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Short Communication

# A bounded stochastic optimal semi-active control

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**Abstract**

A bounded stochastic optimal semi-active control strategy for magneto-rheological/electro-rheological (MR/ER) dampers is developed from the previously proposed strategy by considering the constraint of semi-active control forces. The control force is separated into a semi-active part and passive part incorporated in the uncontrolled system. The control system is converted into an averaged control system for energy processes by using the stochastic averaging method. The bounded and dynamic constraint of semi-active control forces and a performance index are formulated to constitute a bounded stochastic optimal semi-active control problem. Then the dynamical programming equation is established by applying the stochastic dynamical programming principle to the control problem. The bounded optimal semi-active control forces are obtained from solving this equation based on the variation method. A bounded optimal active control law given is implementable by MR/ER dampers. Finally, an example of controlled and stochastically excited nonlinear system is studied to illustrate the efficiency and chattering attenuation of the developed bounded control.


## 1. Introduction

The vibration control of structural systems using semi-active smart devices has been an active research subject in recent years [1]. Magneto-rheological (MR) and electro-rheological (ER) fluid dampers as semi-active smart control devices have attractive features such as mechanical simplicity, reliability, especially small power source requirement and the capacity to provide large controllable damping forces. Extensive theoretical and experimental researches have been done on the dynamic behavior and potential control application of MR/ER dampers. The control efficacy of an MR/ER damper depends much on the control strategy used. Several semi-active control strategies have been proposed and some of them have been compared with each other through a numerical study [2]. The clipping treatment is incorporated in those control strategies to ensure the commanded control forces implementable, which could reduce the control efficacy. The dynamic loading such as wind and earthquake ground motion acting on structural systems is random in nature. A semi-active stochastic optimal control strategy for MR/ER dampers without clipping has been proposed recently [3,4]. In those studies, control devices are assumed to have sufficient capacity of large control forces. However, the control force produced by a control device, for example, MR/ER damper is always bounded. The bounded

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implementation of an unbounded optimal control force will reduce the control efficiency due to the much energy consumption. The stochastic optimal bounded active controls have been proposed [5,6], which belong to the catalog of bang–bang control. The bang–bang control force jumps frequently between maximum and minimum values so that the control efficiency is not high. The discontinuous control force with high-frequency commutation can cause strong chattering and then deteriorate the control performance [7].

In the present communication, a bounded stochastic optimal semi-active control strategy is developed from the previous work by considering the bounded and dynamic constraint of semi-active control forces. The bounded optimal semi-active control law is determined and applied to a nonlinear system to show the control efficacy and chattering attenuation.

## 2. Bounded optimal semi-active control problem

Consider a semi-actively controlled, stochastically excited and dissipated Hamiltonian system, as many engineering structures are modeled in vibration control study. The differential equations of motion of the system is

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial H}{\partial Q_i} - c_{ij}\frac{\partial H}{\partial P_j} + f_{ik}W_k(t) + b_{ir}u_r,$$

$$i,j = 1,2,\ldots,n, \quad k = 1,2,\ldots,m, \quad r = 1,2,\ldots,n_u, \tag{1}$$

where $Q_i$ and $P_i$ are generalized displacement and momentum, respectively, $H = H(\mathbf{Q}, \mathbf{P})$ is the Hamiltonian generally representing total energy of the system, $c_{ij} = c_{ij}(\mathbf{Q}, \mathbf{P})$ is damping coefficient, $f_{ik}(\mathbf{Q}, \mathbf{P})$ is excitation amplitude, $W_k(t)$ $(k = 1, 2,\ldots,m)$ are random processes with zero mean and correlation function $R_{kk'}(t)$, $\mathbf{Q} = [Q_1, Q_2,\ldots,Q_n]^{\mathrm{T}}$, $\mathbf{P} = [P_1, P_2,\ldots,P_n]^{\mathrm{T}}$, $u_r$ represents the semi-active control force, and $b_{ir}$ is control placement coefficient. The control force produced by an MR/ER damper is bounded so that $|u_r| \leqslant u'_{br}$ in terms of symmetric bounded force, where $u'_{br}$ is positive constant.

