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Short Communication

Comparison of some approaches to stability of a sdof system under random parametric excitation

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Abstract

We give a rigorous comparison of the asymptotic stochastic averaging method and the moment Lyapunov exponent approach in assessing the mean-square stability through an example of a single-degree-of-freedom system under small periodic parametric excitation with random phase modulation. The comparison is based on an efficient numerical procedure to determine the regions of mean-square stability.

1. Introduction

Consider the system, governed by the following dimensionless equation:

$$\ddot{y} + 2\beta\dot{y} + \Omega^2[1 + \xi(t)]y = 0, \quad t > 0, \tag{1}$$

where β is a small viscous damping constant, Ω is a natural frequency of the system, and $\xi(t)$ is a random parametric excitation. Such a system is often encountered in mechanics (see, e.g., Refs. [1–3]). The random excitation $\xi(t)$ is of the form

$$\xi(t) = \lambda \sin[\theta + \omega t + \alpha w(t)], \tag{2}$$

where $w(t)$ is a standard Wiener process, θ is a uniformly distributed on $[0, 2\pi]$ random variable independent of $w(t)$ and λ, ω, α are deterministic parameters. Recently, the random excitation (2) with either the sine or cosine function have been intensively used in many investigations (see, e.g., Refs. [2–11]) because it is a realistic model of random fluctuations. For example, systems (1), (2) are used in certain problems of aeroelasticity [12] and in structural dynamical problems with travelling loadings, having certain imperfect spatial periodicity [4]. The stability for systems (1), (2) are a topic of considerable interest. There are different notions of the stability for dynamical systems with random excitations. The mean-square asymptotic stability for system (1) implies that the second moments of the solutions of Eq. (1) tend to zero as $t \to \infty$. This notion plays an important role in applications (see a discussion in Ref. [2]).

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Suppose that the excitation amplitude λ is small compared with unity. Using the asymptotic averaging method in the mean-square stability the following approximate formula for the critical excitation amplitude was obtained in Refs. [4,5]:

$$\lambda_1 = \frac{4\beta}{\Omega}\left(1+\frac{\delta_1^2}{4\beta^2}\right)^{1/2}\left(1+\frac{4\alpha^2(4\beta^2+\beta\alpha^2-\delta_1^2)}{\beta(16\beta^3+4\beta\delta_1^2+4\beta^2\alpha^2+\alpha^2\delta_1^2)}\right)^{1/2}, \tag{3}$$

where $\delta_1 = 2\Omega - \omega$. It means that system (1) is mean-square stable if $\lambda < \lambda_1$ and unstable if $\lambda > \lambda_1$. On the other hand, the following approximate formula for the critical excitation amplitude follows from Eqs. (8), (19), (31), (37) of Ref. [9]:

$$\lambda_2 = \left(\frac{8\beta[(2+\delta_2\omega)^2+\alpha^4\delta_2^2/4][(2-\delta_2\omega)^2+\alpha^4\delta_2^2/4]}{\Omega^4\delta_2^4\alpha^2(4+\delta_2^2\omega^2+\alpha^4\delta_2^2/4)}\right)^{1/2}, \tag{4}$$

where $\delta_2 = 1/(\Omega^2-\beta^2)^{1/2}$.

This formula was obtained by the moment Lyapunov exponent method.

In the note we give a rigorous comparison of these approximations near the principal parametric resonant value $\omega = 2\Omega$ using numerical computations. An efficient numerical method is applied for this purpose which is based on a closure of hierarchy for the second moments. It is shown that formulae (3) and (4) are good approximations in the case of narrow-band excitation (α is small) and wide-band excitation (α is large), respectively.

2. Hierarchy for the second moments

According to Eq. (1) the vector

$$\mathbf{x}(t) := \{(\dot{y}(t))^2, y(t)\dot{y}(t), y^2(t)\}^{\mathrm{T}}$$

satisfies the following equation in R^3:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \xi(t)\mathbf{B}\mathbf{x}, \quad t>0, \tag{5}$$

where

$$\mathbf{A} = \begin{pmatrix} -4\beta & -2\Omega^2 & 0 \\ 1 & -2\beta & -\Omega^2 \\ 0 & 2 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & -2\Omega^2 & 0 \\ 0 & 0 & -\Omega^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The solution of Eq. (5) is a functional of the Wiener process and therefore we shall write $\mathbf{x}(t) = \mathbf{x}(t; w(s))$. Using the Cameron–Martin formula for the density of the Wiener measure under translation [13,14] we deduce that for all nonrandom γ,

$$E[\exp\{\mathrm{i}\gamma w(t)\}\mathbf{x}(t; w(s))] = \exp\left\{-\frac{\gamma^2 t}{2}\right\}E[\mathbf{x}(t; w(s)+\mathrm{i}\gamma s)], \quad \mathrm{i} = \sqrt{-1}. \tag{6}$$

