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Short Communication

Large deflections of a cantilever beam subjected to a follower force

B.S. Shvartsman

International University Audentes, Tondi 55, 11316 Tallinn, Estonia

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Abstract

The large-deflection problem of a non-uniform spring-hinged cantilever beam under a tip-concentrated follower force is considered. The angle of inclination of the force with respect to the deformed axis of the beam remains unchanged during deformation. The mathematical formulation of this problem yields a nonlinear two-point boundary-value problem which is reduced to an initial-value problem by change of variables. The resulting problem can be solved without iterations. It is shown that there exist no critical loads in the Euler sense (divergence) for any flexural-stiffness distribution and angle of inclination of the follower force. The load–displacement characteristics of a uniform cantilever under a follower force normal to the deformed beam axis are presented.

1. Introduction

Stability of structures subjected to follower compressive loads has been treated by many researchers [1,2]. It is well known that the type of instability (flutter or divergence) of beams under follower compressive forces depends on the end elastic restraints and other parameters. A uniform cantilever beam subjected to a tip force that remains tangent to the beam axis during deformation (Beck's problem) is a classical example of the follower force stability problems. In this case, there exist no critical loads in the Euler sense and stability analysis calls for dynamic approach [1–3]. The applicability of static and dynamic stability criteria to uniform and non-uniform cantilever columns under tip-concentrated subtangential follower forces was studied in Refs. [4,5]. Kounadis [6] showed that a spring-hinged uniform cantilever subjected to a follower compressive force applied to its free end can exhibit only flutter instability. It is of interest to assess the validity of this statement for non-uniform cantilevers.

On the other hand, Argyris and Symeonidis [7] performed static geometrically nonlinear analysis of cantilevers subjected to follower loads by the finite-element method and found the critical flutter loads. Rao et al. [8–10] studied in detail large deflections of uniform and non-uniform cantilever beams under tip rotational loads using the elliptic-function solution and the shooting method. In Ref. [11], the bending problem of a flexible cantilever under a distributed follower load was studied by a direct method.

E-mail address: bshwar@hot.ee.

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In the present paper, the large-deflection problem of a non-uniform spring-hinged cantilever beam under a tip-concentrated follower force is considered. The angle of inclination of the force with respect to the deformed axis of the beam is assumed to be constant. The mathematical formulation of this problem yields a nonlinear two-point boundary-value problem, which is reduced to an initial-value problem by change of variables. The advantage of this approach is that the problem can be solved without iterations. Since the solution of the initial-value problem is unique, divergence instability does not occur. Therefore, the elastic cantilever beam in question can lose stability only by flutter. In particular, if the follower load is tangential, the rectilinear shape of the non-uniform cantilever beam is the only possible equilibrium configuration. The load–displacement characteristics of a uniform cantilever beam under a follower tip force normal to the beam axis are given.

2. Formulation of the problem

We consider a rectilinear non-uniform spring-hinged cantilever beam having length L, rotational spring constant c and flexural rigidity $EI(s)$ subjected to a concentrated follower force P applied to its free end (Fig. 1). The angle of inclination of the force with respect to the deformed axis of the beam α is kept constant. The arc length measured from the free end and the slope of the centroidal axis of the beam are denoted by s and $\varphi(s)$, respectively. Using the Euler–Bernoulli law of bending states, we obtain the nonlinear differential equation governing the behavior of the beam

$$(EI\varphi')' + P\sin(\varphi + \alpha - \varphi(0)) = 0 \tag{1}$$

subject to the boundary conditions

$$\varphi'(0) = 0, \quad EI(L)\varphi'(L) + c\varphi(L) = 0. \tag{2}$$

The angle $\alpha = \pi/2$ corresponds to the rotational force acting in the normal direction to the deformed axis of the beam [7–10] and the angle $\alpha = 0$ corresponds to the tangential follower force [1–6]. For $c \to \infty$, the spring-hinged end becomes fixed and the boundary conditions (2) are simplified:

$$\varphi'(0) = 0, \quad \varphi(L) = 0. \tag{3}$$

Once the slope $\varphi(s)$ has been found, the Cartesian coordinates of the beam axis are readily determined from the relations

$$x(s) = \int_s^L \cos\varphi \, d\tilde{s}, \quad y(s) = \int_s^L \sin\varphi \, d\tilde{s}. \tag{4}$$