The semi-active control force $u_r$ can be separated into passive and semi-active components as $u_r = u_{pr} + u_{sr}$. The passive control force component $u_{pr}$ is combined with the uncontrolled system to form a passive control system. The stochastic averaging method [8] can be first applied to the system to yield Itô stochastic differential equations. For instance, in the case of integrable Hamiltonian system, the averaged Itô equations are derived as follows [3]:

$$\mathrm{d}H_i = \left[m_i(H) + \left\langle \frac{\partial H_i}{\partial P_j} b_{jr}u_{sr} \right\rangle\right]\mathrm{d}t + \sigma_{ik}(H)\,\mathrm{d}B_k(t),$$

$$i,j = 1,2,\ldots,n, \quad k = 1,2,\ldots,m, \quad r = 1,2,\ldots,n_u, \tag{2}$$

where $H_i$ $(i = 1, 2,\ldots,n)$ are independent integrals of motion generally representing modal vibration energies, $\mathbf{H} = [H_1, H_2,\ldots,H_n]^{\mathrm{T}}$, $\langle\ \rangle$ denotes the averaging operation, $B_k(t)$ $(k = 1, 2,\ldots,m)$ are independent unit Wiener processes, $m_i(\mathbf{H})$ and $\sigma_{ik}(\mathbf{H})$ are drift and diffusion coefficients, respectively. The semi-active control force is constrained, for instance, by the MR/ER damper as

$$|u_{sr}| \leqslant u_{br},$$

$$u_{sr} = -F_r\,\mathrm{sgn}\left(b_{ir}\frac{\partial H}{\partial P_i}\right), \quad F_r \geqslant 0, \tag{3}$$

according to the Bingham model, where $u_{br}$ is the control force bound and $F_r$ is the damper yielding force dependent on applied voltage. The control system (1) is converted into averaged control system (2) so that the response control of (1) can be performed by the energy control of (2) and the dimension of control system is reduced from $2n$ to $n$.

The stochastic optimal control of system (2) with constraint (3) is to minimize a performance index such as

$$J = \lim_{t_f \to \infty} \frac{1}{t_f}\int_0^{t_f} L(\mathbf{H}(t), \mathbf{u}_s(t))\,\mathrm{d}t, \tag{4}$$

in the case of infinite time-interval ergodic control, where $L(\mathbf{H}, \mathbf{u}_s)$ is a continuous differential convex function and $\mathbf{u}_s = [u_{s1}, u_{s2}, \ldots, u_{sn_u}]^{\mathrm{T}}$. System (2), performance index (4) and constraint (3) constitute a bounded stochastic optimal semi-active control problem.

## 3. Bounded optimal semi-active control law

By applying the stochastic dynamical programming principle [9] to system (2) and performance index (4), the following dynamical programming equation is derived:

$$\lambda = \min_{\mathbf{u}_s \in U} \left\{ L(\mathbf{H}, \mathbf{u}_s) + \left[ m_i(\mathbf{H}) + \left\langle b_{jr} u_{sr} \frac{\partial H_i}{\partial P_j} \right\rangle \right] \frac{\partial V}{\partial H_i} + \frac{1}{2} \sigma_{ik}(\mathbf{H}) \sigma_{jk}(\mathbf{H}) \frac{\partial^2 V}{\partial H_i \partial H_j} \right\}, \tag{5}$$

where $V$ is called the value function, $\lambda$ the constant and $U$ the domain of semi-active control force $u_{sr}$. Based on the variation method, minimizing the right-hand side of Eq. (5) yields

$$\frac{\partial}{\partial u_{r_1}} \left[ L(\mathbf{H}, \mathbf{u}_s) + \left\langle b_{jr} u_{sr} \frac{\partial H_i}{\partial P_j} \right\rangle \frac{\partial V}{\partial H_i} \right] = 0, \quad \delta u_{r_1} \neq 0,$$

$$r, r_1 = 1, 2, \ldots, n_u, \quad i, j = 1, 2, \ldots, n, \tag{6}$$

while noting constraint (3). The bounded optimal semi-active control law is determined as follows: when $|u_{sr}| < u_{br}$ and $u_{sr}\,\mathrm{sgn}(b_{ir}\partial H/\partial P_i) < 0$, $u_{sr}$ is obtained by solving Eq. (6); when $|u_{sr}| \geqslant u_{br}$ and $u_{sr}\mathrm{sgn}(b_{ir}\partial H/\partial P_i) < 0$, $|u_{sr}| = u_{br}$; when $u_{sr}\,\mathrm{sgn}(b_{ir}\partial H/\partial P_i) \geqslant 0$, $u_{sr} = 0$. Obviously, the longer the time for $|u_{sr}| \leqslant u_{br}$ and $u_{sr}\,\mathrm{sgn}(b_{ir}\partial H/\partial P_i) \leqslant 0$ is, the better the control effectiveness using MR/ER dampers is.

