Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 305 (2007) 365–377
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Improved eigenvalues for combined dynamical systems using a modified finite element discretization scheme

Philip D. Cha[a,*], Xiang Zhou[b]

[a]Department of Engineering, Harvey Mudd College, Claremont, CA 91711, USA
[b]School of Aerospace, Tsinghua University, Beijing 100084, China

Received 30 September 2005; received in revised form 17 May 2006; accepted 20 March 2007
Available online 4 June 2007

Abstract

New approaches are presented to discretize an arbitrarily supported linear structure carrying various lumped attachments. Specifically, the exact eigendata, i.e., the exact natural frequencies and mode shapes, of the linear structure without the lumped attachments are first used to modify its finite element mass and stiffness matrix so that the eigensolutions of the discretized system coincide with the exact modes of vibration. This is achieved by identifying a set of minimum changes in the finite element system matrices and enforcing certain constraint conditions. Once the updated matrices for the linear structure are found, the finite element assembling technique is then used to include the lumped attachments by adding their parameters to the appropriate elements in the modified mass and stiffness matrices. Numerical experiments show that for the same number of elements, the proposed scheme returns higher natural frequencies that are substantially more accurate than those given by the finite element model. Alternatively, the proposed discretization scheme allows one to efficiently and accurately determine the higher natural frequencies of a combined system without increasing the number of elements in the finite element model.

1. Introduction

Frequency analysis of combined dynamical systems consisting of a linear structure carrying any number of lumped attachments has been studied extensively over the years, and hence only a few selected recent references are given here [1–25]. Commonly used analytical approaches include the assumed-modes method [21,25], the Lagrange multipliers formalism [9,16,18,20], dynamic Green's function approach [10,17,19], Laplace transform with respect to the spatial variable approach [8,24], and the analytical-and-numerical-combined method [12,22]. However, due to their complexity, these methods have been used less than the finite element method.

Highly accurate and detailed models are required to analyze and predict the dynamical behavior of complex structures. With the advent of digital computers, new methods of analysis have been developed, especially the finite element method (FEM). Because the finite element method is a numerical procedure, after an analysis

*Corresponding author.
E-mail address: philip_cha@hmc.edu (P.D. Cha).

doi:10.1016/j.jsv.2007.03.086

has been performed, its accuracy must be assessed. If the accuracy criteria are not satisfied, the finite element model must be refined by using more elements until a sufficient accuracy is reached.

In this paper, a modified approach is proposed that can be effectively used to obtain the natural frequencies of a combined system consisting of a linear structure carrying various lumped attachments. To obtain the higher natural frequencies of such a system by using finite element method, one typically refines the mesh of the linear structure until the accuracy criteria are satisfied. While conceptually straightforward, this approach of refining the mesh to determine the higher natural frequencies is costly and time consuming. The slow convergence can be attributed to the fact that many elements are often needed to model the linear structure itself so that the higher natural frequencies of the discretized linear structure match well with the exact solutions.

To expedite convergence and to obtain sufficiently accurate results with the least cost, a new scheme is introduced to improve the finite element mass and stiffness matrices of the linear structure such that the eigendata of the updated finite element model of the linear structure coincide with the exact eigensolution. Once the system matrices of the linear structure have been updated, the finite element assembling technique is exploited and used to account for the lumped attachments. To compute the natural frequencies of the combined system, one then solves the generalized eigenvalue problem associated with the newly assembled mass and stiffness matrices of the combined structure. Numerical experiments showed that by applying the proposed discretization scheme, one can use a coarse mesh to obtain the natural frequencies, including the higher ones, of a combined system accurately, instead of using the traditional approach of refining the mesh and performing a potentially costly reanalysis to obtain the higher natural frequencies.

2. Theory

Berman and Nagy [26] developed a method that used test data to improve the analytical mass and stiffness matrices of a structure. The method yields a set of minimum changes in the system matrices such that the eigensolutions coincide with the test measurements. In this paper, the same approach is employed to compute the eigenvalues of a combined system consisting of a linear structure carrying lumped attachments. In particular, the exact eigendata of the linear structure are first used to modify its finite element mass and stiffness matrices. Once the system matrices of the linear structure have been updated, one can easily include the lumped attachments by exploiting the finite element assembling technique, and determine the eigenvalues of the combined system by solving a generalized eigenvalue problem.

Consider a combined system consisting of an arbitrarily supported linear structure carrying various lumped attachments. Assume the linear structure has been discretized and possesses N generalized coordinates. Let $[M_0]$ and $[K_0]$ denote the finite element mass and stiffness matrices (both of size $N \times N$) of the linear structure, respectively. The eigensolutions of the associated generalized eigenvalue problem correspond to the modes of vibration of the discretized linear structure. As N approaches infinity, the modes of vibration of the finite element model for the linear structure approach the exact eigensolution. Suppose the exact eigendata of the linear structure are known. Then they can be used to improve or update the mass and stiffness matrices of the linear structure such that the modified system returns eigensolutions that are exact even for finite N.

To find the updated mass matrix $[M]$ of the linear structure, the following objective function is minimized:

$$J_M = \|[M_0]^{-1/2}([M] - [M_0])[M_0]^{-1/2}\| + \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{ij}([U]^{\mathrm{T}}[M]U] - [I])_{ij}, \tag{1}$$

where $\|[A]\|$ denotes the sum of the squares of all elements of matrix $[A]$, λ_{ij} denotes the Lagrange multiplier that is used to enforce the orthogonality of the eigenvectors with respect to the updated mass matrix, and $[U]$ is the exact modal matrix of the linear structure (of size $N \times N$), whose elements are obtained from the exact eigenfunctions of the linear structure. Eq. (1) is differentiated with respect to the elements of $[M]$ and set to 0, and the undetermined Lagrange multipliers are obtained by enforcing the constraint equation, i.e., $[U]^{\mathrm{T}}[M][U] = [I]$. The minimization procedure results in the expression for the updated mass matrix as follows (see Ref. [27] for detailed derivation):

$$[M] = [M_0] + [M_0][U][m]^{-1}([I] - [m])[m]^{-1}[U]^{\mathrm{T}}[M_0], \tag{2}$$

where

$$[m] = [U]^{\mathrm{T}}[M_0][U]. \tag{3}$$

Note that the computation of $[M_0]^{-1/2}$, which appears in Eq. (1), is not needed in the final expression of Eq. (2).

