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Short Communication

On the convergence of the Volterra-series representation of the Duffing's oscillators subjected to harmonic excitations

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Abstract

This paper is concerned with the investigation of the convergence issue of the Volterra-series representations of the Duffing's oscillator subjected to harmonic inputs. A simple criterion is proposed to determine the upper limit of the magnitude of the harmonic inputs. A comparison between the new proposed criterion and a criterion suggested by Tomlinson et al. [G.R. Tomlinson, G Manson, G.M. Lee, A simple criterion for establishing an upper limit to the harmonic excitation level of the Duffing oscillator using the Volterra series. *Journal of Sound and Vibration* 190 (1996) 751–762] was carried out. The results show that the new criterion can provide a more accurate prediction about the convergence of the Volterra-series representation of the Duffing's oscillator.

1. Introduction

In engineering many dynamical systems have nonlinear components, which cannot simply be described by a linear model. For example, vibration components with clearances [1] and motion limiting stops [2] or vibration components with fatigue damage [3], which cause abrupt changes in the stiffness and damping coefficients, represent a significant proportion of these systems. To investigate such nonlinear systems, nonlinear oscillators have been widely adopted, among which the Duffing's oscillator is the most well-known one. It has been widely used to model a single-degree-of-freedom system with nonlinear stiffness [4].

The Volterra-series approach [5,6] is a powerful tool for the analysis of nonlinear systems, which extends the familiar concept of the convolution integral for linear systems to a series of multidimensional convolution integrals. The Fourier transforms of the Volterra kernels, called generalised frequency response functions (GFRFs) [7], are an extension of the linear frequency response function (FRF) to the nonlinear case. If a differential equation or difference equation is available for a nonlinear system, the GFRFs can be determined using the algorithm in Refs. [8,9]. Recently, a novel concept known as nonlinear output frequency response functions (NOFRFs) was proposed by the authors [10]. The concept can be considered to be an alternative extension of the classical FRF for linear systems to the nonlinear case. NOFRFs are one-dimensional

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functions of frequency, which allow the analysis of nonlinear systems to be implemented in a manner similar to the analysis of linear systems.

Theoretically, nonlinear systems such as the Duffing's oscillator need to be expressed using an infinite Volterra series, however, in practice a truncated series can be used provided the number of terms included can provide an accurate approximation to the response of the system. Therefore, the Volterra-series representation is required to be convergent, and a divergent Volterra series can not be used to describe the responses of nonlinear systems. Toward the convergence issue of the Volterra-series representation of the Duffing's oscillators subjected to harmonic inputs, a few researchers [4,11,12] have put forward some criterions to determine the upper limit of the magnitude of the harmonic inputs. In the present study, based on the concept of NOFRFs, a new method is proposed to study the convergence issue of the Volterra-series representation of the Duffing's oscillator. A new simple criterion is derived to determine the upper limit of the magnitude of the harmonic inputs, under which the Volterra-series representation is absolutely convergent. A comparison between the new proposed criterion and the criterion suggested in Ref. [4] has been carried out. The results show that the new criterion can provide a more accurate prediction about the convergence of the Volterra-series representation of the Duffing's oscillator.

2. Nonlinear output frequency response functions

The definition of NOFRFs is based on the Volterra-series theory of nonlinear systems. Consider the class of nonlinear systems, which are stable at zero equilibrium and which can be described in the neighbourhood of the equilibrium by the Volterra series

$$y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i)\, \mathrm{d}\tau_i, \tag{1}$$

where $y(t)$ and $u(t)$ are the output and input of the system, $h_n(\tau_1, \ldots, \tau_n)$ is the nth-order Volterra kernel. Lang and Billings [7] derived an expression for the output frequency response of this class of nonlinear systems to a general input. The result is

$$\begin{cases} Y(\mathrm{j}\omega) = \sum\limits_{n=1}^{\infty} Y_n(\mathrm{j}\omega) \quad \text{for } \forall\omega, \\ Y_n(\mathrm{j}\omega) = \frac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1+\cdots+\omega_n=\omega} H_n(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_n) \prod\limits_{i=1}^{n} U(\mathrm{j}\omega_i)\, \mathrm{d}\sigma_{n\omega}. \end{cases} \tag{2}$$

In Eq. (2), $Y(\mathrm{j}\omega)$ is the spectrum of the system output, $Y_n(\mathrm{j}\omega)$ represents the nth-order output frequency response of the system

$$H_n(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n)\, \mathrm{e}^{-(\omega_1\tau_1+\cdots+\omega_n\tau_n)j}\, \mathrm{d}\tau_1, \ldots, \mathrm{d}\tau_n \tag{3}$$

is the nth-order GFRF [3], and

$$\int_{\omega_1+\cdots+\omega_n=\omega} H_n(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_n) \prod_{i=1}^{n} U(\mathrm{j}\omega_i)\, \mathrm{d}\sigma_{n\omega}$$

denotes the integration of $H_n(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_n)\prod_{i=1}^{n} U(\mathrm{j}\omega_i)$ over the n-dimensional hyper-plane $\omega_1 + \cdots + \omega_n = \omega$.