Let $L(\mathbf{H}, \mathbf{u}_s) = g(\mathbf{H}) + \langle \mathbf{u}_s^{\mathrm{T}} \mathbf{R} \mathbf{u}_s \rangle$, in which $g(\mathbf{H}) \geqslant 0$ and $\mathbf{R}$ is a positive-definite symmetric matrix. The bounded optimal semi-active control forces obtained are

$$u_{sr}^* = \begin{cases} -F_r, & |F_r| < u_{br}, F_r\,\mathrm{sgn}(b_{ir}\dot{Q}_i) > 0, \\ -u_{br}\,\mathrm{sgn}(F_r), & |F_r| \geqslant u_{br}, F_r\,\mathrm{sgn}(b_{ir}\dot{Q}_i) > 0, \\ 0, & F_r\,\mathrm{sgn}(b_{ir}\dot{Q}_i) \leqslant 0, \end{cases}$$

$$F_r = \frac{1}{2} R_{rr_1}^{-1} b_{jr_1} \frac{\partial H_i}{\partial P_j} \frac{\partial V}{\partial H_i}, \quad r = 1, 2, \ldots, n_u. \tag{7}$$

In the case of $H_i = H_i(Q_i, P_i)$ $(i = 1, 2, \ldots, n)$, by taking $\mathbf{R}$ as a diagonal matrix and $g(\mathbf{H})$ such that $\partial V/\partial H_1 = \partial V/\partial H_2 = \cdots = \partial V/\partial H_n \geqslant 0$, there exists

$$F_r\,\mathrm{sgn}\left( b_{ir} \frac{\partial H}{\partial P_i} \right) = \frac{1}{2R_{rr}} b_{jr} \dot{Q}_j \frac{\partial V}{\partial H_j} \mathrm{sgn}(b_{ir}\dot{Q}_i) = \frac{1}{2R_{rr}} \frac{\partial V}{\partial H_1} |b_{ir}\dot{Q}_i| \geqslant 0, \tag{8}$$

which implies that the second equation of constraint (3) holds all the time. Thus, semi-active MR/ER dampers can perform the active optimal control. The bounded optimal semi-active control forces (7) become

$$u_{sr}^* = \begin{cases} -F_r, & |F_r| < u_{br}, \\ -u_{br}\,\mathrm{sgn}(F_r), & |F_r| \geqslant u_{br}, \end{cases}$$

$$F_r = \frac{1}{2R_{rr}} \frac{\partial V}{\partial H_1} b_{ir} \dot{Q}_i. \tag{9}$$

The value function $V$ can be obtained by substituting the bounded optimal semi-active control forces (7) or (9) into dynamical programming Eq. (5) and solving this equation.

## 4. Bounded stochastic optimal semi-active control of a nonlinear system

To illustrate the application and efficacy of the bounded stochastic optimal semi-active control strategy, consider the following controlled and stochastically excited nonlinear system, which can model many structural vibrations such as the large-amplitude vibration of an upside-down pendulum subjected to horizontal support motion:

$$\ddot{X} + c_0\dot{X} + aX + bX^3 = W(t) + u, \tag{10}$$

where $a$, $b$ and $c_0$ are the constants, $W(t)$ the Gaussian white noise with intensity $2D$ and $u$ the control force produced by an MR/ER damper. The passive control force component of the damper, $u_p = c_1\dot{X}$, is combined with the damping force of the uncontrolled system, $c_0\dot{X}$, to form the damping force of passively controlled system, $c\dot{X}$. Let $X = Q$ and $\dot{X} = P$. Eq. (10) is rewritten as

$$\dot{Q} = P, \quad \dot{P} = -aQ - bQ^3 - cP + u_s + W(t). \tag{11}$$

The semi-active control force constraint is

$$|u_s| \leqslant u_b; \quad u_s = -F\,\mathrm{sgn}(\dot{Q}), \quad F \geqslant 0, \tag{12}$$