Once $[M]$ has been computed using Eq. (2), an updated stiffness matrix $[K]$ of the linear structure can be obtained by minimizing yet another objective function of the form

$$J_K = \|[M]^{-1/2}([K] - [K_0])[M]^{-1/2}\| + \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{Kij}([K][U] - [M][U][\Lambda])_{ij} + \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{0ij}\left([U]^{\mathrm{T}}[K][U] - [\Lambda]\right)_{ij} + \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{Sij}([K] - [K]^{\mathrm{T}})_{ij}, \tag{4}$$

where $[\Lambda]$ denotes a diagonal matrix consisting of the exact eigenvalues of the linear structure. Here, the Lagrange multipliers are used to enforce the generalized eigenvalue problem, the orthogonality of the eigenvectors with respect to the updated stiffness matrix, and the stiffness symmetry. Eq. (4) is differentiated with respect to the elements of $[K]$ and set to 0. Using the constraint equations, i.e., $[K][U] = [M][U][\Lambda]$, $[U]^{\mathrm{T}}[K][U] = [\Lambda]$ and $[K] = [K]^{\mathrm{T}}$, to eliminate the undetermined Lagrange multipliers yields the following expression for the updated stiffness matrix (see Ref. [28] for detailed derivation):

$$[K] = [K_0] + ([\Delta] + [\Delta]^{\mathrm{T}}), \tag{5}$$

where

$$[\Delta] = \tfrac{1}{2}[M][U]([U]^{\mathrm{T}}[K_0][U] + [\Lambda])[U]^{\mathrm{T}}[M] - [K_0][U][U]^{\mathrm{T}}[M]. \tag{6}$$

Note that Eq. (6) requires only simple matrix multiplications.

Eqs. (2) and (5) lead to updated mass and stiffness matrices whose eigensolutions coincide with the exact eigendata of the linear structure.The proposed modification scheme returns an updated model without iteration, and requires only matrix multiplications. Once these updated matrices are obtained, the lumped attachments are added to the updated system matrices to form the global mass and stiffness matrices $[\mathscr{M}]$ and $[\mathscr{K}]$. Finally, the natural frequencies of the combined assembly are obtained by solving the following generalized eigenvalue problem:

$$[\mathscr{K}]\bar{\mathbf{q}} = \omega^2[\mathscr{M}]\bar{\mathbf{q}}, \tag{7}$$

where $\bar{\mathbf{q}}$ denotes the vector of the amplitudes of the generalized coordinates for the combined system.

3. Results

In Fig. 1 is shown a combined system consisting of an arbitrarily supported linear structure carrying various lumped attachments, including a grounded translational spring of stiffness k_1 at x_1, a lumped mass m_1 at x_2, a damped oscillator of parameters c, m_2 and k_2 with a rigid body degree of freedom at x_3, a grounded torsional spring of stiffness k_t at x_4, and an oscillator of parameters m_3 and k_3 with no rigid body degree of freedom at x_5. To validate the proposed discretization scheme, the natural frequencies of a combined system consisting of a simply supported beam or a fixed–free beam carrying various lumped attachments will be considered. In order to apply Eqs. (2) and (5), matrices $[M_0]$, $[K_0]$, $[U]$, and $[\Lambda]$ are required. The finite element mass and stiffness matrices, $[M_0]$ and $[K_0]$, of the beam can be easily obtained by superimposing the individual element matrices (see Appendix A for the element matrices), and enforcing the appropriate boundary conditions at the ends. Matrices $[U]$ and $[\Lambda]$ can be assembled directly from the exact modes of vibration once the boundary conditions for the beam are specified. For a simply supported beam, its normalized (with respect to the mass

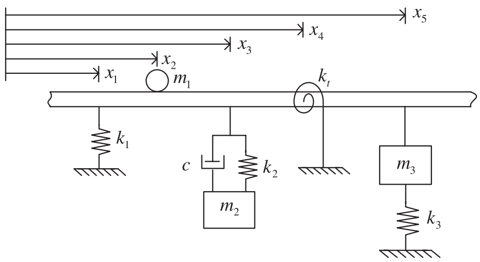

Fig. 1. A combined system consisting of an arbitrarily supported linear structure carrying various lumped attachments, including a grounded translational spring of stiffness k_1 at x_1, a lumped mass m_1 at x_2, an oscillator of parameters m_2 and k_2 with a rigid body degree of freedom at x_3, a grounded torsional spring of stiffness k_t at x_4, and an oscillator of parameters m_3 and k_3 with no rigid body degree of freedom at x_5.

per unit length, ρ, of the beam) eigenfunctions, $v_i(x)$, and eigenvalues, λ_i, for $i = 1, \ldots, N$, are given by

$$v_i(x) = \sqrt{\frac{2}{\rho L}} \sin\left(\frac{i\pi x}{L}\right), \tag{8}$$

$$\lambda_i = (i\pi)^4 \frac{EI}{\rho L^4}, \tag{9}$$

where E, I, and L denote the Young's modulus, the area moment of inertia of the cross section, and the length of the beam, respectively. For a fixed–free beam, its normalized eigenfunctions and eigenvalues are given by

$$v_i(x) = \frac{1}{\sqrt{\rho L}} \left[\cos \beta_i x - \cosh \beta_i x + \frac{\sin \beta_i L - \sinh \beta_i L}{\cos \beta_i L + \cosh \beta_i L} (\sin \beta_i x - \sinh \beta_i x)\right], \tag{10}$$

where $\beta_i L$ satisfies the following transcendental equation:

$$\cos \beta_i L \cosh \beta_i L = -1 \tag{11}$$

and

$$\lambda_i = (\beta_i L)^4 \frac{EI}{\rho L^4}. \tag{12}$$

For a beam element, its generalized coordinates consist of the lateral displacement and the angular rotation (or slope) at the nodes [29]. Hence, if the simply supported or fixed–free beam is discretized into n equal finite elements, there is a total of $N = 2n$ generalized coordinates. Moreover, to assemble the exact modal matrix, $[U]$, of the linear structure, the lateral deflection and the slope at each node must be specified. Fortunately, knowing the exact eigenfunctions $v_i(x)$ of beam, its slope at any point x can be easily determined by taking the derivative of $v_i(x)$ with respect to x, i.e.,

$$\theta_i(x) = \frac{\mathrm{d}}{\mathrm{d}x}[v_i(x)]. \tag{13}$$

Once the exact lateral displacements and angular rotations at the nodes have been computed, matrix $[U]$ can be easily assembled, where the elements of the ith column of $[U]$ are obtained by evaluating the ith eigenfunction and its derivative at the appropriate node locations. Finally, the ith element of the diagonal matrix $[\Lambda]$ is given by λ_i.

Updating the stiffness matrix requires only matrix multiplications. Updating the mass matrix, however, requires the inversion of the matrix $[m]$. Nevertheless, the additional computation needed to invert the $N \times N$ matrix is a relatively small price to pay for the ability to obtain the higher natural frequencies or eigenvalues that are nearly identical to the exact results, as will be illustrated.