The new concept of the NOFRFs recently proposed by Lang and Billings [10] is defined as

$$G_n(\mathrm{j}\omega) = \frac{\int_{\omega_1+\cdots+\omega_n=\omega} H_n(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_n)\prod_{i=1}^{n} U(\mathrm{j}\omega_i)\, \mathrm{d}\sigma_{n\omega}}{\int_{\omega_1+\cdots+\omega_n=\omega} \prod_{i=1}^{n} U(\mathrm{j}\omega_i)\, \mathrm{d}\sigma_{n\omega}} \tag{4}$$

under the condition that

$$U_n(\mathrm{j}\omega) = \int_{\omega_1+\cdots+\omega_n=\omega} \prod_{i=1}^{n} U(\mathrm{j}\omega_i)\, \mathrm{d}\sigma_{n\omega} \neq 0. \tag{5}$$

Note that $G_n(j\omega)$ is valid over the frequency range of $U_n(j\omega)$, which can be determined using the algorithm in Ref. [7].

By introducing the NOFRFs $G_n(j\omega)$, $n = 1, \ldots, \infty$, Eq. (2) can be written as

$$Y(j\omega) = \sum_{n=1}^{\infty} Y_n(j\omega) = \sum_{n=1}^{\infty} G_n(j\omega) U_n(j\omega), \tag{6}$$

which is similar to the description of the output frequency response for linear systems. The NOFRFs reflect a combined contribution of the system and the input to the system output frequency response behaviour.

When system (1) is subject to a harmonic input [13]

$$u(t) = A\cos(\omega_F t + \beta), \tag{7}$$

it can be derived that the output spectrum $Y(j\omega)$ of nonlinear systems can be expressed as

$$Y(jk\omega_F) = \sum_{n=1}^{\infty} G^H_{k+2(n-1)}(jk\omega_F) A_{k+2(n-1)}(jk\omega_F) \quad (k = 0, 1, \ldots, \infty), \tag{8}$$

where

$$A_n(j(-n+2k)\omega_F) = \frac{1}{2^n} C_n^k |A|^n e^{j(-n+2k)\beta}, \tag{9}$$

$$G_n^H(j(-n+2k)\omega_F) = H_n(\overbrace{j\omega_F, \ldots, j\omega_F}^{k}, \overbrace{-j\omega_F, \ldots, -j\omega_F}^{n-k}). \tag{10}$$

Consider the nonlinear systems described by the Duffing's oscillator

$$m\ddot{y} + c\dot{y} + k_1 y + k_3 y^3 = u(t), \tag{11}$$

where m, c, k_1 and k_3 are the parameters of the mass, damping and stiffness of the system, respectively. Denote $\omega_0 = \sqrt{k_1/m}$, $\mu = c/(2\omega_0)$, $\varepsilon = k_3/k_1$, $u_0(t) = (1/m)u(t)$ Eq. (11) can be rewritten as

$$\ddot{y} + 2\mu\omega_0\dot{y} + \omega_0^2 y + \varepsilon\omega_0^2 y^3 = u_0(t). \tag{12}$$

Using the results in Ref. [4] and Eq. (10), the NOFRFs of the Dunffing's oscillator under a harmonic input can be obtained. The results show that all even order NOFRFs are zero; the expressions of the other NOFRFs are

$$G_1^H(j\omega_F) = H_1(j\omega_F) = \frac{1}{-\omega_F^2 + j2\mu\omega_0\omega_F + \omega_0^2} \tag{13}$$

and

$$G_n^H(j(-n+2k)\omega_F) = -\frac{\varepsilon\omega_0^2}{C_n^k} G_1^H(j(-n+2k)\omega_F) \times \left(\sum_{\substack{n_1+n_2+n_3=n \\ L_1+L_2+L_3=k}} C_{n_1}^{L_1} C_{n_2}^{L_2} C_{n_3}^{L_3} \begin{pmatrix} G_{n_1}^H(j(-n_1+2L_1)\omega_F) \\ \times G_{n_2}^H(j(-n_2+2L_2)\omega_F) \\ \times G_{n_2}^H(j(-n_3+2L_3)\omega_F) \end{pmatrix} \right) \quad (n = 3, 5, \ldots). \tag{14}$$

3. Convergence of the NOFRFs of Duffing's oscillator

Substituting (9) into (8) yields

$$Y(jk\omega_F) = \sum_{n=1}^{\infty} G^H_{k+2(n-1)}(jk\omega_F) C^{n+k-1}_{k+2(n-1)} \left|A/2\right|^{k+2(n-1)} e^{jk\beta} \quad (k = 0, 1, \ldots, \infty). \tag{15}$$

Obviously,

$$\left|Y(\mathrm{j}k\omega_F)\right| \leqslant \sum_{n=1}^{\infty}\left|G^H_{k+2(n-1)}(\mathrm{j}k\omega_F)\frac{C^{n+k-1}_{k+2(n-1)}}{2^{k+2(n-1)}}\right||A|^{k+2(n-1)} \quad (k=0,1,\ldots,\infty). \tag{16}$$

Consider the ordinary power series $g(x)=\sum_{n=1}^{\infty}a_n x^n$, which converges absolutely for $|x|<\rho$, where the radius of convergence is given by [14]

$$\rho = \left(\lim\left(\sup_{n\to\infty}|a_n|^{1/n}\right)\right)^{-1}. \tag{17}$$