where $u_b$ is the semi-active control force bound and $F$ the damper yielding force. Applying the stochastic averaging method to system (11) yields the averaged Itô equation

$$\mathrm{d}H = \left[m(H) + \left\langle \frac{\partial H}{\partial P} u_s \right\rangle\right]\mathrm{d}t + \sigma(H)\,\mathrm{d}B(t), \tag{13}$$

where $B(t)$ is the unit Wiener process

$$\begin{aligned} H &= \frac{1}{2}\left(P^2 + aQ^2 + \frac{1}{2}bQ^4\right), \quad m(H) = D - cG(H), \quad \sigma^2(H) = 2DG(H), \\ G(H) &= \frac{2}{T(H)}\int_{-A}^{A}\sqrt{2H - aq^2 - bq^4/2}\,\mathrm{d}q, \\ T(H) &= 2\int_{-A}^{A}\frac{\mathrm{d}q}{\sqrt{2H - aq^2 - bq^4/2}}, \\ A &= \sqrt{[(a^2 + 4bH)^{1/2} - a]/b}. \end{aligned} \tag{14}$$

For the infinite time-interval ergodic control, take the performance index (4) with

$$L(H, u_s) = g(H) + \langle Ru_s^2 \rangle, \quad g(H) = s_0 + s_1H + s_2H^2 + s_3H^3, \tag{15}$$

where $R > 0$, $s_0$, $s_1$, $s_2$ and $s_3$ are constants. Then the bounded optimal semi-active control force (7) is reduced to

$$u_s^* = \begin{cases} -\dfrac{1}{2R}\dfrac{\mathrm{d}V}{\mathrm{d}H}\dot{X}, & \left|\dfrac{1}{2R}\dfrac{\mathrm{d}V}{\mathrm{d}H}\dot{X}\right| < u_b, \quad \dfrac{\mathrm{d}V}{\mathrm{d}H} > 0, \\ -u_b\,\mathrm{sgn}(\dot{X}), & \left|\dfrac{1}{2R}\dfrac{\mathrm{d}V}{\mathrm{d}H}\dot{X}\right| \geqslant u_b, \quad \dfrac{\mathrm{d}V}{\mathrm{d}H} > 0, \\ 0, & \dfrac{\mathrm{d}V}{\mathrm{d}H} \leqslant 0. \end{cases} \tag{16}$$

Note that $G(H)$ in Eq. (14) is the rapidly increasing function of $H$. It is always possible to select $s_0$, $s_1$, $s_2$, $s_3$, $R$ and $\lambda$ so that $\mathrm{d}V/\mathrm{d}H \geqslant 0$. In this case, the bounded optimal semi-active control force (16) is simplified into

$$u_s^* = \begin{cases} -\dfrac{1}{2R}\dfrac{\mathrm{d}V}{\mathrm{d}H}\dot{X}, & \left|\dfrac{1}{2R}\dfrac{\mathrm{d}V}{\mathrm{d}H}\dot{X}\right| < u_b, \\ -u_b\,\mathrm{sgn}(\dot{X}), & \left|\dfrac{1}{2R}\dfrac{\mathrm{d}V}{\mathrm{d}H}\dot{X}\right| \geqslant u_b. \end{cases} \tag{17}$$

The $\mathrm{d}V/\mathrm{d}H$ is obtained by solving the following equation:

$$\frac{1}{2}\sigma^2(H)\frac{\mathrm{d}^2V}{\mathrm{d}H^2} + m(H)\frac{\mathrm{d}V}{\mathrm{d}H} + m_u(H) + g(H) = \lambda, \tag{18}$$

where

$$m_u(H) = Ru_b^2\frac{T_1(H)}{T(H)} - 4u_b\frac{x_{cr}(H)}{T(H)}\frac{\mathrm{d}V}{\mathrm{d}H} - \frac{1}{4R}[G(H) - G_1(H)]\left(\frac{\mathrm{d}V}{\mathrm{d}H}\right)^2,$$

$$\begin{aligned} T_1(H) &= 2\int_{-x_{cr}}^{x_{cr}} \frac{\mathrm{d}q}{\sqrt{2H - aq^2 - bq^4/2}}, \\ G_1(H) &= \frac{2}{T(H)}\int_{-x_{cr}}^{x_{cr}} \sqrt{2H - aq^2 - bq^4/2}\,\mathrm{d}q, \\ x_{cr} &= \sqrt{[(a^2 + 2bH(2H - \dot{x}_{cr}^2))^{1/2} - a]/b}, \\ \lambda &= s_0 + D\left.\frac{\mathrm{d}V}{\mathrm{d}H}\right|_{H=0}. \end{aligned} \tag{19}$$