In all the following numerical examples, the first 10 natural frequencies of various combined systems are first obtained by discretizing the linear structure into 100 equal elements. For all practical purposes, these natural frequencies can be considered exact. In Tables 1 and 2 are listed the exact and the finite element

Table 1
The first 10 natural frequencies of a uniform simply supported Euler–Bernoulli beam, obtained exactly and using the finite element method (FEM) with 100 equal elements

ω_i	Exact	FEM ($n=100$)
ω_1	9.86960e + 00	9.86960e + 00
ω_2	3.94784e + 01	3.94784e + 01
ω_3	8.88264e + 01	8.88264e + 01
ω_4	1.57914e + 02	1.57914e + 02
ω_5	2.46740e + 02	2.46740e + 02
ω_6	3.55306e + 02	3.55306e + 02
ω_7	4.83611e + 02	4.83611e + 02
ω_8	6.31655e + 02	6.31656e + 02
ω_9	7.99438e + 02	7.99441e + 02
ω_{10}	9.86960e + 02	9.86967e + 02

All of the natural frequencies are normalized by dividing by $\sqrt{EI/(\rho L^4)}$.

Table 2
The first 10 natural frequencies of a uniform fixed–free Euler–Bernoulli beam, obtained exactly and using the finite element method (FEM) with 100 equal elements

ω_i	Exact	FEM ($n=100$)
ω_1	3.51602e + 00	3.51602e + 00
ω_2	2.20345e + 01	2.20345e + 01
ω_3	6.16972e + 02	6.16972e + 01
ω_4	1.20902e + 02	1.20902e + 02
ω_5	1.99860e + 02	1.99860e + 02
ω_6	2.98556e + 02	2.98556e + 02
ω_7	4.16991e + 02	4.16991e + 02
ω_8	5.55165e + 02	5.55166e + 02
ω_9	7.13079e + 02	7.13081e + 02
ω_{10}	8.90732e + 02	8.90736e + 02

(with 100 elements) natural frequencies of a simply supported and a fixed–free beam, respectively. Note the excellent agreement between the two sets of natural frequencies, thus justifying the assumption that the finite element results with 100 elements can be considered exact.

To illustrate the utility of the proposed discretization schemes, the first 10 natural frequencies of a combined system consisting of a beam carrying lumped attachments are obtained by using the finite element method and the proposed discretization scheme, whereby the beam is discretized into 5 equal elements, i.e., $n=5$. To gauge the accuracy of the natural frequencies obtained by using the proposed discretization scheme and the finite element method, a relative error parameter is introduced as follows:

$$\varepsilon_i = \frac{\omega_i - \omega_{i\text{ex}}}{\omega_{i\text{ex}}}, \tag{14}$$

where $\omega_{i\text{ex}}$ denotes the "exact" ith natural frequency (obtained by using the finite element method and discretizing the beam into 100 equal elements), and ω_i represents the ith natural frequency for $n=5$, obtained by using the finite element method or the new discretization scheme. The smaller the $|\varepsilon_i|$, the more accurate the natural frequency estimates are.

Consider a simply supported beam carrying a lumped mass of $m=0.1\rho L$ at $x_a=0.2L$. In Table 3 are shown the first 10 natural frequencies of the system given by the finite element method with $n=100$ and with $n=5$, and the natural frequencies from the proposed discretization scheme derived by updating the stiffness and the mass matrices. When using the finite element method with the beam discretized into 5 elements with 10 generalized coordinates, only the first 4 natural frequencies are within 2.11% of the exact results, while the other natural frequencies are in error by over 10%. When using the new discretization scheme, on the other

Table 3
The first 10 natural frequencies of a uniform simply supported Euler–Bernoulli beam carrying a lumped mass of $0.1\rho L$ at $x_a = 0.2L$

ω_i	FEM, $n = 100$	FEM, $n = 5$ (ε_i)	New scheme, $n = 5$ (ε_i)
ω_1	9.5410e + 00	9.5419e + 00 (9.34e − 05)	9.5410e + 00(9.73e − 07)
ω_2	3.6414e + 01	3.6457e + 01 (1.20e − 03)	3.6414e + 01 (3.04e − 05)
ω_3	8.3071e + 01	8.3588e + 01 (6.23e − 03)	8.3079e + 01 (1.08e − 04)
ω_4	1.5434e + 02	1.5759e + 02 (2.10e − 02)	1.5436e + 02 (1.15e − 04)
ω_5	2.4674e + 02	2.7386e + 02 (1.10e − 01)	2.4674e + 02 (−4.22e − 07)
ω_6	3.4559e + 02	3.8643e + 02 (1.18e − 01)	3.4588e + 02 (8.41e − 04)
ω_7	4.5501e + 02	5.4738e + 02 (2.03e − 01)	4.5622e + 02 (2.66e − 03)
ω_8	6.0350e + 02	7.9189e + 02 (3.12e − 01)	6.0517e + 02 (2.77e − 03)
ω_9	7.8678e + 02	1.0950e + 03 (3.92e − 01)	7.8812e + 02 (1.70e − 03)
ω_{10}	9.8697e + 02	1.2550e + 03 (2.72e − 01)	9.8696e + 02 (−6.74e − 06)

Table 4
The first 10 natural frequencies of a uniform simply supported Euler–Bernoulli beam carrying a grounded torsional spring $k_t = 0.1EI/L$ at $x_a = 0.4L$

ω_i	FEM, $n = 100$	FEM, $n = 5$ (ε_i)	New scheme, $n = 5$ (ε_i)
ω_1	9.8791e + 00	9.8801e + 00 (1.07e − 04)	9.8790e + 00 (8.89e − 07)
ω_2	3.9544e + 01	3.9609e + 01 (1.66e − 03)	3.9543e + 01 (1.49e − 06)
ω_3	8.8892e + 01	8.9599e + 01 (7.95e − 03)	8.8891e + 01 (6.20e − 07)
ω_4	1.5792e + 02	1.6156e + 02 (2.30e − 02)	1.5792e + 02 (−1.17e − 07)
ω_5	2.4684e + 02	2.7400e + 02 (1.10e − 01)	2.4684e + 02 (−4.82e − 08)
ω_6	3.5532e + 02	3.9533e + 02 (1.12e − 01)	3.5531e + 02 (−8.50e − 07)
ω_7	4.8368e + 02	5.7572e + 02 (1.90e − 01)	4.8367e + 02 (−1.49e − 06)
ω_8	6.3172e + 02	8.1767e + 02 (2.94e − 01)	6.3172e + 02 (−2.66e − 06)
ω_9	7.9945e + 02	1.1003e + 03 (3.76e − 01)	7.9944e + 02 (−4.41e − 06)
ω_{10}	9.8707e + 02	1.2552e + 03 (2.72e − 01)	9.8706e + 02 (−6.63e − 06)

hand, all 10 natural frequencies are within 0.28% of the exact results. Moreover, note that the magnitudes of all errors ε_i when using the new discretization scheme are at least one order of magnitude smaller than those from the finite element method, even though in both approaches the beam is discretized using $n = 5$. This implies that for the same number of elements, the new scheme returns frequencies that are more accurate than those obtained using the finite element method. Numerical experiments show that this important result is observed for all cases considered.