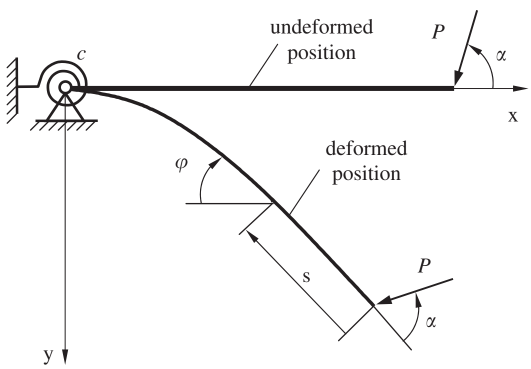

Fig. 1. Cantilever beam under a follower force.

3. Method of solution

Nonlinear two-point boundary-value problems similar to that formulated above are usually solved by iterative methods. For example, according to the shooting method, the nonlinear two-point boundary-value problem (1), (2) is reduced to a set of initial-value problems and the unknown initial value is then determined iteratively [9,10]. It is well known that the convergence of the iterative procedure depends upon how close the initial guess values are to the solution sought for. Moreover, in the general case, similar boundary-value problems admit multiple solutions [12]. It can be shown, however, that the problem formulated above can be solved without iterations.

We introduce the new variable

$$z(s) = \varphi(s) + \alpha - \varphi(0). \tag{5}$$

As a result, the boundary-value problem (1), (2) is reduced to the initial-value problem

$$(EIz')' + P\sin(z) = 0, \tag{6}$$

$$z(0) = \alpha, \quad z'(0) = 0 \tag{7}$$

with the supplementary condition

$$EI(L)z'(L) + c(z(L) - \alpha + \varphi(0)) = 0. \tag{8}$$

It should be noted that in the case of a uniform cantilever ($EI(s) = \text{const}$), Eqs. (6) and (7) describe also the motion of a simple pendulum from the rest position at an angle α (Kirchhoff's kinetic analogy) if the arc length s is interpreted as time and $z(s)$ as the angle that measures the deviation of the pendulum from a vertical line. In particular, Beck's problem ($\alpha = 0$) has the unique solution $z(s) = \varphi(s) \equiv 0$ [3], which corresponds to the equilibrium state of the pendulum at the lowest point. For $\alpha \neq 0$, the maximum deviation of the pendulum $z(L) = -\alpha$ corresponds to the half of the oscillation period of the pendulum. Using formulas (3) and (5), we infer that in this case the maximum tip slope of the rigidly fixed cantilever is given by $\varphi(0) = \alpha - z(L) = 2\alpha$.

Introducing the notation

$$z_1 = z, \quad z_2 = EI(s)z' \tag{9}$$

we reduce problem (6), (7) to the normal system of nonlinear differential equations

$$z_1' = z_2/EI(s), \quad z_2' = -P\sin(z_1), \tag{10}$$

$$z_1(0) = \alpha, \quad z_2(0) = 0. \tag{11}$$

System (10), (11) can be integrated over a given interval $s \subset [0; L]$ by a standard numerical method. The values of $\varphi(s)$ are calculated by the formula

$$\varphi(s) = z_1(s) - z_1(L) - z_2(L)/c, \tag{12}$$

which follows from Eqs. (5), (8), and (9). For the fixed cantilever ($c \to \infty$), this formula becomes

$$\varphi(s) = z_1(s) - z_1(L). \tag{13}$$

Thus, in contrast to the shooting method the problem considered is solved without iterations. For a continuous function $EI(s)$ and any value of c, the solution of problem (6), (7) is unique. Therefore, there exist no points of static instability for any flexural-stiffness distribution along the beam. In particular, if the follower force is tangential ($\alpha = 0$), the straight configuration is the only equilibrium configuration of the beam and there is no critical load in the Euler sense (divergence). It follows that the elastic cantilever beam in question can exhibit only flutter instability. This conclusion generalizes the known results for uniform ($EI(s) = \text{const}$) spring-hinged and fixed cantilevers [1–6].

