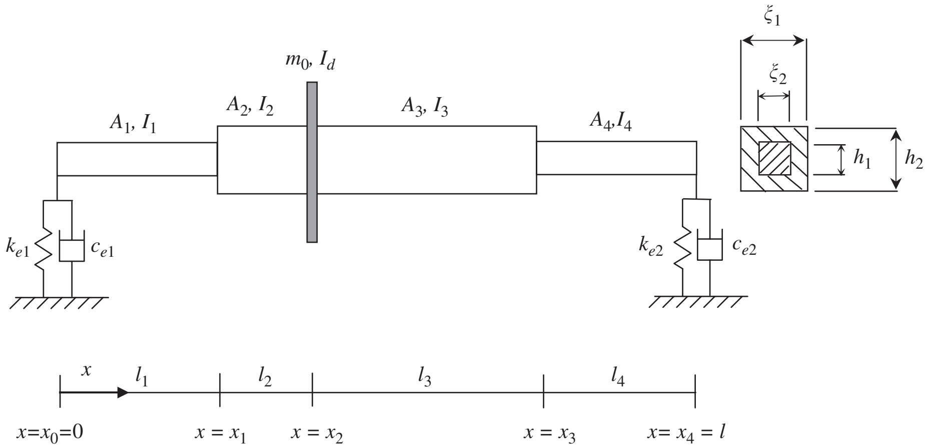

Fig. 1. Example 1, non-homogeneous three-stepped Timoshenko beam with a lumped mass and external non-proportional damping (see Ref. [7]).

Table 1
Example case 1 geometrical parameters

Segment p	Length l_p (m)	Height h_p (m)	Width ξ_p (m)
1	0.3	0.04	0.04
2	0.2	0.06	0.06
3	0.4	0.06	0.06
4	0.3	0.04	0.04

Table 2
Example case 1 structural properties

Elements	Properties	Data
Beam	Young's modulus E (GN/m^2)	200
	Shear modulus G (GN/m^2)	80
	Poisson's ratio ν	0.3
	Shear coefficient κ	$\kappa = \frac{10(1+\nu)}{12+11\nu}$
	Density ρ (kg/m^3)	8000
Concentrated mass	Mass m_0 (kg)	20
	Mass moment of inertia I_0 (kg m^2)	3.333×10^{-4}
Supporting joints (2 identical)	Stiffness $k_{e1} = k_{e2}$ (MN/m)	2
	Damping $c_{e1} = c_{e2}$ (Ns/m)	10

4.1. Example 1

As first numerical example, a three-stepped Timoshenko beam with external constraints is considered (lumped inertial, stiffness and damping elements) as proposed in Ref. [7]. Clearly, lumped damping elements form a generalized damping distribution. Geometrical and structural properties of the beam, shown in Fig. 1, are summarized in Tables 1 and 2, respectively. The minimum number of elements needed is 4, due to the lumped mass added between the segments 2 and 3.

Since the matrix dimensions are constant (i.e. 4×4), the present approach allows to avoid the increase of computational effort that affects those methods in which the dimensions grow proportionally to the number P of segments [9].

The continuity of displacement, rotation, moment and shear in sections x_1, x_2 and x_3 is guaranteed by the $\mathbf{B}_{wi}$ matrices in Eq. (21), while the external stiffness and damping constraints are taken into account by the boundary condition matrices $\mathbf{B}_w^{(0)}$ and $\mathbf{B}_w^{(l)}$: shear contributions due to the flexible supports are included in the free boundary condition matrix Eq. (27).

In Table 3, the first five eigenvalues, obtained by setting to zero the determinant of the coefficient matrix of the system in Eq. (26), are compared with those computed in Ref. [7]: the results are identical within the adopted numerical precision. Identical results are also obtained for a simpler numerical example, referred to as 'numerical example 1' in Ref. [7], not reported here.

Most of the computational effort is due to the zero finding routine in the complex domain involved in Eq. (26). This problem has been solved by means of the classical Secant Method applied to a real function of the complex variable s [15].