In Table 4 are shown the first 10 natural frequencies of a simply supported beam with a grounded torsional spring of stiffness $k_t = 0.1EI/L$ attached at $x_a = 0.4L$. Note that using the new discretization scheme, the magnitudes of all ε_i are three orders of magnitude or more smaller than those for the finite element method. In addition, for the fifth and higher natural frequencies, the corresponding $|\varepsilon_i|$ from the finite element method exceed 10%, while those from the new scheme are less than 6.64×10^{-4}%. Thus, to obtain accurate higher natural frequencies one needs to use a very fine mesh when applying the finite element method. Here, excellent agreement to the exact results is obtained by using the newly developed discretization method with only 5 elements. Incidentally, larger values of m and k_t for the combined systems of Tables 3 and 4 were also considered. The results showed that the new method consistently returns natural frequencies that are more accurate, even the higher ones, implying that the new discretization scheme is applicable even for larger lumped masses or torsional spring constants.

Consider now a simply supported beam carrying an undamped oscillator with no rigid body degree of freedom at $x_a = 0.8L$. The oscillator parameters are $m = 0.02\rho L$ and $k = 0.5EI/L^3$. In Table 5 are shown the first 10 natural frequencies of the combined system. When using the finite element method with $n = 5$, the largest relative error exceeds 38%, while using the new discretization method with $n = 5$, the magnitudes for all ε_i are less than 3.63×10^{-2}%. Note that for all natural frequencies, the proposed scheme returns $|\varepsilon_i|$ that

Table 5
The first 10 natural frequencies of a uniform simply supported Euler–Bernoulli beam carrying an undamped oscillator with no rigid body degree of freedom at $x_a = 0.8L$

ω_i	FEM, $n = 100$	FEM, $n = 5$ (ε_i)	New scheme, $n = 5$ (ε_i)
ω_1	9.8194e + 00	9.8204e + 00 (1.04e − 04)	9.8193e + 00 (3.86e − 08)
ω_2	3.8797e + 01	3.8857e + 01 (1.54e − 03)	3.8797e + 01 (1.60e − 06)
ω_3	8.7339e + 01	8.7987e + 01 (7.42e − 03)	8.7339e + 01 (7.65e − 06)
ω_4	1.5693e + 02	1.6043e + 02 (2.23e − 02)	1.5693e + 02 (8.86e − 06)
ω_5	2.4674e + 02	2.7386e + 02 (1.10e − 01)	2.4674e + 02 (−4.22e − 07)
ω_6	3.5296e + 02	3.9299e + 02 (1.13e − 01)	3.5298e + 02 (5.15e − 05)
ω_7	4.7559e + 02	5.6722e + 02 (1.93e − 01)	4.7570e + 02 (2.37e − 04)
ω_8	6.2206e + 02	8.0885e + 02 (3.00e − 01)	6.2228e + 02 (3.62e − 04)
ω_9	7.9495e + 02	1.0984e + 03 (3.82e − 01)	7.9513e + 02 (2.32e − 04)
ω_{10}	9.8697e + 02	1.2550e + 03 (2.72e − 01)	9.8696e + 02 (−6.74e − 06)

The oscillator parameters are $m = 0.02\rho L$ and $k = 0.5EI/L^3$.

Table 6
The first 10 natural frequencies of a uniform simply supported Euler–Bernoulli beam carrying an undamped oscillator with a rigid body degree of freedom at $x_a = 0.4L$

ω_i	FEM, $n = 100$	FEM, $n = 5$ (ε_i)	New scheme, $n = 5$ (ε_i)
ω_1	3.1292e + 00	3.1292e + 00 (2.72e − 08)	3.1291e + 00 (1.62e − 06)
ω_2	9.9709e + 00	9.9720e + 00 (1.09e − 04)	9.9709e + 00 (−7.20e − 07)
ω_3	3.9487e + 01	3.9553e + 01 (1.66e − 03)	3.9487e + 01 (−2.60e − 08)
ω_4	8.8830e + 01	8.9536e + 01 (7.94e − 03)	8.8830e + 01 (−6.84e − 08)
ω_5	1.5792e + 02	1.6156e + 02 (2.30e − 02)	1.5792e + 02 (−1.76e − 07)
ω_6	2.4674e + 02	2.7386e + 02 (1.10e − 01)	2.4674e + 02 (−4.21e − 07)
ω_7	3.5531e + 02	3.9532e + 02 (1.13e − 01)	3.5531e + 02 (−8.75e − 07)
ω_8	4.8361e + 02	5.7558e + 02 (1.90e − 01)	4.8361e + 02 (−1.62e − 06)
ω_9	6.3166e + 02	8.1749e + 02 (2.94e − 01)	6.3165e + 02 (−2.76e − 06)
ω_{10}	7.9944e + 02	1.1003e + 02 (3.76e − 01)	7.9943e + 02 (−4.43e − 06)

The oscillator parameters are $m = 0.1\rho L$ and $k = 1.0EI/L^3$.

are three or more orders of magnitude smaller than those given by the finite element method. The previous observation implies that for $n = 5$ only, the new scheme can be used to obtain the higher natural frequencies accurately, while the finite element method cannot.

The first 10 natural frequencies of a simply supported beam carrying an undamped oscillator with a rigid body degree of freedom at $x_a = 0.4L$ are given in Table 6. The oscillator parameters are $m = 0.1\rho L$ and $k = 1.0EI/L^3$. When using the new method, the magnitude for all $|\varepsilon_i|$ are less than 4.44×10^{-4}%, implying that with 5 elements only, the proposed discretization scheme yields natural frequencies that are nearly exact. In contrast, only the first five natural frequencies returned by the finite element method have relative errors less than 2.31%.

From the results in Tables 3–6 it is concluded that the new discretization scheme enables one to obtain the natural frequencies accurately, even the higher ones. This is in contrast to the finite element method, where the lower natural frequencies are predicted well, but the higher natural frequency estimates are poor. Thus, the proposed scheme can be used to obtain the higher natural frequencies accurately with fewer elements than is required when using the finite element method.