It is worth noting that the problems of flexible cantilever beams under inclined tip dead forces (conservative problems) admit multiple equilibrium solutions [12].

Table 1
Slope and coordinates of a cantilever loaded by a normal follower force ($\alpha = \pi/2$)

$\tilde{P}$	$\varphi(0)$	$x(0)/L$	$y(0)/L$
2	55.48	0.7674	0.5738
4	101.78	0.3428	0.7862
8	157.94	−0.0739	0.6336
13.75	180.00	0.0000	0.4570
16	177.56	0.0910	0.4212
24	140.04	0.4348	0.1588
36	55.64	0.2855	−0.4546

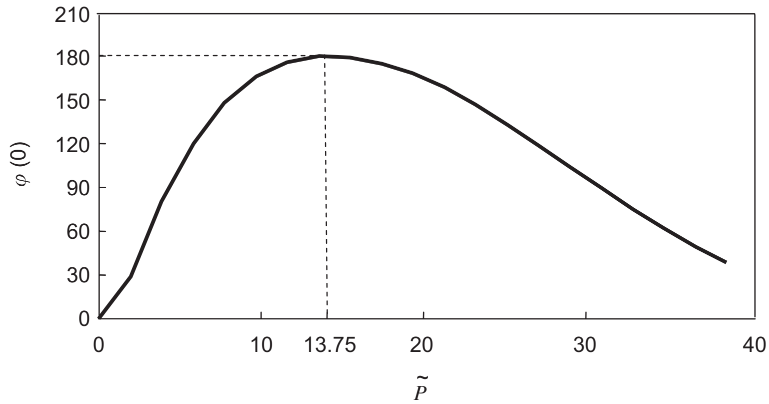

Fig. 2. Tip slope versus normal follower force for a cantilever.

4. Numerical results

Using the method of solution outlined above, we studied the behavior of a uniform cantilever subjected to a tip follower force P acting in the normal direction to the deformed axis of the beam ($\alpha = \pi/2$) [7–9]. Eqs. (6) and (7) were integrated numerically by the fourth-order Runge–Kutta method with a fixed step size equal to $0.05L$, and integrals in Eq. (4) were evaluated numerically using Simpson's rule. The values of the tip coordinates $x(0)$ and $y(0)$ and slope $\varphi(0)$ (in degrees) of the beam are listed in Table 1 for various values of the load parameter $\tilde{P} = PL^2/EI$. The results given in Table 1 were compared with the solutions obtained for the step size equal to $0.1L$. The discrepancy between these solutions was found to be within 0.1%. These results are in a good agreement with the elliptic-function solution given in Ref. [8] and numerical solution obtained by the shooting method in Ref. [9]. Fig. 2 shows the tip slope versus the load parameter $\tilde{P}$ varying from 0 to 38.3 for which, as shown in Ref. [7], flutter occurs. The maximum value $\varphi(0) = 2\alpha = \pi$ is reached for $\tilde{P} = 13.75$. This value corresponds to the half of the oscillation period of the pendulum ($z(L) = -z(0) = -\alpha = -\pi/2$).

Similar approach was performed for various angles of inclination α of the follower force and stiffness distributions $EI(s)$.

5. Conclusions

A direct method for the large-deflection problem of a non-uniform spring-hinged cantilever beam under a tip follower force is proposed. It is shown that, for any flexural-stiffness distribution, the elastic cantilever beam can lose stability only by flutter. The load–displacement characteristics of the beam loaded by a force normal to the beam axis are obtained. The direct numerical method considered is simple, provides high accuracy of calculations, and involves less computational time compared to the shooting method. The direct method can easily be extended to similar problems of curved cantilever beams.

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