Table 3
Comparison of eigenvalues s_i (rad/s) of example case 1

Mode (i)	Present study	[7]
1	$-6.8390e{-}02+2.2241e+02i$	$-6.8390e{-}02+2.2241e+02i$
2	$-1.1037e+00+7.3241e+02i$	$-1.1037e+00+7.3241e+02i$
3	$-2.9635e+00+1.7357e+03i$	$-2.9635e+00+1.7357e+03i$
4	$-3.6033e+00+3.0197e+03i$	$-3.6033e+00+3.0197e+03i$
5	$-2.8679e+00+5.1358e+03i$	$-2.8679e+00+5.1358e+03i$

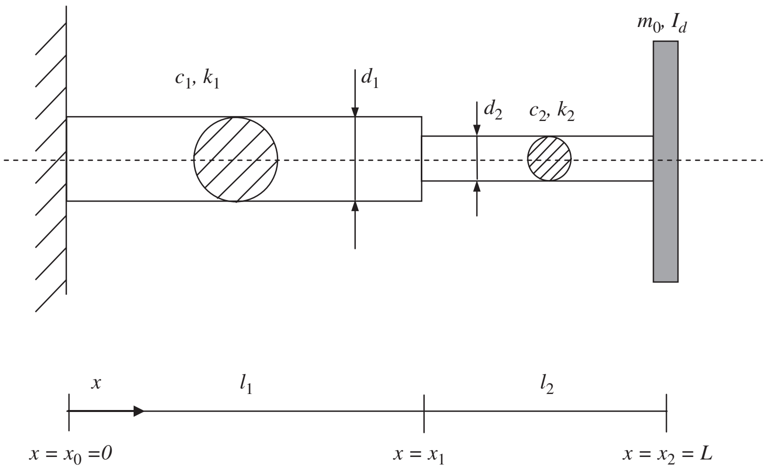

Fig. 2. Example 2, non-homogeneous Timoshenko beam with non-proportional internal damping.

Table 4
Example case 2 geometrical parameters

Segment p	Length l_p (m)	Diameter d_p (m)
1	0.3	0.05
2	0.2	0.03

Table 5
Example case 2 structural properties

Elements	Properties	Data	
		Beam 1	Beam 2
Beam	Young's modulus k_p (GN/m^2)	210	180
	Shear modulus G_p (GN/m^2)	80.769	69.231
	Internal damping (Voigt model) c_p (MN s/m^2)	3.15	0.9
	Poisson's ratio v	0.3	
	Shear coefficient κ	$\kappa = \frac{6(1+v)}{7+6v}$	
	Density ρ (kg/m^3)	7800	
Concentrated mass	Mass m_0 (kg)	1.034	
	Mass moment of inertia I_d (kg m^2)	3.634×10^{-4}	

Table 6
Eigenvalues s_i of example case 2

Mode (i)	Eigenvalue s_i (rad/s)	Equivalent modal parameters, Eq. (33)	
		$\tilde{\omega}_i$ (rad/s)	$\tilde{\zeta}_i$ (%)
1	$-3.893642e+000 \pm 7.997645e+002$i	$7.9977e+002$	0.4868
2	$-4.663169e+001 \pm 3.136414e+003$i	$3.1367e+003$	1.4866
3	$-5.228142e+002 \pm 9.983012e+003$i	$9.9967e+003$	5.2299
4	$-1.283141e+003 \pm 1.656462e+004$i	$1.6614e+004$	7.7231
5	$-3.155200e+003 \pm 2.739635e+004$i	$2.7577e+004$	11.4412

4.2. Example 2

This example deals with a two-stepped Timoshenko beam of total length L, clamped at one end and with an added concentrated inertial element at the other one (see Fig. 2 and Table 4). For each segment p a different internal damping coefficient c_p, defined in Eq. (13), has been assumed, as reported in Table 5.

The free vibration analysis, performed by means of the proposed method, produces the first five poles of the system reported in Table 6. In order to quantify the damping level, the following equivalent 'proportional' damping modal parameters are defined:

$$\text{equivalent natural frequency } \tilde{\omega}_n = ||s_n||, \tag{33a}$$

$$\text{equivalent damping ratio } \tilde{\zeta}_n = \frac{-\mathrm{Re}[s_n]}{\tilde{\omega}_n}. \tag{33b}$$

As discussed in Ref. [11,12] in case of generalized damping distributions, the modal damping ratio defined in Eq. (33b) does not correspond to a modal damping ratio in the classical sense, and it does not provide any information about the modal asynchronous behaviour: in fact, a relevant non-stationary modal displacement may correspond to small values of it (see for example Fig. 2 in Ref. [11]).

