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The variational iteration method for nonlinear oscillators with discontinuities

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Abstract

In this paper, He's variational iteration method (VIM) is applied to nonlinear oscillators with discontinuities. We illustrate that the VIM is very effective and convenient and does not require linearization or small perturbation. Contrary to the conventional methods, in VIM, only one iteration leads to high accuracy of the solutions. Moreover, we show that the obtained approximate solutions are valid for the whole solution domain and the approximations are uniformly valid not only for small parameters, but also for very large parameters.

1. Introduction

This paper considers the following general nonlinear oscillators with discontinuities [1]:

$$u'' + f(u, u', u'') = 0, \tag{1}$$

with initial conditions $u(0) = A$ and $u'(0) = 0$. Here f is a known discontinuous function.

If there is no small parameter in the equation, the traditional perturbation methods cannot be applied directly. Recently, considerable attention has been directed towards the analytical solutions for nonlinear equations without possible small parameters. The traditional perturbation methods have many shortcomings, and they are not valid for strongly nonlinear equations. To overcome the shortcomings, many new techniques have appeared in open literature, for example, d-perturbation method [2,3], energy balance method [4,5], variational iteration method (VIM) [6–11], homotopy perturbation method [12–19], bookkeeping parameter perturbation method [20], just to name a few, a review on some recently developed nonlinear analytical methods can be found in detail in Refs. [21–23]. The homotopy perturbation method [17] and the modified Lindstedt–Poincare method [24–27] were first applied to the nonlinear oscillators with discontinuities, and the first-order approximate solution is of high accuracy. New interpretations of homotopy perturbation method were also discussed by He and others [28–30].

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The VIM was first proposed by He [31] and used to give approximate solutions of the problem of seepage flow in porous media with fractional derivatives. In this paper, we will show how to solve nonlinear oscillators with discontinuous terms by the VIM, which leads to a very rapid convergence of the solution series, in the most cases only one iteration leads to high accuracy of the solution, providing an effective and convenient mathematical tool for nonlinear equations. The VIM is useful to obtain exact and approximate solutions of linear and nonlinear differential equations. There is no need of linearization or discretization, and large computational work and round-off errors is avoided. It has been used to solve effectively, easily and accurately a large class of nonlinear problems with approximation [32,33]. Applications of VIM to the nonlinear oscillators can be found in Refs. [34,35].

2. Solution procedures

We re-write Eq. (1) in the following form:

$$u'' + \Omega^2 u = F(u), \quad F(u) = \Omega^2 u - f(u). \tag{2}$$

We consider that the angular frequency of the oscillator is Ω, and we choose the trial function $u_0(t) = A \cos \Omega t$. The angular frequency Ω is identified with the physical understanding that no secular terms should appear in $u_1(t)$, which leads to

$$\int_0^T \cos \Omega t \left[\Omega^2 u_0 - f(u_0)\right] \mathrm{d}t = 0, \quad T = \frac{2\pi}{\Omega}. \tag{3}$$

From this equation, Ω can easily be found. It should be specially pointed out that the more accurate the identification of the multiplier, the more faster the approximations converge to its exact solution, and for this reason, we identify the multiplier from Eq. (2) rather than Eq. (1).

According to the VIM, we can construct a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left\{u_n''(\tau) + \Omega^2 u_n(\tau) - \tilde{F}_n\right\} \mathrm{d}\tau, \tag{4}$$

where λ is a general Lagrange multiplier [36], which can be identified optimally via the variational theory [37], the subscript n denotes the nth-order approximation, $\tilde{F}_n$ is considered as a restricted variation [38], i.e., $\delta\tilde{F}_n = 0$. Under this condition, its stationary conditions of the above correction functional can be written as follows:

$$\begin{aligned} &\lambda''(\tau) + \Omega^2 \lambda(\tau) = 0, \\ &\lambda(\tau)|_{\tau=t} = 0, \\ &1 - \lambda'(\tau)|_{\tau=t} = 0. \end{aligned} \tag{5}$$

The Lagrange multiplier, therefore, can be readily identified by

$$\lambda = \frac{1}{\Omega} \sin \Omega(\tau - t), \tag{6}$$

which leads to following iteration formula:

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{\Omega} \sin \Omega(\tau - t)\left\{u_n''(\tau) + f_n\right\} \mathrm{d}\tau. \tag{7}$$

As we will see in the forthcoming illustrative examples, we always stop at the first-order approximation, and the obtained approximate and accurate solution is valid for the whole solution domain.