Therefore, a radius of convergence ρ_k can be found for the Volterra-series representation (15) using (16) as

$$|A|<\rho_k = \left(\lim\left(\sup_{n\to\infty}\left|G^H_{k+2(n-1)}(\mathrm{j}k\omega_F)\frac{C^{n+k-1}_{k+2(n-1)}}{2^{k+2(n-1)}}\right|^{1/(k+2(n-1))}\right)\right)^{-1} \quad (k=0,1,\ldots,\infty). \tag{18}$$

Eq. (18) can be used to determine the upper bound on the amplitude of the harmonic input, below which the Volterra-series representation of the kth harmonic component of system (1) is absolutely convergent. Moreover, it can be seen that

$$|A|<\rho=\min(\rho_0,\rho_1,\ldots) \tag{19}$$

can be defined as the convergence radius for the Volterra-series representation of the response of the nonlinear system subjected to a harmonic input.

The present study is to determine the convergence radius of the Volterra series for the Duffing's oscillator. However, it is impossible to directly use Eqs. (18) and (19) in practice. An alternative approach is to find a gain bound series F such that, for all k,

$$F_n \geqslant \left|G^H_n(\mathrm{j}(2k-n)\omega_F)\frac{C^k_n}{2^n}\right|, \tag{20}$$

then the convergence radius is defined as

$$|A|<\rho=\left(\lim\left(\sup_{n\to\infty}(F_n)^{1/n}\right)\right)^{-1} \tag{21}$$

within which the Volterra-series representation of the Duffing's oscillators subjected to a harmonic input converge absolutely.

Following this idea, a method is established to determine the convergence radius for the Volterra-series representation of the Duffing's oscillators subjected to harmonic inputs.

Proposition 1. *Denote*

$$\lambda = \max_{k=1,\ldots,\infty}\left(\left|G^H_1(\mathrm{j}(2k-1)\omega_F)\right|\right), \tag{22}$$

then the NOFRFs expressed in Eq. (10) satisfy

$$\left|G^H_n(\mathrm{j}(-n+2k)\omega_F)\right| \leqslant P_{(n,k)}\left(\lambda\varepsilon\omega_0^2\right)^{(n-1)/2}\left|G_1(\mathrm{j}\omega_F)\right|^n \quad (n=1,3,5,\ldots), \tag{23}$$

where $P_{(1,1)}=P_{(3,1)}=P_{(3,3)}=1$ *and*

$$P_{(n,k)}=\frac{1}{C^k_n}\sum_{\substack{n_1+n_2+n_3=n\\ L_1+L_2+L_3=k}} C^{L_1}_{n_1}C^{L_2}_{n_2}C^{L_3}_{n_3}P_{(n_1,L_1)}P_{(n_2,L_2)}P_{(n_3,L_3)}. \tag{24}$$

Proof of Proposition 1. Clearly, for $n=1$,

$$\left|G_1(\mathrm{j}\omega_F)\right| = P_{(1,1)}\left(\lambda\varepsilon\omega_0^2\right)^{(1-1)/2}\left|G_1(\mathrm{j}\omega_F)\right| = \left|G_1(\mathrm{j}\omega_F)\right| \tag{25}$$

and for $n=3$, it is known from Eq. (14) that

$$G^H_3(\mathrm{j}\omega_F) = -\varepsilon\omega_0^2 G^H_1(-\mathrm{j}\omega_F)G^H_1(\mathrm{j}\omega_F)G^H_1(\mathrm{j}\omega_F)G^H_1(\mathrm{j}\omega_F), \tag{26}$$

$$G_3^H(\mathrm{j}3\omega_F) = -\varepsilon\omega_0^2 G_1^H(\mathrm{j}\omega_F)G_1^H(\mathrm{j}\omega_F)G_1^H(\mathrm{j}\omega_F)G_1^H(\mathrm{j}3\omega_F). \tag{27}$$

Obviously,

$$\left|G_3^H(\mathrm{j}\omega_F)\right| \leqslant \lambda\varepsilon\omega_0^2\left|G_1^H(\mathrm{j}\omega_F)\right|^3 = P_{(3,1)}\lambda\varepsilon\omega_0^2\left|G_1^H(\mathrm{j}\omega_F)\right|^3, \tag{28}$$

$$\left|G_3^H(\mathrm{j}3\omega_F)\right| \leqslant \lambda\varepsilon\omega_0^2\left|G_1^H(\mathrm{j}\omega_F)\right|^3 = P_{(3,3)}\lambda\varepsilon\omega_0^2\left|G_1^H(\mathrm{j}\omega_F)\right|^3. \tag{29}$$

Therefore, the proposition holds for $n = 1$ and 3.