Particular attention must be paid to the added concentrated inertial element (a rigid body with mass m_0 and mass moment of inertia I_d), noticing that the definition of the modal parameters given in Eqs. (28) allows to take into account Dirac distributions for lumped parameters. Since in this case the concentrated inertial element is placed at the free end of the beam where the shear is null, i.e. $T = \kappa GA[\varphi - \phi^{\mathrm{I}}] = 0$, it follows that $\varphi(L) = \phi^{\mathrm{I}}(L)$. So, using Eqs. (28) it is possible to express the additional term Δa due to the concentrated inertial element as a function of ϕ:

$$\Delta a = 2[m_0 s \phi^2(L) + I_d s(\phi^{\mathrm{I}}(L))^2]. \tag{34}$$

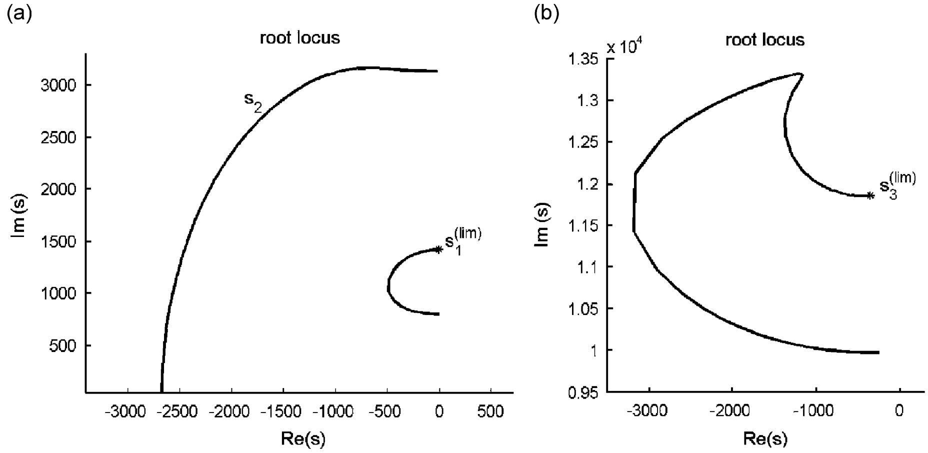

Fig. 3. Root loci for example 2 (internal damping): c_1 increases and c_2 does not vary (non-proportional damping); (a) first and second mode; (b) third mode.

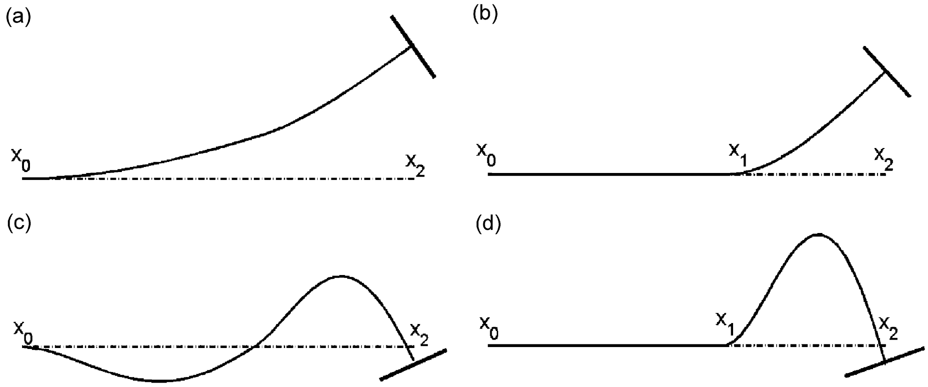

Fig. 4. Example 2: modal displacement corresponding to a pair of complex eigenfunctions; left column $c_1 = 1.05\,\mathrm{MNs/m^2}$ and $c_2 = 0.9\,\mathrm{MNs/m^2}$; right column $c_1 \to \infty$; (a) and (b) first mode; (c) and (d) third mode.