3. Applications

In order to assess the advantages and the accuracy of the VIM, we will consider the following three examples.

Example 1. Let us consider the following nonlinear oscillators with discontinuities [1]:

$$u'' + u + \varepsilon u|u| = 0, \tag{8}$$

with initial conditions $u(0) = A$ and $u'(0) = 0$.

Here the discontinuous function is $f(u) = u + \varepsilon u|u|$. From Eq. (3), we can determine the angular frequency:

$$\int_0^T \cos \Omega t\left[\Omega^2 A \cos \Omega t - (A \cos \Omega t + \varepsilon A \cos \Omega t|A \cos \Omega t|)\right] \mathrm{d}t = 0, \quad T = \frac{2\pi}{\Omega}. \tag{9}$$

Noting that $|\cos \Omega t| = \cos \Omega t$ when $-\pi/2 \leqslant \Omega t \leqslant \pi/2$, and $|\cos \Omega t| = -\cos \Omega t$ when $\pi/2 \leqslant \Omega t \leqslant 3\pi/2$, so we write Eq. (9) in the following form:

$$\int_{-\pi/2\Omega}^{\pi/2\Omega} \left[\left(\Omega^2 - 1\right) A \cos^2 \Omega t - \varepsilon A^2 \cos^3 \Omega t\right] \mathrm{d}t + \int_{\pi/2\Omega}^{3\pi/2\Omega} \left[\left(\Omega^2 - 1\right) A \cos^2 \Omega t + \varepsilon A^2 \cos^3 \Omega t\right] \mathrm{d}t = 0. \tag{10}$$

From the above equation, one can easily conclude that

$$\Omega = \sqrt{1 + \frac{8}{3\pi}\varepsilon A}. \tag{11}$$

We re-write Eq. (7) in the following form:

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{\Omega} \sin \Omega(\tau - t)\left\{u_n''(\tau) + u_n(\tau) + \varepsilon u_n(\tau)\left|u_n(\tau)\right|\right\} \mathrm{d}\tau. \tag{12}$$

By the above iteration formula, we can calculate the first-order approximation:

$$u_1(t) = \begin{cases} A \cos \Omega t + \int_0^t \frac{1}{\Omega} \sin \Omega(\tau - t)\left\{\left(1 - \Omega^2\right) A \cos \Omega \tau + \varepsilon A^2 \cos^2 \Omega \tau\right\} \mathrm{d}\tau, & -\frac{\pi}{2} \leqslant \Omega t \leqslant \frac{\pi}{2}, \\ A \cos \Omega t + \int_0^t \frac{1}{\Omega} \sin \Omega(\tau - t)\left\{\left(1 - \Omega^2\right) A \cos \Omega \tau - \varepsilon A^2 \cos^2 \Omega \tau\right\} \mathrm{d}\tau, & \frac{\pi}{2} \leqslant \Omega t \leqslant \frac{3\pi}{2}, \end{cases} \tag{13}$$

which yields

$$u_1(t) = \begin{cases} A \cos \Omega t + \frac{1}{2\Omega} A\left(\Omega^2 - 1\right) t \sin \Omega t + \frac{\varepsilon A^2}{6\omega^2}(\cos 2\Omega t + 2 \cos \Omega t) - \frac{\varepsilon A^2}{2\Omega^2}, & -\frac{\pi}{2} \leqslant \Omega t \leqslant \frac{\pi}{2}, \\ A \cos \Omega t + \frac{1}{2\Omega} A\left(\Omega^2 - 1\right) t \sin \Omega t - \frac{\varepsilon A^2}{6\omega^2}(\cos 2\Omega t + 2 \cos \Omega t) + \frac{\varepsilon A^2}{2\Omega^2}, & \frac{\pi}{2} \leqslant \Omega t \leqslant \frac{3\pi}{2}, \end{cases} \tag{14}$$

where the angular frequency Ω is defined as Eq. (11).