Consider now the case where the linear structure consists of a uniform fixed–free Euler–Bernoulli beam. In Table 7 are shown the first 10 natural frequencies of a fixed–free beam carrying a grounded translational spring of stiffness $k = 1.2EI/L^3$ at $x_a = 0.4L$. Note that all $|\varepsilon_i|$ obtained when using the new scheme are three or more orders of magnitude smaller than those for the finite element method (for $i > 1$ they are five or more orders of magnitude smaller). Note also that the natural frequencies obtained with the new method are nearly identical to those obtained with the finite element method with $n = 100$. Thus, instead of solving a 200×200

Table 7
The first 10 natural frequencies of a uniform fixed–free Euler–Bernoulli beam carrying a grounded translational spring $k = 1.2EI/L^3$ at $x_a = 0.4L$

ω_i	FEM, $n = 100$	FEM, $n = 5$ (ε_i)	New scheme, $n = 5$ (ε_i)
ω_1	3.5517e + 00	3.5518e + 00 (1.36e − 05)	3.5517e + 00 (−1.66e − 07)
ω_2	2.2085e + 01	2.2096e + 01 (5.02e − 04)	2.2085e + 01 (−2.00e − 09)
ω_3	6.1708e + 01	6.1930e + 01 (3.59e − 03)	6.1708e + 01 (−2.52e − 08)
ω_4	1.2090e + 02	1.2232e + 02 (1.17e − 02)	1.2090e + 02 (−1.01e − 07)
ω_5	1.9987e + 02	2.0303e + 02 (1.58e − 02)	1.9987e + 02 (−2.77e − 07)
ω_6	2.9856e + 02	3.3727e + 02 (1.29e − 01)	2.9856e + 02 (−6.18e − 07)
ω_7	4.1699e + 02	4.9327e + 02 (1.83e − 01)	4.1699e + 02 (−1.20e − 06)
ω_8	5.5517e + 02	7.1534e + 02 (2.89e − 01)	5.5517e + 02 (−2.13e − 06)
ω_9	7.1308e + 02	1.0162e + 03 (4.25e − 01)	7.1308e + 02 (−3.52e − 06)
ω_{10}	8.9074e + 02	1.4948e + 03 (6.78e − 01)	8.9073e + 02 (−5.49e − 06)

Table 8
The first 10 natural frequencies of a uniform fixed–free Euler–Bernoulli beam carrying an undamped oscillator with no rigid body degree of freedom at $x_a = 0.6L$

ω_i	FEM, $n = 100$	FEM, $n = 5$ (ε_i)	New scheme, $n = 5$ (ε_i)
ω_1	3.3048e + 00	3.3048e + 00 (1.01e − 05)	3.3048e + 00 (1.81e − 07)
ω_2	1.9780e + 01	1.9787e + 01 (3.30e − 04)	1.9781e + 01 (1.88e − 05)
ω_3	5.8134e + 01	5.8310e + 01 (3.04e − 03)	5.8138e + 01 (7.60e − 05)
ω_4	1.1748e + 02	1.1864e + 02 (9.84e − 03)	1.1750e + 02 (1.51e − 04)
ω_5	1.7984e + 02	1.8213e + 02 (1.27e − 02)	1.8002e + 02 (1.03e − 03)
ω_6	2.9779e + 02	3.3690e + 02 (1.31e − 01)	2.9781e + 02 (6.73e − 05)
ω_7	3.8581e + 02	4.5956e + 02 (1.91e − 01)	3.8702e + 02 (3.14e − 03)
ω_8	5.3418e + 02	7.0158e + 02 (3.13e − 01)	5.3547e + 02 (2.42e − 03)
ω_9	7.0015e + 02	1.0032e + 03 (4.33e − 01)	7.0179e + 02 (2.34e − 03)
ω_{10}	8.3247e + 02	1.4835e + 03 (7.82e − 01)	8.4109e + 02 (1.03e − 02)

The oscillator parameters are $m = 0.2\rho L$ and $k = 0.5EI/L^3$.

generalized eigenvalue problem using the finite element method when $n = 100$, one can achieve practically the same results by inverting a 10×10 matrix and then solving a generalized eigenvalue problem of size 10×10 using the new scheme with $n = 5$ only.

Consider now a fixed–free beam carrying an undamped oscillator with no rigid body degree of freedom at $x_a = 0.6L$. The oscillator parameters are $m = 0.2\rho L$ and $k = 0.5EI/L^3$. In Table 8 are shown the first 10 natural frequencies of the combined system. Using the proposed method, the largest relative error is less than 1.04%, while using the finite element method with $n = 5$, the largest relative error exceeds 78%. Note also that all ε_i given by updating the mass and stiffness matrices are substantially reduced compared to those for the finite element method.

In Table 9 are shown the first 10 natural frequencies of a fixed–free beam carrying an undamped oscillator with a rigid body degree of freedom at $x_a = 0.4L$. The oscillator parameters are $m = 1.0\rho L$ and $k = 1.0EI/L^3$. Using the new technique, all $|\varepsilon_i|$ are less than 3.53×10^{-4}%, while using the finite element method, the largest ε exceeds 42%. The results of Table 9 clearly demonstrate the utility of the newly developed discretization scheme.

Consider a fixed–free beam carrying a damped oscillator (with parameters $m = 0.5\rho L$, $k = 1.0EI/L^3$ and $c = 0.2\sqrt{EI\rho/L^2}$) with a rigid body degree of freedom at $x_a = 0.8L$, whose governing equations are given by

$$[\mathscr{M}]\ddot{\mathbf{p}} + [\mathscr{C}]\dot{\mathbf{p}} + [\mathscr{K}]\mathbf{p} = \mathbf{0}. \tag{15}$$

Matrices $[\mathscr{M}]$, $[\mathscr{C}]$, and $[\mathscr{K}]$ are the $(N + 1) \times (N + 1)$ global mass, damping, and stiffness matrices of the combined system (the mass and stiffness matrices of the beam have already been modified by using the newly

Table 9
The first 10 natural frequencies of a uniform fixed–free Euler–Bernoulli beam carrying an undamped oscillator with a rigid body degree of freedom at $x_a = 0.4L$

ω_i	FEM, $n = 100$	FEM, $n = 5$ (ε_i)	New scheme, $n = 5$ (ε_i)
ω_1	9.8879e − 01	9.8879e − 01 (3.31e − 07)	9.8879e − 01 (1.74e − 06)
ω_2	3.5483e + 00	3.5484e + 00 (3.00e − 05)	3.5484e + 00 (1.63e − 05)
ω_3	2.2077e + 01	2.2088e + 01 (5.01e − 04)	2.2077e + 01 (9.35e − 08)
ω_4	6.1706e + 01	6.1928e + 01 (3.59e − 03)	6.1706e + 01 (−4.11e − 08)
ω_5	1.2090e + 02	1.2232e + 02 (1.17e − 02)	1.2090e + 02 (−1.01e − 07)
ω_6	1.9986e + 02	2.0303e + 02 (1.58e − 02)	1.9986e + 02 (−2.77e − 07)
ω_7	2.9856e + 02	3.3727e + 02 (1.30e − 01)	2.9856e + 02 (−6.17e − 07)
ω_8	4.1699e + 02	4.9326e + 02 (1.83e − 01)	4.1699e + 02 (−1.21e − 06)
ω_9	5.5517e + 02	7.1534e + 02 (2.89e − 01)	5.5517e + 02 (−2.14e − 06)
ω_{10}	7.1308e + 02	1.0162e + 03 (4.25e − 01)	7.1308e + 02 (−3.52e − 06)