In order to show the effect of a non-proportional damping distribution on the system poles, the distributed internal damping density on the second segment ($l_1 \leqslant x \leqslant L$) is assumed to be constant, $c_2 = 0.9\,\mathrm{MNs/m^2}$, while on the first segment ($0 \leqslant x \leqslant l_1$) it varies from a low level ($c_1 = 1.05\,\mathrm{MNs/m^2}$) to infinity (non-proportional damping limit case). So, different levels of non-proportionality can be obtained by increasing the damping on the first segment only. Fig. 3 shows the root loci for the first three modes of the beam under the effect of non-proportional internal damping. Due to the particular choice of the damped segment lengths, the second mode becomes overdamped at high values of c_1, while the first and third mode curves never reach the real axis but tend respectively to $s_1^{(\mathrm{lim})}$ and $s_3^{(\mathrm{lim})}$ for $c_1 \to \infty$.

The asymptotic behaviour of the root loci of the first and third mode can be explained considering that as $c_1 \to \infty$, the clamped–free beam under analysis tends to transform into a clamped–free beam of total length l_2 as shown for a similar example in Ref. [11], where the same clamped–free beam with a different damping variation gives completely different root loci.

As a consequence, the first mode of the initial configuration tends to the first mode of the limit constraint set-up (see Fig. 4), the second mode vanishes and the third mode of the initial configuration tends to the second mode of the limit constraint set-up.

In order to verify the accuracy of the proposed method, the values $s_1^{(\mathrm{lim})}$ and $s_3^{(\mathrm{lim})}$ have been successfully compared with the first two eigenvalues of the clamped–free beam of total length l_2 (with internal damping c_2) obtained by the proposed method.

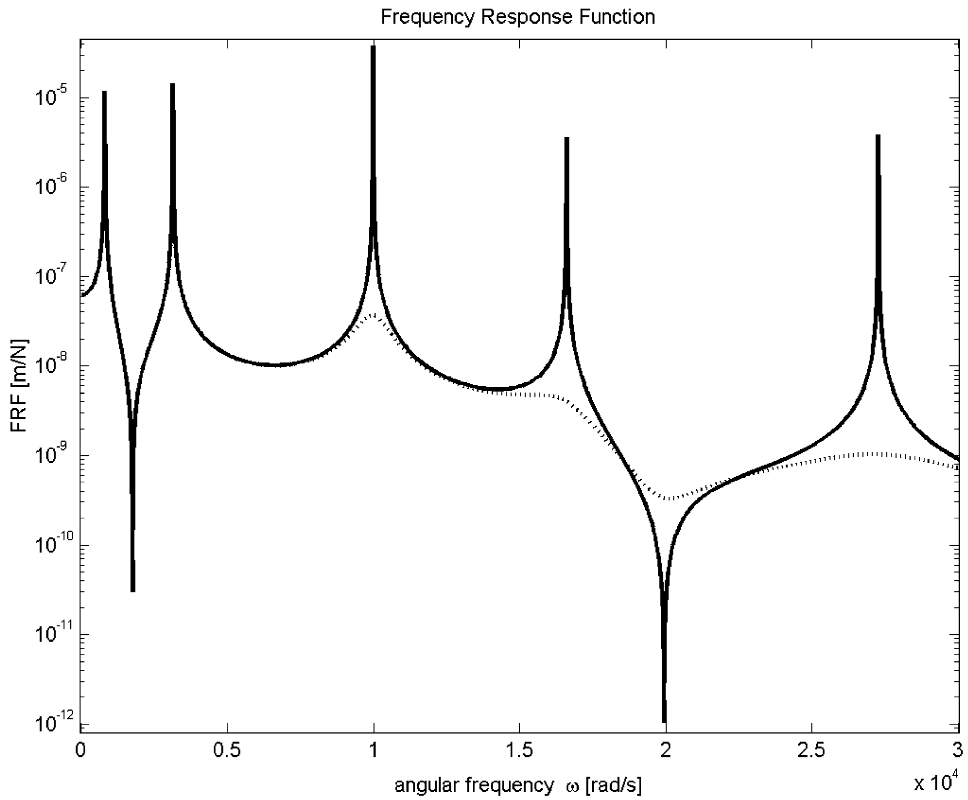

Fig. 5. Receptance modulus $\left|H_{x,x_f}(\omega)\right|$ of example 2 with $x = l_1/2$ (measurement point) and $x_f = l_1 + l_2/2$ (forced point). Five modes are considered for both the undamped system (solid line) and internally damped system (dotted line).