The above results are in good agreement with the results obtained by the homotopy perturbation reported in Ref. [17].

In order to compare with traditional perturbation solution, we write Nayfeh's result [1]:

$$u = A \cos\left(1 + \frac{4}{3\pi}\varepsilon A\right) t + \cdots, \tag{15}$$

which is valid only for small parameter.

Example 2. Considering the following nonlinear oscillator with discontinuities [17]:

$$\ddot{u} + \beta u^3 + \varepsilon u|u| = 0, \tag{16}$$

with initial conditions $u(0) = A$ and $u'(0) = 0$.

Here the discontinuous function is $f(u) = \beta u^3 + \varepsilon u|u|$. Now we begin with the initial approximation $u_0(t) = A\cos\Omega t$. From Eq. (3), we can determine the angular frequency easily as

$$\int_0^T \cos\,\Omega t\left[\Omega^2 A\,\cos\,\Omega t - \beta(A\,\cos\,\Omega t)^3 - \varepsilon A\,\cos\,\Omega t|A\,\cos\,\Omega t|\right]\mathrm{d}t = 0, \quad T = \frac{2\pi}{\Omega}. \tag{17}$$

Similar to Example 1, we have

$$\int_{-\pi/2\Omega}^{\pi/2\Omega}\left[\Omega^2 A\cos^2\Omega t - \beta A^3\cos^4\Omega t - \varepsilon A^2\cos^3\Omega t\right]\mathrm{d}t + \int_{\pi/2\Omega}^{3\pi/2\Omega}\left[\Omega^2 A\cos^2\Omega t - \beta A^3\cos^4\Omega t + \varepsilon A^2\cos^3\Omega t\right]\mathrm{d}t = 0. \tag{18}$$

For $\varepsilon = 0$, we obtain

$$\Omega = \sqrt{\frac{3}{4}\beta A^2 + \frac{8}{3\pi}\varepsilon A}, \tag{19}$$

and its period is given by

$$T = \frac{2\pi}{\Omega} = \frac{2\pi}{\sqrt{\frac{3}{4}\beta A^2 + \frac{8}{3\pi}\varepsilon A}}. \tag{20}$$

In the case in which $\varepsilon = 0$, its period can be written as

$$T = \frac{4\pi}{A\sqrt{3\beta}} = 7.25\beta^{-1/2}A^{-1}. \tag{21}$$

Its exact period can be readily obtained, and it reads [39]

$$T_{\mathrm{ex}} = 7.4164\beta^{-1/2}A^{-1}. \tag{22}$$

Therefore, the maximal relative error is less than 2.2% for $\beta > 0$.
We re-write Eq. (7) in the following form:

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{\Omega}\sin\,\Omega(\tau - t)\left\{u_n''(\tau) + \beta u_n^3(\tau) + \varepsilon u_n(\tau)\left|u_n(\tau)\right|\right\}\mathrm{d}\tau. \tag{23}$$

By the above iteration formula, we can calculate the first-order approximation:

$$u_1(t) = \begin{cases} A\,\cos\,\Omega t + \int_0^t \frac{1}{\Omega}\sin\,\Omega(\tau - t)\left\{u_0''(\tau) + \beta u_0^3(\tau) + \varepsilon u_0^2\right\}\mathrm{d}\tau, & -\frac{\pi}{2}\leqslant\Omega t\leqslant\frac{\pi}{2}, \\ A\,\cos\,\Omega t + \int_0^t \frac{1}{\Omega}\sin\,\Omega(\tau - t)\left\{u_0''(\tau) + \beta u_0^3(\tau) - \varepsilon u_0^2\right\}\mathrm{d}\tau, & \frac{\pi}{2}\leqslant\Omega t\leqslant\frac{3\pi}{2}. \end{cases} \tag{24}$$