The oscillator parameters are $m = 1.0\rho L$ and $k = 1.0EI/L^3$.

developed scheme), and

$$\mathbf{p} = \begin{bmatrix} \mathbf{q} \\ y \end{bmatrix}, \tag{16}$$

where $\mathbf{q}$ is the vector of generalized coordinates for the beam, and y denotes the vertical displacement of the damped oscillator. Because damping is present, the state vector approach [30] is used to determine the eigenvalues. By Introducing

$$\mathbf{z} = \begin{bmatrix} \dot{\mathbf{p}} \\ \mathbf{p} \end{bmatrix}, \tag{17}$$

Eq. (15) becomes

$$[A]\dot{\mathbf{z}} - [B]\mathbf{z} = \mathbf{0}, \tag{18}$$

where matrices $[A]$ and $[B]$ are given by

$$[A] = \begin{bmatrix} [0] & [\mathscr{M}] \\ [\mathscr{M}] & [\mathscr{C}] \end{bmatrix} \quad \text{and} \quad [B] = \begin{bmatrix} [\mathscr{M}] & [0] \\ [0] & -[\mathscr{K}] \end{bmatrix}. \tag{19}$$

Eq. (18) leads to the following $2(N+1) \times 2(N+1)$ generalized eigenvalue problem:

$$[B]\bar{\mathbf{z}} = \mu[A]\bar{\mathbf{z}}. \tag{20}$$

Because the system is damped, the complex eigenvalues μ rather than the natural frequencies will be investigated. For damped systems with complex eigenvalues, the relative error in eigenvalues is defined as

$$\varepsilon_i = \frac{|\mu_i - \mu_{i\text{ex}}|}{|\mu_{i\text{ex}}|}, \tag{21}$$

where $\mu_{i\text{ex}}$ and μ_i denote, respectively, the exact (obtained by using the finite element method and discretizing the beam into 100 equal elements) and the complex ith eigenvalues for $n = 5$ (obtained by using the finite element method or the new discretization scheme), and $|a|$ denotes the modulus of the complex number a. In Table 10 are shown the first 10 eigenvalues of the combined system. Note how well the eigenvalues obtained by using the new approach with $n = 5$ track those obtained by using the finite element method with $n = 100$ (all ε_i are within $3.66 \times 10^{-4}\%$), illustrating the accuracy of the proposed scheme. Thus, in this example, rather than solving a 402×402 generalized eigenvalue problem that is required of the finite element method when the beam is discretized into 100 equal elements (where $N = 200$), one only needs to invert a 10×10 matrix and then solve a generalized eigenvalue problem of size 22×22 when using the new method with $n = 5$ (where $N = 10$), which leads to dramatic increase in computational efficiency.

Table 10
The first 10 eigenvalues of a uniform fixed–free Euler–Bernoulli beam carrying a damped oscillator with a rigid body degree of freedom at $x_a = 0.8L$

μ_i	FEM, $n = 100$	FEM, $n = 5$ (ε_i)	New scheme, $n = 5$ (ε_i)
μ_1	−1.3740e − 01 + j1.2926e + 00	−1.3740e − 01 + j1.2926e + 00 (4.49e − 08)	−1.3740e − 01 + j1.2927e + 00 (1.10e − 06)
μ_2	−2.7285e − 01 + j3.8148e + 00	−2.7286e − 01 + j3.8148e + 00 (1.56e − 05)	−2.7285e − 01 + j3.8148e + 00 (3.66e − 07)
μ_3	−1.9920e − 03 + j2.2035e + 01	−1.9941e − 03 + j2.2046e + 01 (5.00e − 04)	−1.9920e − 03 + j2.2035e + 01 (4.95e − 09)
μ_4	−6.2473e − 02 + j6.1702e + 01	−6.3564e − 02 + j6.1923e + 01 (3.59e − 03)	−6.2473e − 02 + j6.1702e + 01 (5.92e − 08)
μ_5	−1.6548e − 01 + j1.2091e + 02	−1.7444e − 01 + j1.2233e + 02 (1.17e − 02)	−1.6548e − 01 + j1.2091e + 02 (1.88e − 07)
μ_6	−1.4426e − 01 + j1.9986e + 02	−1.4870e − 01 + j2.0302e + 02 (1.58e − 02)	−1.4426e − 01 + j1.9986e + 02 (3.53e − 07)
μ_7	−3.7274e − 02 + j2.9855e + 02	−3.1564e − 02 + j3.3727e + 02 (1.30e − 01)	−3.7274e − 02 + j2.9856e + 02 (6.38e − 07)
μ_8	−5.6681e − 03 + j4.1699e + 02	−1.0394e − 02 + j4.9326e + 02 (1.83e − 01)	−5.6681e − 03 + j4.1699e + 02 (1.21e − 06)
μ_9	−1.0181e − 01 + j5.5517e + 02	−1.0793e − 01 + j7.1534e + 02 (2.89e − 01)	−1.0181e − 01 + j5.5516e + 02 (2.19e − 06)
μ_{10}	−1.9645e − 01 + j7.1308e + 02	−1.0207e − 01 + j1.0162e + 03 (4.25e − 01)	−1.9645e − 01 + j7.1308e + 02 (3.65e − 06)

The oscillator parameters are $m = 0.5\rho L$, $k = 1.0EI/L^3$ and $c = 0.2\sqrt{EI\rho/L^2}$. All of the eigenvalues are normalized by dividing by $\sqrt{EI/(\rho L^4)}$.

Table 11
The first 10 natural frequencies of a uniform fixed–free Euler–Bernoulli beam carrying a grounded translational spring, a lumped mass, an undamped oscillator with a rigid body degree of freedom, and a grounded torsional spring at $x_{a1} = 0.2L$, $x_{a2} = 0.4L$, $x_{a3} = 0.6L$ and $x_{a4} = 0.8L$, respectively

ω_i	FEM, $n = 100$	FEM, $n = 5$ (ε_i)	New scheme, $n = 5$ (ε_i)
ω_1	2.1960e + 00	2.1960e + 00 (1.69e − 07)	2.1962e + 00 (5.78e − 05)
ω_2	4.2689e + 00	4.2690e + 00 (2.09e − 05)	4.2750e + 00 (1.42e − 03)
ω_3	2.0168e + 01	2.0175e + 01 (3.08e − 04)	2.0182e + 01 (6.57e − 04)
ω_4	5.8341e + 01	5.8505e + 01 (2.82e − 03)	5.8358e + 01 (3.02e − 04)
ω_5	1.1782e + 02	1.1913e + 02 (1.11e − 02)	1.1784e + 02 (1.49e − 04)
ω_6	1.8016e + 02	1.8217e + 02 (1.11e − 02)	1.8035e + 02 (1.04e − 03)
ω_7	2.9862e + 02	3.3702e + 02 (1.28e − 01)	2.9864e + 02 (7.76e − 05)
ω_8	3.8644e + 02	4.6828e + 02 (2.12e − 01)	3.8765e + 02 (3.12e − 03)
ω_9	5.3494e + 02	6.8615e + 02 (2.83e − 01)	5.3626e + 02 (2.47e − 03)
ω_{10}	7.0018e + 02	1.0153e + 03 (4.50e − 01)	7.0183e + 02 (2.35e − 03)

The system parameters are $k_1 = 0.8EI/L^3$, $m_1 = 0.2\rho L$, $k_2 = 0.5EI/L^3$, $m_2 = 0.1\rho L$, and $k_t = 1.0EI/L$.