Assume that proposition also holds for all values n up to $N-2$ with $N \geqslant 3$. Consider the case of $n = N$ below. Substituting the cases of $n = n_1$, n_2 and n_3 $(n_1 + n_2 + n_3 = N)$ and (22) into (14) yields

$$\begin{aligned}
\left|G_N^H(\mathrm{j}(-N+2k)\omega_F)\right| &\leqslant \varepsilon\omega_0^2 \frac{1}{C_N^k}\left|G_1^H(\mathrm{j}(-N+2k)\omega_F)\right| \\
&\times\left(\sum_{\substack{n_1+n_2+n_3=N\\L_1+L_2+L_3=k}} C_{n_1}^{L_1}C_{n_2}^{L_2}C_{n_3}^{L_3}\left(\begin{array}{l}\left|P_{(n_1,L_1)}\left(\lambda\varepsilon\omega_0^2\right)^{(n_1-1)/2}\left|G_1(\mathrm{j}\omega_F)\right|^{n_1}\right|\\ \quad\times\left|P_{(n_2,L_2)}\left(\lambda\varepsilon\omega_0^2\right)^{(n_2-1)/2}\left|G_1(\mathrm{j}\omega_F)\right|^{n_2}\right|\\ \quad\times\left|P_{(n_3,L_3)}\left(\lambda\varepsilon\omega_0^2\right)^{(n_3-1)/2}\left|G_1(\mathrm{j}\omega_F)\right|^{n_3}\right|\end{array}\right)\right) \\
&\leqslant \lambda\varepsilon\omega_0^2\left(\left(\lambda\varepsilon\omega_0^2\right)^{(N-3)/2}\left|G_1(\mathrm{j}\omega_F)\right|^N \frac{1}{C_N^k}\sum_{\substack{n_1+n_2+n_3=N\\L_1+L_2+L_3=k}} P_{(n_1,L_1)}P_{(n_2,L_2)}P_{(n_3,L_3)}C_{n_1}^{L_1}C_{n_2}^{L_2}C_{n_3}^{L_3}\right) \\
&= P_{(N,k)}\left(\lambda\varepsilon\omega_0^2\right)^{(N-1)/2}\left|G_1(\mathrm{j}\omega_F)\right|^N,
\end{aligned} \tag{30}$$

that is, the proposition holds for $n = N$. Therefore Proposition 1 is proved. □

Proposition 2.

$$P_{(n,k)} = \frac{1}{C_n^k}\sum_{\substack{n_1+n_2+n_3=n\\L_1+L_2+L_3=k}} C_{n_1}^{L_1}C_{n_2}^{L_2}C_{n_3}^{L_3}P_{(n_1,L_1)}P_{(n_2,L_2)}P_{(n_3,L_3)} \leqslant 1 \quad (n = 1, 3, 5, \ldots). \tag{31}$$

Proof of Proposition 2. This proposition clearly holds for $n = 1$ and $n = 3$. For $n = 5$, it is known from Eq. (24) that

$$\begin{aligned}
P_{(5,k)} &= \frac{1}{C_5^k}\sum_{\substack{n_1+n_2+n_3=5\\L_1+L_2+L_3=k}} C_{n_1}^{L_1}C_{n_2}^{L_2}C_{n_3}^{L_3}P_{(n_1,L_1)}P_{(n_2,L_2)}P_{(n_3,L_3)} \\
&= \frac{1}{C_5^k}\sum_{\substack{n_1+n_2+n_3=5\\L_1+L_2+L_3=k}} C_{n_1}^{L_1}C_{n_2}^{L_2}C_{n_3}^{L_3} \leqslant \frac{1}{C_5^k}C_5^k = 1.
\end{aligned} \tag{32}$$

Therefore, this proposition also holds for $n = 5$.

Assume that proposition also holds for all values n up to $N-2$ with $N \geqslant 3$. Consider the case of $n = N$ below. Substituting the cases of $n = n_1$, n_2 and n_3 $(n_1 + n_2 + n_3 = N)$ into (24) yields

$$\begin{aligned}
P_{(N,k)} &= \frac{1}{C_N^k}\sum_{\substack{n_1+n_2+n_3=N\\L_1+L_2+L_3=k}} C_{n_1}^{L_1}C_{n_2}^{L_2}C_{n_3}^{L_3}P_{(n_1,L_1)}P_{(n_2,L_2)}P_{(n_3,L_3)} \\
&\leqslant \frac{1}{C_N^k}\sum_{\substack{n_1+n_2+n_3=N\\L_1+L_2+L_3=k}} C_{n_1}^{L_1}C_{n_2}^{L_2}C_{n_3}^{L_3} \leqslant \frac{1}{C_N^k}C_N^k = 1,
\end{aligned} \tag{33}$$

Thus the proposition holds for $n = N$. Therefore Proposition 2 is proved. □

From Eqs. (23) and (31), it can be deduced that

$$\left|G_N^H(\mathrm{j}(2k-N)\omega_F)\frac{C_N^k}{2^N}\right| \leqslant \left(\lambda\varepsilon\omega_0^2\right)^{(N-1)/2}\left|G_1(\mathrm{j}\omega_F)\right|^N\frac{C_N^k}{2^N}$$
$$\leqslant \left(\lambda\varepsilon\omega_0^2\right)^{(N-1)/2}\left|G_1(\mathrm{j}\omega_F)\right|^N\frac{C_N^{(N+1)/2}}{2^N} \quad (N=1,3,5,\ldots). \tag{34}$$