To evaluate the efficacy of the bounded stochastic optimal semi-active control (17), substituting $u_s^*$ (17) into Eq. (13) yields

$$\mathrm{d}H = \bar{m}(H)\,\mathrm{d}t + \sigma(H)\,\mathrm{d}B(t),$$

$$\bar{m}(H) = m(H) - 4u_b\frac{x_{cr}(H)}{T(H)} - \frac{1}{2R}[G(H) - G_1(H)]\frac{\mathrm{d}V}{\mathrm{d}H}. \tag{20}$$

A stationary probability density solution to the Fokker–Planck–Kolmogorov equation associated with Itô equation (20) can be expressed as

$$p_s(H) = C_s \exp\left\{-\int_0^H \frac{-2\bar{m}(y) + \mathrm{d}\sigma^2(y)/\mathrm{d}y}{\sigma^2(y)}\,\mathrm{d}y\right\}, \tag{21}$$

where $C_s$ is a normalization constant. The response of the semi-actively controlled system can be predicted by using Eq. (21). The mean system energy and the mean-square bounded optimal semi-active control force are, respectively,

$$E[H_s] = \int_0^\infty Hp_s(H)\,\mathrm{d}H, \tag{22}$$

$$E[u_s^{*2}] = \int_0^\infty p_s(H)\mathrm{d}H\left\{u_b^2\frac{T_1(H)}{T(H)} + \frac{1}{4R^2}[G(H) - G_1(H)]\left(\frac{\mathrm{d}V}{\mathrm{d}H}\right)^2\right\}, \tag{23}$$

where $E[\cdot]$ denotes the expectation operation. The mean energy of the corresponding passively controlled system, $E[H_p]$, can be obtained in the same way by eliminating the $u_s^*$-caused terms. Then the control effectiveness and efficiency of the bounded stochastic optimal semi-active control strategy can be measured by the following performance criteria:

$$K_s = \frac{E[H_p] - E[H_s]}{E[H_p]} \times 100\%, \quad \mu_s = \frac{\mathrm{K}_s}{E[u_s^{*2}]/(2D)}. \tag{24}$$

The criterion $K_s$ represents the percentage reduction in mean system energies of the bounded-optimal-semi-actively controlled system to passively controlled system. The criterion $\mu_s$ represents the relative reduction per unit of the normalized mean-square bounded optimal semi-active control force. The higher $K_s$ and $\mu_s$ indicate, respectively, the better control effectiveness and control efficiency.

Since the comparison between the stochastic optimal semi-active control strategy and LQG control strategy has been studied in Ref. [3], the present study focuses mainly on the developed bounded stochastic optimal semi-active control strategy affected by the control force bound and compared with the bang–bang semi-active control strategy. Numerical results are obtained for system (10) with the following parameter values: $a = 1$, $b = 0.2$, $c = 0.2$, $D = 0.3$, $u_b = 1$, $R = 0.6$, $s_1 = s_3 = 0$, $s_2 = 1.5$, $\mathrm{d}V(0)/\mathrm{d}H = 2.5$ unless otherwise mentioned. Figs. 1, 3 and 5 show that the control effectiveness of the bounded stochastic optimal semi-active control strategy ($K_s$) and bang–bang semi-active control strategy ($K_b$) for various control force bounds ($u_b$), excitation intensities ($D$) and nonlinear stiffness coefficients ($b$), respectively. The effectiveness of the bounded stochastic optimal semi-active control is close to that of the bang–bang semi-active control. However,

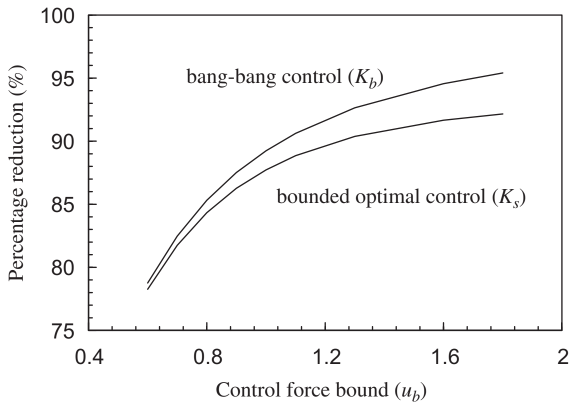


Fig. 1. Control effectiveness versus control force bound.