Finally, consider a uniform fixed–free Euler–Bernoulli beam carrying a grounded spring, a lumped mass, an undamped oscillator with a rigid body degree of freedom, and a grounded torsional spring at $x_{a1} = 0.2L$, $x_{a2} = 0.4L$, $x_{a3} = 0.6L$, and $x_{a4} = 0.8L$, respectively. The system parameters are $k_1 = 0.8EI/L^3$, $m_1 = 0.2\rho L$, $k_2 = 0.5EI/L^3$, $m_2 = 0.1\rho L$, and $k_t = 1.0EI/L$. In Table 11 are shown the first 10 natural frequencies of the combined assembly. Note that all ε_i obtained when using the new scheme are less than 3.13×10^{-1}%, while the ε_i for $i > 6$ obtained when using the finite element method all exceed 12%. The results demonstrate that the new method remains applicable when the beam is carrying multiple lumped attachments.

A few words regarding the eigenvectors are warranted. The modal assurance criterion (MAC), defined by Allemang and Brown [31], is often used to compare two eigenvectors. It is easy to apply and does not require the mass and stiffness matrices. The MAC for the jth eigenvector is defined as

$$\gamma_j = \frac{(\phi_{1j}^{\mathrm{T}}\phi_{2j})^2}{(\phi_{1j}^{\mathrm{T}}\phi_{1j})(\phi_{2j}^{\mathrm{T}}\phi_{2j})}, \tag{22}$$

where ϕ_{1j} corresponds to the jth "exact" eigenvector (obtained by using the finite element method when the beam is divided into 100 uniform elements), and ϕ_{2j} denotes the jth eigenvector obtained with the finite element method or the proposed scheme with $n = 5$. The value of γ_j is bounded between 0 and 1. A value of 1

Table 12
The modal assurance criteria of the first 10 eigenvectors of a uniform fixed–free Euler–Bernoulli beam carrying a grounded translational spring, a lumped mass, an undamped oscillator with a rigid body degree of freedom, and a grounded torsional spring at $x_{a1} = 0.2L$, $x_{a2} = 0.4L$, $x_{a3} = 0.6L$ and $x_{a4} = 0.8L$, respectively

γ_i	FEM, $n = 5$	New scheme, $n = 5$
γ_1	9.99999e−01	9.99999e−01
γ_2	9.99999e−01	9.99989e−01
γ_3	9.99999e−01	9.99981e−01
γ_4	9.99999e−01	9.99976e−01
γ_5	9.99968e−01	9.99984e−01
γ_6	9.99733e−01	9.99884e−01
γ_7	9.94427e−01	9.99989e−01
γ_8	9.88037e−01	9.99862e−01
γ_9	9.58339e−01	9.98826e−01
γ_{10}	9.53496e−01	9.97767e−01

The system parameters are identical to those of Table 11.

Table 13
The CPU times required to generate the results of Tables 3–11

CPU time	FEM, $n = 100$	New scheme, $n = 5$
Table 3	4.61E−01	3.20E−02
Table 4	4.23E−01	3.61E−02
Table 5	4.29E−01	3.61E−02
Table 6	5.13E−01	3.20E−02
Table 7	4.35E−01	2.00E−02
Table 8	3.97E−01	2.40E−02
Table 9	4.79E−01	2.00E−02
Table 10	2.13E + 01	2.00E−02
Table 11	4.59E − 01	3.20E−02

implies a perfect correlation, and a γ_j of 0 indicates that the two eigenvectors are uncorrelated. To compute γ_j, the two eigenvectors must be of the same size. Thus, the elements in the eigenvectors must contain the translational and rotational displacements at the same node locations. In Table 12 are shown the γ_j for the first 10 eigenvectors of shown in Table 11. Note that γ_1 to γ_4 are closer to 1 when using the finite element method, and γ_5 to γ_{10} are closer to 1 when using the new discretization scheme. Numerical studies showed that the modal assurance criteria for almost all of the eigenvectors are nearly 1, and tables of modal assurance criteria for the other examples will not be presented.

The modified finite element discretization scheme allows one to compute all of the natural frequencies accurately without refining the mesh of the linear structure. To demonstrate the computation advantage one gains by using the proposed method, the MATLAB command *cputime* is utilized to obtain the CPU time. For the finite element method with 100 elements, the CPU time includes the time needed to assemble the global finite element system matrices, $[\mathscr{K}]$ and $[\mathscr{M}]$, and to solve a 200×200 generalized eigenvalue problem, assuming the linear structure is undamped and does not carry an oscillator with a rigid body degree of freedom. For the new discretization scheme with 5 elements, the cpu time includes the time needed to assemble the finite element system matrices $[K_0]$ and $[M_0]$, determine the exact $[U]$ and $[\Lambda]$ matrices, invert $[m]$ (of size 10 by 10), compute the system matrices of Eqs. (2) and (5), add the lumped attachments, and solve a 10 by 10 generalized eigenvalue problem. The required CPU times to generate the results of Tables 3–11 are shown in Table 13. Note that in all of the numerical experiments considered, the CPU times of the new scheme are at least an order of magnitude smaller. Moreover, for a linear structure carrying a damped oscillator, the CPU time required for the proposed discretization scheme is three orders of magnitude smaller. Numerical experiments clearly illustrate the computational efficiency of the proposed method.

In this paper, a new discretization scheme is proposed that can be used to obtain the eigenvalues of a combined system consisting of a linear structure carrying lumped attachments. For the same number of elements, the proposed scheme returns estimates of the higher eigenvalues that are substantially more accurate than those given by the finite element method. The new discretization algorithm allows one to determine the higher eigenvalues accurately without having to refine the mesh of the linear structure, which is required when using the finite element approach, if the same order of accuracy is needed.