Define the gain bound series F as

$$F_n = \left(\lambda\varepsilon\omega_0^2\right)^{(n-1)/2}\left|G_1(\mathrm{j}\omega_F)\right|^n\frac{C_n^{(n+1)/2}}{2^n} \quad (n=1,3,5,\ldots) \tag{35}$$

then the convergence radius of the Duffing's oscillator is determined as

$$|A|<\rho = \left(\lim\left(\sup_{n\to\infty}\left(\left(\lambda\varepsilon\omega_0^2\right)^{(n-1)/2}\left|G_1(\mathrm{j}\omega_F)\right|^n\frac{C_n^{(n+1)/2}}{2^n}\right)^{1/n}\right)\right)^{-1}$$
$$= \frac{1}{\left|G_1(\mathrm{j}\omega_F)\right|\sqrt{\lambda\varepsilon\omega_0^2}}\left(\lim\left(\sup_{n\to\infty}\left(\left(\lambda\varepsilon\omega_0^2\right)^{-1/2}\frac{C_n^{(n+1)/2}}{2^n}\right)^{1/n}\right)\right)^{-1}$$
$$= \frac{1}{\left|G_1(\mathrm{j}\omega_F)\right|\sqrt{\lambda\varepsilon\omega_0^2}}\left(\lim\left(\sup_{n\to\infty}\left(\left(\lambda\varepsilon\omega_0^2\right)^{-1/2}\frac{3}{4}\frac{5}{6}\cdots\frac{n}{n+1}\right)^{1/n}\right)\right)^{-1}$$
$$= \frac{1}{\left|G_1(\mathrm{j}\omega_F)\right|\omega_0\sqrt{\lambda\varepsilon}}. \tag{36}$$

The criterion defined by expression (36) provides a simple approach to determine the maximal magnitude of the input excitation level, under which the response of the Duffing's oscillator can be described using a convergent Volterra series.

3. Numerical studies and discussions

Actually, a few researchers have investigated the convergence issue of using the Volterra series to describe the responses of the Duffing's oscillators subjected to the harmonic inputs. Through analysing the Volterra-series representation of the first-order harmonic, Tomlinson, Manson and Lee [4] derived a simple criterion, which can provide an estimation of the magnitude of the harmonic input below which the Volterra-series representation is convergent. The criterion is defined as

$$1>6\varepsilon\omega_0^2(A/2)^2\left|H_1(\mathrm{j}\omega_F)\right|^3 \Rightarrow A<\left(\tfrac{3}{2}\varepsilon\omega_0^2\left|H_1(\mathrm{j}\omega_F)\right|^3\right)^{-2}. \tag{37}$$

Based on a ratio test procedure, Chatterjee and Vyas [11] proposed an algorithm to compute the critical value of the harmonic input magnitude for the non-dimensional Duffing's oscillators. The idea of the ratio test procedure is essentially the same as Tomlinson's. Numerical studies have showed that criterion by the algorithm is in close agreement with the real situation. However, the ratio test procedure used to find the critical value needs iterative computation over a large number of values of the ratios, therefore it is time consuming.

Through comparing (36) and (37), it can be found that there is a great similarity between the criterion proposed here and the criterion proposed by Tomlinson et al. [4], and the only significant difference is the replacement of $|G_1^H(\mathrm{j}\omega_F)|$ by λ in Eq. (36). In the following numerical case studies, it can be seen that, by introducing λ to the definition of the criterion, the effects of the sub-resonances on the convergence of the Volterra-series representation can be eliminated.

The parameters of the considered Duffing's oscillator in the numerical study are

$$\mu = 0.005, \quad \omega_0 = 20\pi, \quad \varepsilon = 0.01\omega_0^4.$$

Fig. 1 shows the two criterions calculated using (36) and (37), respectively for the Duffing's oscillator. Obviously, apart from the frequency range around $\frac{1}{3}\omega_0$, there are no big differences between the criterions calculated by the two methods.

According to the criterions defined by both (36) and (37), from Fig. 1, it can be predicted that, when $A = 0.008$, the Volterra-series representations are always convergent no matter whatever the frequency of the harmonic input may be, and when $A = 0.1$, if the frequency of the harmonic input is around ω_0, the Volterra-series representations will be divergent. But, at the case of $A = 100$, only when the frequency of the harmonic input is larger than $2.3\omega_0$, the response of the Duffing's oscillator can be described using a convergent Volterra series.

Figs. 2–4 show the decompositions of the first and third harmonic components using NOFRFs when $A = 0.008$, 0.1 and 100, respectively. It is worth denoting here that, in all figures, only the NOFRFs up to 13th order are considered.

Denote $|Y_{(N,k)}| = |G_N^H(\mathrm{j}k\omega_F)A_N(\mathrm{j}k\omega_F)|$, from Fig. 2, it can be seen that, in the case of $A = 0.008$, at all frequencies, the following relationships hold:

$$\left|Y_{(1,1)}\right| > \left|Y_{(3,1)}\right| > \left|Y_{(5,1)}\right| > \left|Y_{(7,1)}\right| > \left|Y_{(9,1)}\right| > \left|Y_{(11,1)}\right| > \left|Y_{(13,1)}\right| \tag{38}$$

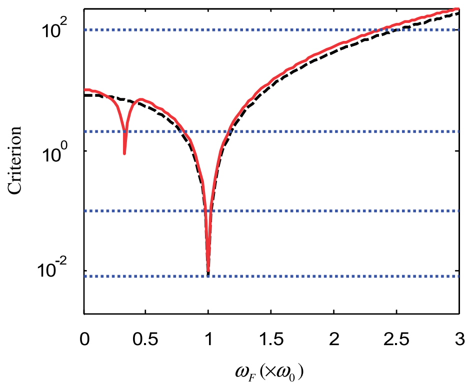

Fig. 1. The criteria calculated using (36) (solid line) and (37) (dashed line), respectively.