Since

$$\sin(\theta+\omega t+\alpha w(t)) = \frac{\exp\{\mathrm{i}\theta+\mathrm{i}\omega t+\mathrm{i}\alpha w(t)\} - \exp\{-\mathrm{i}\theta-\mathrm{i}\omega t-\mathrm{i}\alpha w(t)\}}{2\mathrm{i}},$$

we use Eq. (6) to obtain

$$\begin{aligned}\frac{\mathrm{d}E[\mathbf{x}(t)]}{\mathrm{d}t} = \mathbf{A}E[\mathbf{x}(t)] &+ \frac{\lambda}{2\mathrm{i}}\exp\left\{\frac{-\alpha^2 t}{2}+\mathrm{i}\omega t\right\}\mathbf{B}E[\mathrm{e}^{\mathrm{i}\theta}\mathbf{x}(t; w(s)+\mathrm{i}\alpha s)] \\ &- \frac{\lambda}{2\mathrm{i}}\exp\left\{\frac{-\alpha^2 t}{2}-\mathrm{i}\omega t\right\}\mathbf{B}E[\mathrm{e}^{-\mathrm{i}\theta}\mathbf{x}(t; w(s)-\mathrm{i}\alpha s)],\end{aligned}$$

$$
\begin{aligned}
&\frac{\mathrm{d}E[\mathrm{e}^{\pm \mathrm{i}k\theta}\mathbf{x}(t;w(s)\pm \mathrm{i}k\alpha s)]}{\mathrm{d}t}\\
&\quad = \mathbf{A}E[\mathrm{e}^{\pm \mathrm{i}k\theta}\mathbf{x}(t;w(s)\pm \mathrm{i}k\alpha s)]\\
&\qquad + \frac{\lambda}{2\mathrm{i}}\mathbf{B}\exp\left\{\frac{-\alpha^2 t}{2}\mp k\alpha^2 t+\mathrm{i}\omega t\right\}E[\mathrm{e}^{\pm \mathrm{i}k\theta+\mathrm{i}\theta}\mathbf{x}(t;w(s)\pm \mathrm{i}k\alpha s+\mathrm{i}\alpha s)]\\
&\qquad - \frac{\lambda}{2\mathrm{i}}\mathbf{B}\exp\left\{\frac{-\alpha^2 t}{2}\pm k\alpha^2 t-\mathrm{i}\omega t\right\}E[\mathrm{e}^{\pm \mathrm{i}k\theta-\mathrm{i}\theta}\mathbf{x}(t;w(s)\pm \mathrm{i}k\alpha s-\mathrm{i}\alpha s)],\quad k=1,2,3,\ldots\ . \qquad (7)
\end{aligned}
$$

Let

$$
\mathbf{u}_k(t):=\frac{1}{(2\mathrm{i})^k}\exp\left\{\mathrm{i}k\omega-\frac{k^2\alpha^2 t}{2}\right\}E[\mathrm{e}^{\mathrm{i}k\theta}\mathbf{x}(t;w(s)+\mathrm{i}k\alpha s)],
$$

$$
\mathbf{v}_k(t):=\frac{1}{(-2\mathrm{i})^k}\exp\left\{-\mathrm{i}k\omega-\frac{k^2\alpha^2 t}{2}\right\}E[\mathrm{e}^{-\mathrm{i}k\theta}\mathbf{x}(t;w(s)-\mathrm{i}k\alpha s)],\quad k=1,2,3,\ldots\ .
$$

Then Eqs. (7) lead to the following infinite hierarchy of linear differential equations for the mean $E[\mathbf{x}(t)]$:

$$
\frac{\mathrm{d}E[\mathbf{x}(t)]}{\mathrm{d}t}=\mathbf{A}E[\mathbf{x}(t)]+\lambda\mathbf{B}(\mathbf{u}_1(t)+\mathbf{v}_1(t)),
$$

$$
\frac{\mathrm{d}\mathbf{u}_1}{\mathrm{d}t}=\left(-\frac{\alpha^2}{2}+\mathrm{i}\omega\right)\mathbf{u}_1+\mathbf{A}\mathbf{u}_1+\lambda\mathbf{B}\mathbf{u}_2+\frac{\lambda}{4}\mathbf{B}E[\mathbf{x}(t)],
$$

$$
\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t}=\left(-\frac{\alpha^2}{2}-\mathrm{i}\omega\right)\mathbf{v}_1+\mathbf{A}\mathbf{v}_1+\lambda\mathbf{B}\mathbf{v}_2+\frac{\lambda}{4}\mathbf{B}E[\mathbf{x}(t)],
$$