4. Conclusion

A new finite element discretization scheme is proposed that can be used to accurately and efficiently determine all of the eigenvalues, especially the higher ones, of a linear structure carrying lumped attachments. The finite element mass and stiffness matrices of the linear structure are modified or updated using the exact eigensolutions of the linear structure, such that its finite element model returns modes of vibration that coincide with the exact eigendata. Once the mass and stiffness matrices have been updated, the finite element assembling technique is exploited to include the lumped attachments. Numerical experiments show that with only a few elements, the newly developed discretization scheme returns estimates of the higher eigenvalues that are nearly identical to those obtained by using a finite element model with a very fine mesh. The new method is easy to apply and computationally efficient to use. It can be used for any combination of attachments, and is valid for a combined system that is either undamped or damped.

Appendix A. Mass and stiffness matrices for a beam element

The mass and stiffness matrices for a beam element are:

$$[M_e] = \frac{\rho l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

and

$$[K_e] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix},$$

where ρ represents the mass per unit length of the beam element, l its length, E the Young's modulus, and I the area moment of inertia of the cross section. The vector of generalized coordinates (or nodal displacements) for a beam element are given by $[v_1\ \theta_1\ v_2\ \theta_2]^{\mathrm{T}}$, where (v_1, θ_1) and (v_2, θ_2) denote the deflection and slope at the left and right ends of the beam element, respectively. Finally, to construct the finite element model of an entire beam, individual element matrices are superimposed to obtain the system mass and stiffness matrices.

References

[1] R.E.D. Bishop, D.C. Johnson, *The Mechanics of Vibration*, Cambridge University Press, London, 1960.

[2] J.T. Weissenburger, Effect of local modifications on vibration characteristics of linear systems, *Journal of Applied Mechanics* 35 (1968) 327–332.

[3] R.G. Jacquot, W. Soedel, Vibration of elastic surface systems carrying dynamic elements, *Journal of Acoustical Society of America* 47 (1970) 1354–1358.

[4] R.J. Pomazal, V.W. Snyder, Local modifications of damped linear systems, *American Institute of Aeronautics and Astronautics Journal* 9 (1971) 2216–2221.

[5] R.G. Jacquot, J.D. Gibson, The effect of discrete masses and elastic supports on continuous beam natural frequencies, *Journal of Sound and Vibration* 23 (1972) 237–244.

[6] E.H. Dowell, Free vibration of an arbitrary structure in terms of component modes, *Journal of Applied Mechanics* 39 (1972) 727–732.

[7] J. Hallquist, V.W. Snyder, Linear damped vibratory structures with arbitrary support conditions, *Journal of Applied Mechanics* 40 (1973) 312–313.
[8] R.P. Goel, Free vibrations of a beam-mass system with elastically restrained ends, *Journal of Sound and Vibration* 47 (1976) 9–14.
[9] E.H. Dowell, On some general properties of combined dynamical systems, *Journal of Applied Mechanics* 46 (1979) 206–209.
[10] J.W. Nicholson, L.A. Bergman, Free vibration of combined dynamical system, *Journal of Engineering Mechanics* 112 (1986) 1–13.
[11] L. Ercoli, P.A.A. Laura, Analytical and experimental investigation on continuous beams carrying elastically mounted masses, *Journal of Sound and Vibration* 114 (1987) 519–533.
[12] J.S. Wu, T.L. Lin, Free vibration analysis of a uniform cantilever beam with point masses by an analytical-and-numerical-combined method, *Journal of Sound and Vibration* 136 (1990) 201–213.
[13] H. Abramovich, O. Hamburger, Vibration of a uniform cantilever Timoshenko beam with translational and rotational springs and with a tip mass, *Journal of Sound and Vibration* 154 (1992) 67–80.
[14] R.E. Rossi, P.A.A. Laura, D.R. Avalos, H. Larrondo, Free vibrations of Timoshenko beams carrying elastically mounted, concentrated masses, *Journal of Sound and Vibration* 165 (1993) 209–223.
[15] S. Kukla, B. Posiadała, Free vibrations of beams with elastically mounted masses, *Journal of Sound and Vibration* 175 (1994) 557–564.
[16] M. Gürgöze, On the eigenfrequencies of a cantilever beam with attached tip mass and a spring-mass system, *Journal of Sound and Vibration* 190 (1996) 149–162.
[17] G.G.G. Lueschen, L.A. Bergman, D.M. McFarland, Green's functions for uniform Timoshenko beams, *Journal of Sound and Vibration* 194 (1996) 93–102.
[18] B. Posiadała, Free vibrations of uniform Timoshenko beams with attachments, *Journal of Sound and Vibration* 204 (1997) 359–369.
[19] S. Kukla, Application of Green functions in frequency analysis of Timoshenko beams with oscillators, *Journal of Sound and Vibration* 205 (1997) 355–363.
[20] M. Gürgöze, On the alternative formulations of the frequency equation of a Bernoulli–Euler beam to which several spring–mass systems are attached in-span, *Journal of Sound and Vibration* 217 (1998) 585–595.
[21] P.D. Cha, W.C. Wong, A novel approach to determine the frequency equations of combined dynamical systems, *Journal of Sound and Vibration* 219 (1999) 689–706.
[22] J.S. Wu, H.M. Chou, A new approach for determining the natural frequencies and mode shapes of a uniform beam carrying any number of sprung masses, *Journal of Sound and Vibration* 220 (1999) 451–468.
[23] M. Gürgöze, Alternative formulations of the characteristic equation of a Bernoulli–Euler beam to which several viscously damped spring–mass systems are attached in-span, *Journal of Sound and Vibration* 223 (1999) 666–677.
[24] T.P. Chang, F.I. Chang, M.F. Liu, On the eigenvalues of a viscously damped simple beam carrying point masses and springs, *Journal of Sound and Vibration* 240 (2001) 769–778.
[25] P.D. Cha, Eigenvalues of a linear elastica carrying lumped masses, springs and viscous dampers, *Journal of Sound and Vibration* 257 (2002) 798–808.
[26] A. Berman, E.J. Nagy, Improvement of a large analytical model using test data, *American Institute of Aeronautics and Astronautics* 21 (1983) 1168–1173.
[27] A. Berman, Mass matrix correction using an incomplete set of measured modes, *American Institute of Aeronautics and Astronautics* 17 (1979) 1147–1148.
[28] F.S. Wei, Stiffness matrix correction from incomplete test data, *American Institute of Aeronautics and Astronautics* 18 (1980) 1274–1275.
[29] K.J. Bathe, *Finite Element Procedures*, Prentice-Hall, Inc., New Jersey, 1995.
[30] L. Meirovitch, *Computational Methods in Structural Dynamics*, Sijthoff & Noordhoff, The Netherlands, 1980.
[31] R.J. Allemang, D.L. Brown, A correlation for modal vector analysis, *Proceedings of IMAC I: 1st International Modal Analysis Conference*, Orlando, FL, November 1982, pp. 110–116.