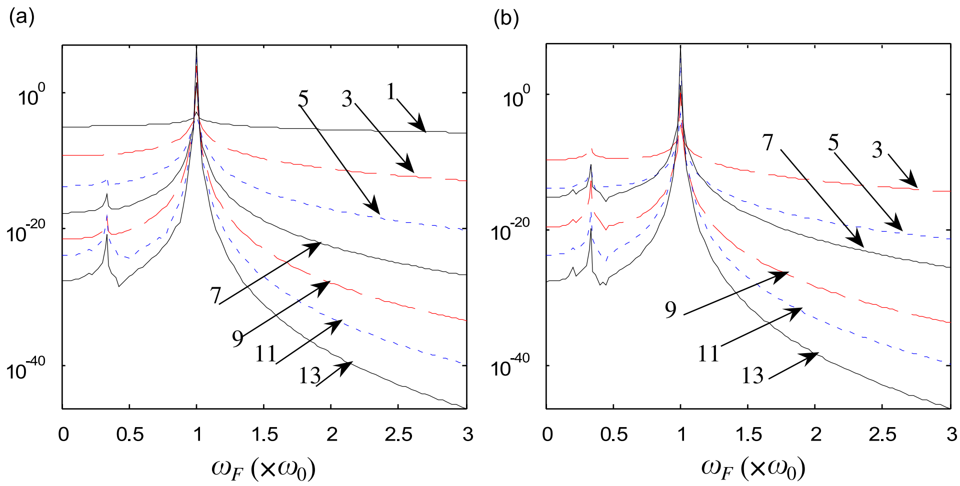

Fig. 2. The decompositions of the first and third harmonics using NOFRFs ($A = 0.008$): (a) the first harmonic, (b) the third harmonic [orders 1, 7, 13—solid line; orders 3, 9—dashed line; orders 5, 11—dotted line].

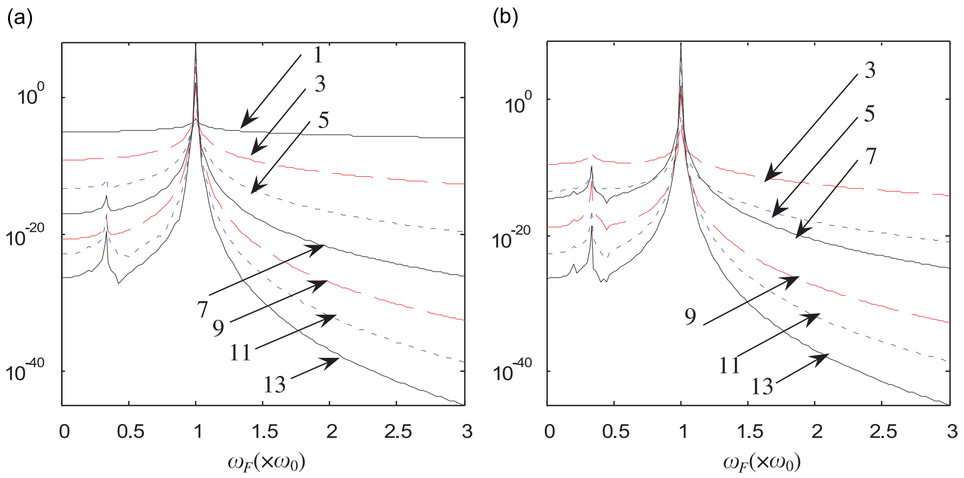

Fig. 3. The decompositions of the first and third harmonics using NOFRFs ($A = 0.1$): (a) the first harmonic, (b) the third harmonic [orders 1, 7, 13—solid line; orders 3, 9—dashed line; orders 5, 11—dotted line].