$$
\frac{\mathrm{d}\mathbf{u}_k}{\mathrm{d}t}=\left(-\frac{k^2\alpha^2}{2}+\mathrm{i}k\omega\right)\mathbf{u}_k+\mathbf{A}\mathbf{u}_k+\lambda\mathbf{B}\mathbf{u}_{k+1}+\frac{\lambda}{4}\mathbf{B}\mathbf{u}_{k-1},
$$

$$
\frac{\mathrm{d}\mathbf{v}_k}{\mathrm{d}t}=\left(-\frac{k^2\alpha^2}{2}-\mathrm{i}k\omega\right)\mathbf{v}_k+\mathbf{A}\mathbf{v}_k+\lambda\mathbf{B}\mathbf{v}_{k+1}+\frac{\lambda}{4}\mathbf{B}\mathbf{v}_{k-1},\quad k=2,3,\ldots\ ,
$$

$$
E[\mathbf{x}(0)]=\mathbf{x}(0),\quad \mathbf{u}_k(0)=\mathbf{v}_k(0)=\mathbf{0},\quad k=1,2,3,\ldots\ . \qquad (8)
$$

Note that for excitation (2) without phase θ the same argument gives the previous hierarchy but with different initial conditions, namely $\mathbf{u}_k(0)=\mathbf{x}(0)/(2\mathrm{i})^k$, $\mathbf{v}_k(0)=\mathbf{x}(0)/(-2\mathrm{i})^k$, $k=1,2,3,\ldots$. If we have the cosine instead of the sine function in excitation (2), then one can easily convince that we obtain again hierarchy (7), where

$$
\mathbf{u}_k(t):=\frac{1}{2^k}\exp\left\{\mathrm{i}k\omega-\frac{k^2\alpha^2 t}{2}\right\}E[\mathrm{e}^{\mathrm{i}k\theta}\mathbf{x}(t;w(s)+\mathrm{i}k\alpha s)],
$$

$$
\mathbf{v}_k(t):=\frac{1}{2^k}\exp\left\{-\mathrm{i}k\omega-\frac{k^2\alpha^2 t}{2}\right\}E[\mathrm{e}^{-\mathrm{i}k\theta}\mathbf{x}(t;w(s)-\mathrm{i}k\alpha s)],\quad k=1,2,3,\ldots\ .
$$

Similar hierarchy was obtained earlier in Ref. [15], when excitation (2) is some Gaussian random process.

A natural way to close hierarchy (8) consists in neglecting the terms $\mathbf{u}_{n+1}$, $\mathbf{v}_{n+1}$ in the equations for $\mathbf{u}_n$, $\mathbf{v}_n$. Then the index n is called the truncation index. After applying this procedure we obtain the closed system of linear differential equations of first order with constant coefficients. It is well-known that we have the asymptotic stability for the system if and only if the matrix of its coefficients has all eigenvalues with negative real parts. For sufficiently large truncation index, the asymptotic stability or instability of this system determines the mean-square stability or instability for system (1). This fact can be proved in a similar way as in Ref. [11], where the random excitation has form $\xi(t)=\lambda\sin[\alpha w(t)]$.

3. Stability regions

The eigenvalues were computed with Mathematica for the matrix of coefficients of the closed system with truncation index $n = 20$. The mean-square stability curves in (ω, λ)—parameter space are shown in Fig. 1 (the solid lines) for different values of the excitation parameter α in the case $\Omega = 1$, $\beta = 0.005$ near the principal resonance frequency $\omega = 2$. The eigenvalues were computed over a 600×1200 grid of equally spaced points ranging from $1.7 \leqslant \omega \leqslant 2.3$ to $0 \leqslant \lambda \leqslant 1.2$. The accuracy of the computations is high: the differences of the real parts of the eigenvalues obtained for the truncation index $n = 20$ and 50 do not exceed 10^{-7}. This fact can be explained by rapid convergence of the closure procedure for the hierarchy (see Ref. [16]). In Fig. 1 we also show the plots of the critical excitation amplitude obtained from Eq. (3) (the dotted lines) and from Eq. (4) (the dashed lines).

It follows from Fig. 1(a) that for $\alpha = 0.05$ the critical curve obtained from the computations coincides with the critical excitation amplitude curve obtained from Eq. (3) but it is inconsistent with the curve obtained from Eq. (4). We have almost full coincidence in the case $\alpha = 0.5$, (Fig. 1(b)). Further growth of the excitation parameter α leads to the coincidence the computations with Eq. (4), but we have noncoincidence with Eq. (3), (Fig. 1(c), (d)). It is interesting to note that we have the coincidence of the computations with Eq. (4) even in the case of not small amplitude λ (see Fig. 1(d)). The computations and Eqs. (3) and (4) both confirm the stabilization effect near the principal resonance frequency with increasing α. Unfortunately, the effect is observed only near this value. In Fig. 2 we display the stability curves based on the computations with truncation index $n = 40$ which show that the destabilizing effect is seen away from the principal resonance frequency. One can also observe the higher-order resonant tongues if $\alpha = 0.05$. These tongues are not observed from Eqs. (3) and (4) because these equations are obtained with assumption $\lambda \ll 1$.