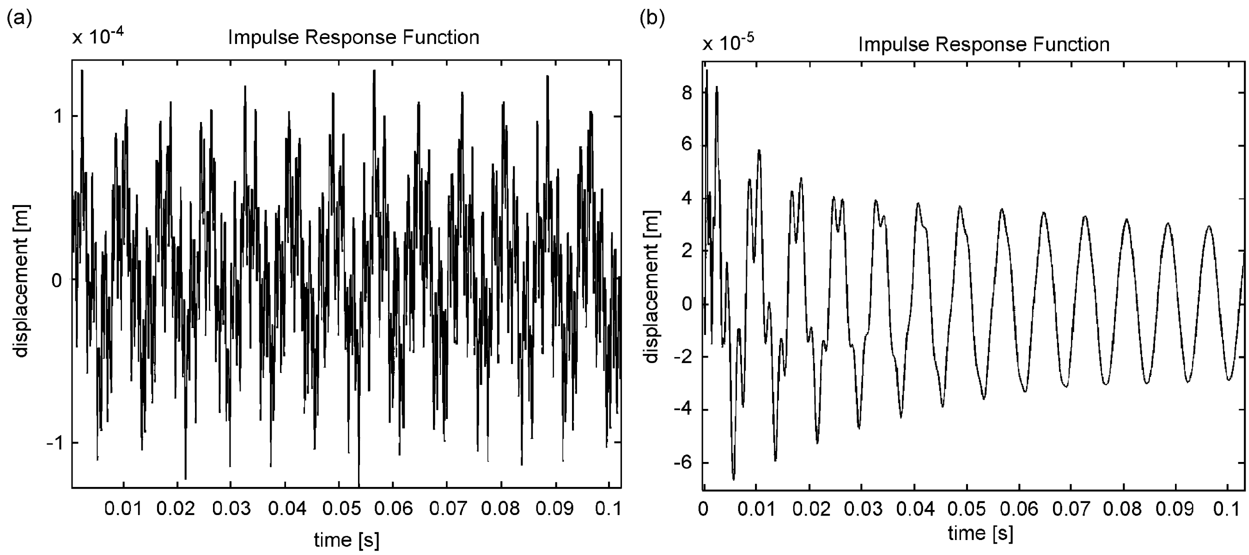

Fig. 6. Impulse response function $\left|h_{x,x_f}(t)\right|$ of example 2 with $x = l_1/2$ (measurement point) and $x_f = l_1 + l_2/2$ (impulse acting point). Five modes are considered in the case of undamped system (a) and internally damped system (b).

Fig. 5 shows the FRFs corresponding to a displacement evaluated at a coordinate $x = l_1/2$ due to a single harmonic force acting at a coordinate $x = l_1 + (l_2/2)$, for the undamped case with solid line, and internal damping case (with system properties reported in Table 5) with dotted line.

The receptance modulus $\left|H_{x,x_f}(\omega)\right|$ in Fig. 5 is obtained by the modal approach using the first five modes. It should be noted that in the undamped case the comparison (not reported here because the FRF curves are superimposed) with a 200 FEM is very good, with the errors on the frequencies of the first five modes within 0.3%. In the internal damping case no comparison was possible, since the adopted FEM is based on a commercial software and is unable to deal with any generalized internal damping model.

In Fig. 6, two IRFs are depicted for the system with the same response and force points as in the previous FRF (first five modes). Fig. 6a shows the response of the undamped system and the contribution of all modes is clear. In Fig. 6b the damped case is considered: by comparing the modal damping ratios in Table 6 it is evident that the first mode is characterized by a lower modal damping than the others. This explains why the contribution of the other modes in the time domain becomes quickly negligible.

5. Conclusions

In this paper an analytical method was developed for the analysis of a class of vibrating continuous systems with generalized viscous damping distributions, and it was applied to the case of Timoshenko beams.