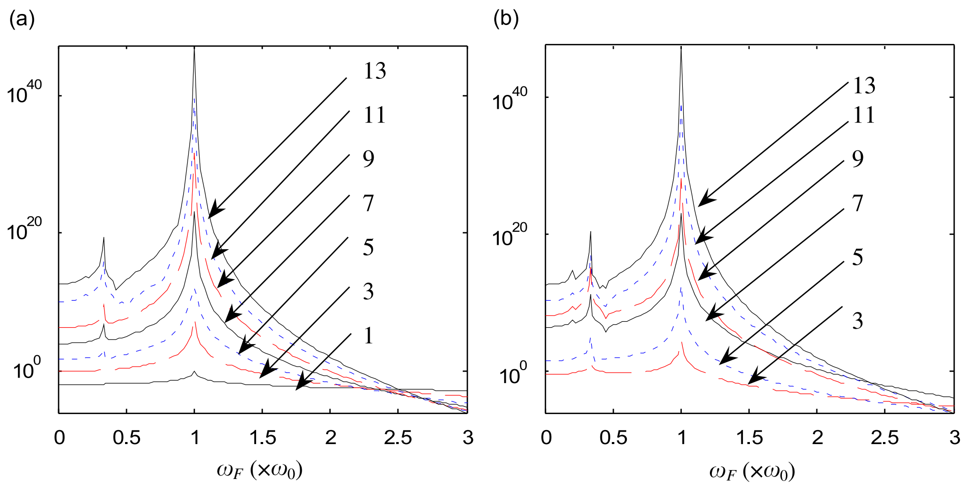

Fig. 4. The decompositions of the first and third harmonics using NOFRFs ($A = 100$): (a) the first harmonic, (b) the third harmonic [orders 1, 7, 13—solid line; orders 3, 9—dashed line; orders 5, 11—dotted line].

and

$$|Y_{(3,3)}| > |Y_{(5,3)}| > |Y_{(7,3)}| > |Y_{(9,3)}| > |Y_{(11,3)}| > |Y_{(13,3)}|. \tag{39}$$

This implies that the Volterra-series representations are always convergent when $A = 0.008$. However, Fig. 3 shows that, when $A = 0.1$, the relationships (38) and (39) are no longer valid at the cases of $\omega_F \approx \omega_0$, which implies the Volterra-series representations are divergent when the frequency of the harmonic input is around ω_0. At the case of $A = 100$, Fig. 4 clearly shows that, for the frequencies $\omega_F < 2.3\omega_0$, the following relationships hold:

$$|Y_{(1,1)}| < |Y_{(3,1)}| < |Y_{(5,1)}| < |Y_{(7,1)}| < |Y_{(9,1)}| < |Y_{(11,1)}| < |Y_{(13,1)}| \tag{40}$$

and

$$|Y_{(3,3)}| < |Y_{(5,3)}| < |Y_{(7,3)}| < |Y_{(9,3)}| < |Y_{(11,3)}| < |Y_{(13,3)}|, \tag{41}$$

therefore, the Volterra-series representations are divergent, and only when $\omega_F > 2.3\omega_0$, the first and third harmonic components can be decomposed using convergent Volterra series. Obviously, above analysis results strictly consist with the results predicted using the criterions defined by (36) and (37).

From Fig. 1, it can be known that, if the magnitude of the harmonic input is above 1, according to the criterion defined by (37), the Volterra-series representation should be convergent at the frequencies around $\frac{1}{3}\omega_0$, on the contrary, according to the criterion defined by (36), the Volterra-series representation is divergent. Figs. 5 and 6 show the decompositions of the first and third harmonic components using the NOFRFs for the case of $A = 2.0$. Clearly, at the frequencies around $\frac{1}{3}\omega_0$, the decompositions of the first and third components using the NOFRFs are divergent. Therefore, criterion (36) can give correct predictions while criterion (37) fails to do so. The frequency $\frac{1}{3}\omega_0$ is actually a sub-resonant frequency of the Duffing's oscillator where the third and the other higher-order NOFRFs can reach a maximum [13]; therefore, the comparison results imply that criterion (37) is not valid when the Duffing's oscillator is running at a sub-resonance region.

Figs. 7(a) and (b) show the amplitudes of the first harmonics and the third harmonics obtained by the NOFRF method and the *Runge–Kutta* method, respectively, where the amplitude of the input is 2 and the

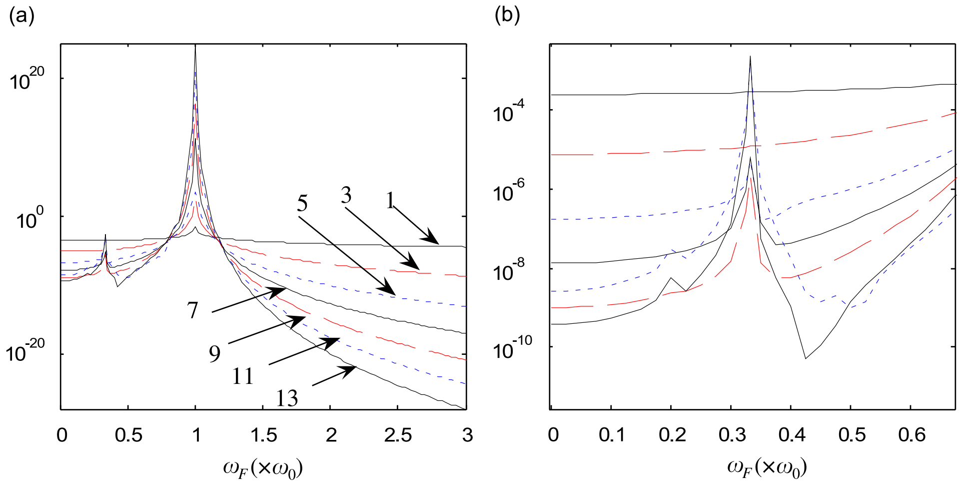

Fig. 5. The decompositions of the first harmonic using NOFRFs together a zoom view around $\omega_F = \frac{1}{3}\omega_0$ ($A = 2$): (a) original view, (b) zoom view [orders 1, 7, 13—solid line; orders 3, 9—dashed line; orders 5, 11—dotted line].