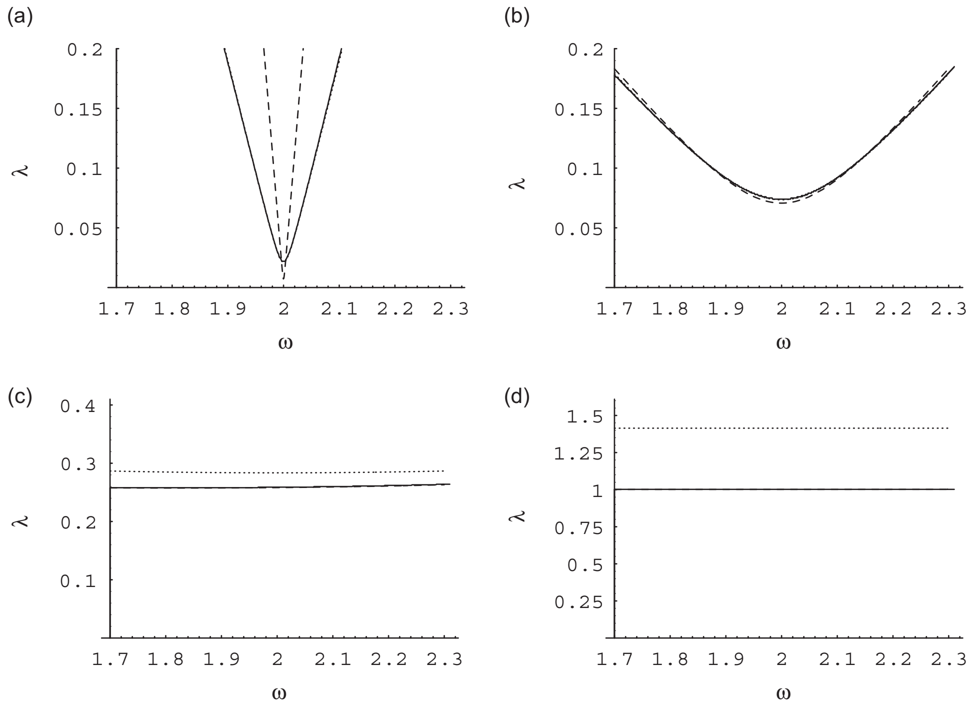

Fig. 1. Mean-square stability curves (the solid lines) to system (1) with excitation (2) near the principal parametric resonance for the values $\Omega = 1$ and $\beta = 0.005$. The dotted and the dashed lines correspond to Eqs. (3) and (4), respectively. In (a) $\alpha = 0.05$; (b) $\alpha = 0.5$; (c) $\alpha = 2$; (d) $\alpha = 10$.

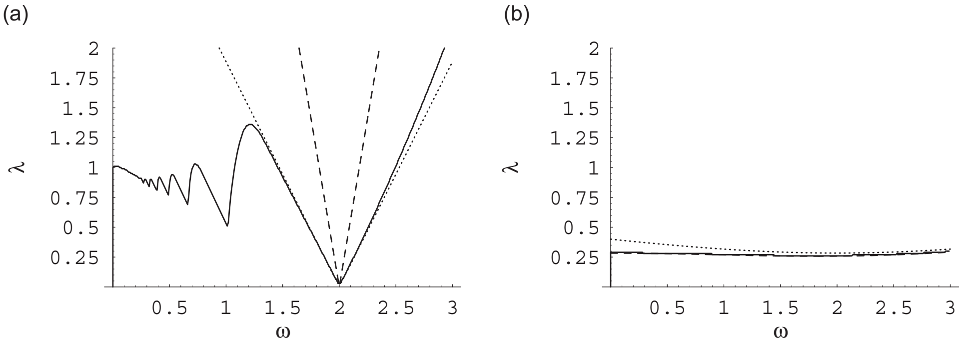

Fig. 2. Mean-square stability curves (the solid lines) to system (1) with excitation (2) for the values $\Omega = 1$, $\beta = 0.005$ and $\omega \in [0, 3]$. The dotted and the dashed lines correspond to Eqs. (3) and (4), respectively. In (a) $\alpha = 0.05$; (b) $\alpha = 2$.

4. Conclusions

In this note, we have investigated the mean-square stability for single-degree-of-freedom system under the periodic parametric excitation with random phase modulation. Two known approximate approaches are compared with precise numerical computations. It is shown that the approach based on Eq. (3) can be considered as a good approximation for the stability curve in the case of narrow-band excitation but the approach based on Eq. (4) gives a good approximation in the case of wide-band excitation.

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