It is based on the modal approach and takes advantage from the orthogonality properties of the eigenfunctions, which were demonstrated for vibrating continuous systems whose equations of motion are characterized by self-adjoint differential operators.

The eigenproblem for systems partitioned in homogeneous substructures was solved by coupling a state-space representation with a transfer matrix technique, carrying high computational efficiency, due in part to the invariance of the dimensions of the matrices involved in the numerical procedures with respect to the number of homogeneous substructures. The numerical results were then successfully validated by means of an FE procedure.

The presented analytical tools enable a complete time and frequency domain study of the effects of generalized damping distributions on stepped Timoshenko beams, and they may be useful to test the accuracy of approximate methods.

Possible applications could regard the analysis and passive control of vibrating elements consisting of non-homogeneous bars, shafts, beams or more complicated systems, such as for example ducts or pipelines, in which the 'proportional' damping assumption generally is not valid to describe the dynamics with sufficient accuracy.

Future work will deal with applications including more complicated damping laws, even involving fractional derivatives and possibly the effects of random or/and moving loads.

Appendix A

The first step to find the expressions of the modal parameter Eqs. (28) consists in writing the operator $\mathbf{A}[\cdot]$ for the Timoshenko beam according to the Voigt internal damping model, using Eqs. (9), (12) and (13):

$$\mathbf{A}[\cdot] = \begin{bmatrix} c\dfrac{\kappa GA}{E} - \dfrac{\partial}{\partial x} cI \dfrac{\partial}{\partial x}(\cdot) & -c\dfrac{\kappa GA}{E}\dfrac{\partial}{\partial x}(\cdot) & \rho I & 0 \\ \dfrac{\partial}{\partial x} c\dfrac{\kappa GA}{E}(\cdot) & -\dfrac{\partial}{\partial x} c\dfrac{\kappa GA}{E}\dfrac{\partial}{\partial x}(\cdot) & 0 & \rho A \\ \rho I & 0 & 0 & 0 \\ 0 & \rho A & 0 & 0 \end{bmatrix} \quad \mathbf{z} = \begin{bmatrix} \varphi \\ \phi \\ s\varphi \\ s\phi \end{bmatrix}. \tag{A1}$$

The resulting inner product over D is:

$$(\mathbf{z}, \mathbf{A}[\mathbf{z}]) = \int_D \left\{ 2(\rho A s\phi^2 + \rho I s\varphi^2) + \frac{c\kappa GA}{E}\,\varphi(\varphi - \phi^{\mathrm{I}}) + \phi\,\frac{\partial}{\partial x}\left[\frac{c\kappa GA}{E}(\varphi - \phi^{\mathrm{I}})\right] - \varphi\,\frac{\partial}{\partial x}(cI\varphi^{\mathrm{I}}) \right\} \mathrm{d}x. \tag{A2}$$

Integrating Eq. (A2) by parts yields:

$$(\mathbf{z},\mathbf{A}[\mathbf{z}]) = 2\int_D (\rho A s\phi^2 + \rho I s\varphi^2)\,\mathrm{d}x + \int_D \frac{c}{E}[\kappa GA(\varphi - \phi^{\mathrm{I}})^2 + EI(\varphi^{\mathrm{I}})^2]\,\mathrm{d}x + \left|\frac{c}{E}[\kappa GA\phi(\varphi - \phi^{\mathrm{I}}) - EI\varphi\varphi^{\mathrm{I}}]\right|_D. \tag{A3}$$

In the case of clamped, pinned, free or sliding boundary conditions, as defined in Eq. (11), Eq. (A3) can be simplified since the last term on the right-hand side vanishes.

The expression of the modal parameter (**z**,**B**[**z**]) can be found taking Eq. (4), pre-multiplying it by $\mathbf{z}^{\mathrm{T}}$ and then integrating it over D, giving:

$$s(\mathbf{z},\mathbf{A}[\mathbf{z}]) + (\mathbf{z},\mathbf{B}[\mathbf{z}]) = 0 \Rightarrow (\mathbf{z},\mathbf{B}[\mathbf{z}]) = -s(\mathbf{z},\mathbf{A}[\mathbf{z}]). \tag{A4}$$

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