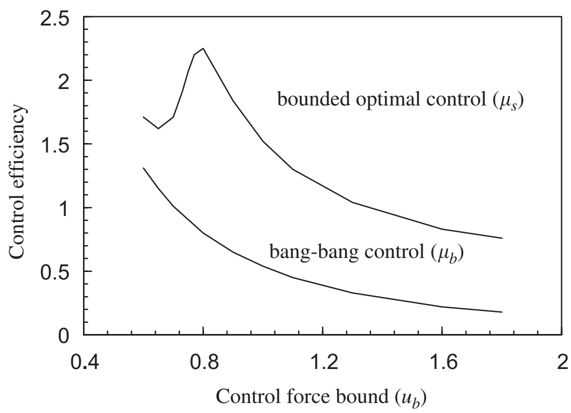


Fig. 2. Control efficiency versus control force bound.

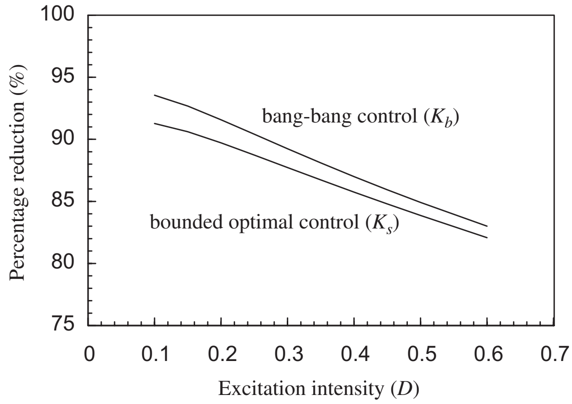


Fig. 3. Control effectiveness versus excitation intensity.

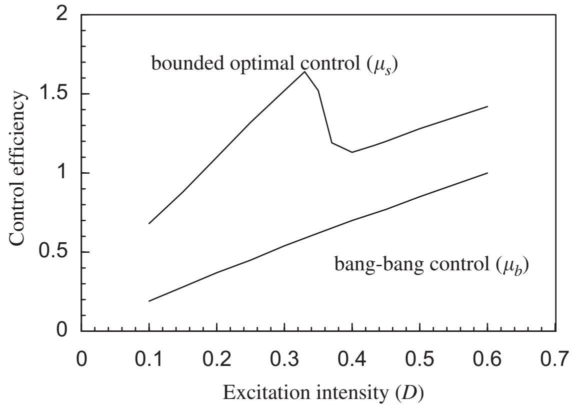


Fig. 4. Control efficiency versus excitation intensity.

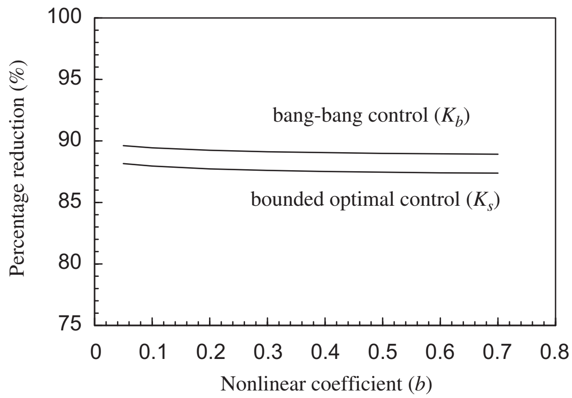


Fig. 5. Control effectiveness versus nonlinear coefficient.

as shown in Figs. 2, 4 and 6, the control efficiency of the bounded stochastic optimal semi-active control strategy ($\mu_s$) is much higher than that of the bang–bang semi-active control strategy ($\mu_b$) for various control force bounds, excitation intensities and nonlinear stiffness coefficients. Also, the maximum control efficiency of the bounded optimal semi-active control strategy can be observed for a certain control force bound. Thus,