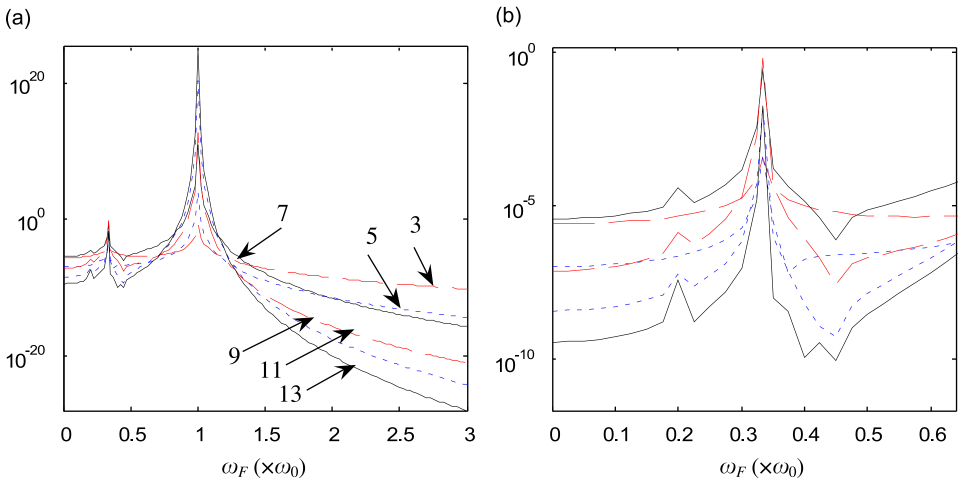

Fig. 6. The decompositions of the third harmonic using NOFRFs together a zoom view around $\omega_F = \frac{1}{3}\omega_0$ ($A = 2$): (a) original view, (b) zoom view [orders 1, 7, 13—solid line; orders 3, 9—dashed line; orders 5, 11—dotted line].

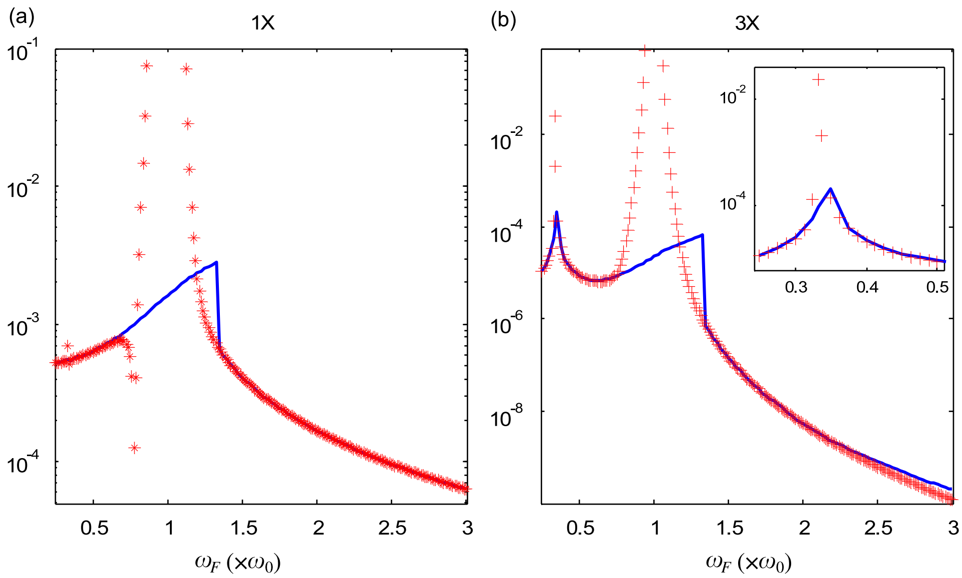

Fig. 7. Comparison between the NOFRF method and the *Runge–Kutta* method (star—NOFRF; $A = 2$): (a) the first harmonic, (b) the third harmonic.

frequency of the sinusoidal input is changed between $0.25\omega_0$ and $3\omega_0$. Obviously, the Volterra-series representation is divergent at the frequencies around ω_0, and the amplitudes of the first harmonic and third harmonic calculated using the NOFRFs are thus significantly deviating from the results obtained by the *Runge–Kutta* method. It can also be seen that, at the frequencies around $\frac{1}{3}\omega_0$, the amplitude of the third harmonic calculated using the NOFRF method cannot match the result obtained by the *Runge–Kutta* method. It implies that, when $A = 2$, at the frequency $\omega_F = \frac{1}{3}\omega_0$, the NOFRF fails in representing the third harmonic because the Volterra-series representation is divergent. This result has validated the effectiveness of criterion (36).

4. Conclusions

In the present study, based on the concept of NOFRFs, a new method is proposed to study the convergence issue of the Volterra-series representation of the Duffing's oscillator. A new simple criterion is deduced to determine the upper limit of the magnitude of the harmonic inputs, under which the Volterra-series representation is absolutely convergent. A comparison study has been carried out between the new proposed criterion and the criterion suggested by other researchers. The results have verified the effectiveness of the new criterion, and showed that the new criterion can give more accurate results about where the Volterra-series representation can be divergent.

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