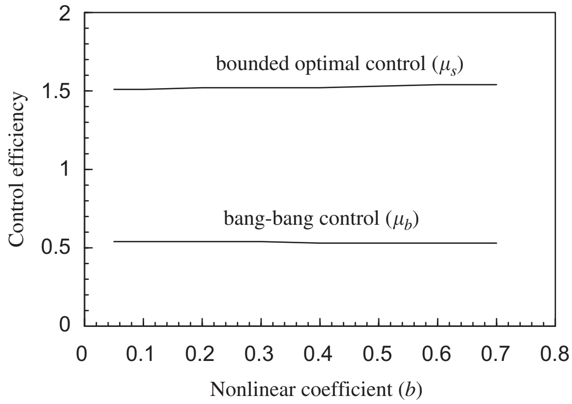


Fig. 6. Control efficiency versus nonlinear coefficient.

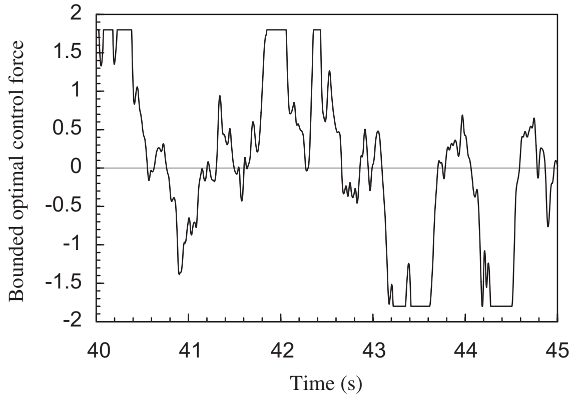


Fig. 7. A bounded stochastic optimal semi-active control force.

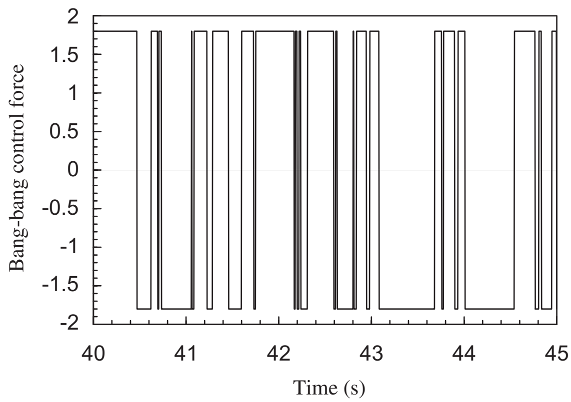


Fig. 8. A bang–bang semi-active control force.

the bounded stochastic optimal semi-active control can achieve the control effectiveness close to the bang–bang semi-active control under less energy consumption due to control force. Samples of the bounded optimal semi-active control force ($u_s^*$) and bang–bang semi-active control force ($u_s^b$) in a certain time interval are given in Figs. 7 and 8, respectively. It is seen that the bang–bang semi-active control force always jumps between maximum and minimum values while the bounded optimal semi-active control force does not so.

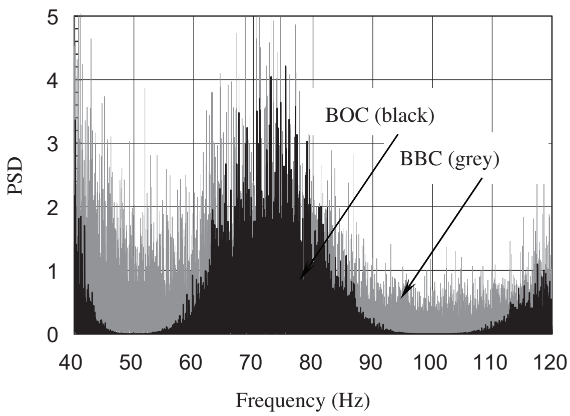


Fig. 9. Power spectral density (PSD) of acceleration response of the semi-actively bang–bang controlled (BBC) and bounded-optimal controlled (BOC) systems.

In this sense, the bounded optimal semi-active control force is more smooth than the bang–bang semi-active control force. Fig. 9 illustrates the power spectral densities of acceleration response under the bounded stochastic optimal semi-active control and bang–bang semi-active control. The high-frequency components of the acceleration response of the bounded-optimal-semi-actively controlled system are less than those of the bang–bang-semi-actively controlled system. Thus, the bounded stochastic optimal semi-active control attenuates the chattering by comparing with the bang–bang semi-active control.

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