Ultimately, we obtain the following results:

$$u_1(t) = \begin{cases} A\,\cos\,\Omega t + \frac{1}{8\Omega^2}A\left\{\left(8\Omega^2 + 6\varepsilon A - 6\beta A^2\right)\cos\,\Omega t + \left(4\beta A^2 - 4A\varepsilon - 4\Omega^2\right)\cos\,2\Omega t \right. \\ \quad \left. + \left(2A\varepsilon - 2A^2\beta\right)\cos\,3\Omega t + A^2\beta\,\cos\,4\Omega t - 4A\varepsilon + 3\beta A^2 - 4\Omega^2\right\}, \\ \quad -\frac{\pi}{2}\leqslant\Omega t\leqslant\frac{\pi}{2}, \\ A\,\cos\,\Omega t + \frac{1}{8\Omega^2}A\left\{\left(8\Omega^2 - 6\varepsilon A - 6\beta A^2\right)\cos\,\Omega t + \left(4\beta A^2 + 4A\varepsilon - 4\Omega^2\right)\cos\,2\Omega t \right. \\ \quad \left. + \left(-2A^2\beta - 2A\varepsilon\right)\cos\,3\Omega t + A^2\beta\,\cos\,4\Omega t + 4A\varepsilon + 3\beta A^2 - 4\Omega^2\right\}, \\ \quad \frac{\pi}{2}\leqslant\Omega t\leqslant\frac{3\pi}{2}. \end{cases} \tag{25}$$

The above results are in the very good agreement with the results obtained by the homotopy perturbation reported in Ref. [17].

Example 3. This example considers the following nonlinear oscillators with discontinuities [24]:

$$u'' + (u) = 0, \tag{26}$$

with initial conditions $u(0) = A$ and $u'(0) = 0$, and sign(u) is defined by

$$(u) = \begin{cases} 1, & u > 0. \\ -1, & u \leqslant 0. \end{cases} \tag{27}$$

Here the discontinuous function is $f(u) = \text{sign}(u)$. There is no small parameter in the equation, so the traditional perturbation methods cannot be applied directly. From Eq. (3), we can determine the angular frequency easily:

$$\int_0^T \cos \Omega t\left[\Omega^2 u_0 - (u_0)\right] \mathrm{d}t = \int_0^T \cos \Omega t\left[\Omega^2 A \cos \Omega t - (A \cos \Omega t)\right] \mathrm{d}t = 0, \quad T = \frac{2\pi}{\Omega}. \tag{28}$$

Noting that $|\cos \Omega t| = \cos \Omega t$ when $-\pi/2 \leqslant \Omega t \leqslant \pi/2$, and $|\cos \Omega t| = -\cos \Omega t$ when $\pi/2 \leqslant \Omega t \leqslant 3\pi/2$, we write Eq. (28) in the form

$$\int_{-\pi/2\Omega}^{\pi/2\Omega} \cos \Omega t\left[\Omega^2 A \cos \Omega t - 1\right] \mathrm{d}t + \int_{\pi/2\Omega}^{3\pi/2\Omega} \cos \Omega t\left[\Omega^2 A \cos \Omega t + 1\right] \mathrm{d}t = 0. \tag{29}$$

From the above equation, the angular frequency can easily be found:

$$\Omega = \frac{2}{\sqrt{\pi A}}, \tag{30}$$

and its approximate period

$$T = \frac{2\pi}{\omega} = \pi\sqrt{\pi A} = 5.56\sqrt{A}. \tag{31}$$

Its exact period can be easily obtained, and is given by [24]:

$$T_{\text{ex}} = 5.66\sqrt{A}. \tag{32}$$

The 1.76% accuracy is remarkably good in view of the first-order approximate solution. We re-write Eq. (7) in the following form:

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{\Omega} \sin \Omega(\tau - t)\left\{u_n''(\tau) + (u_n)\right\} \mathrm{d}\tau. \tag{33}$$

By the above iteration formula, we can calculate the first-order approximation:

$$u_1(t) = \begin{cases} A \cos \Omega t + \int_0^t \frac{1}{\Omega} \sin \Omega(\tau - t)\left\{u_0''(\tau) + 1\right\} \mathrm{d}\tau, & -\frac{\pi}{2} \leqslant \Omega t \leqslant \frac{\pi}{2}, \\ A \cos \Omega t + \int_0^t \frac{1}{\Omega} \sin \Omega(\tau - t)\left\{u_0''(\tau) - 1\right\} \mathrm{d}\tau, & \frac{\pi}{2} \leqslant \Omega t \leqslant \frac{3\pi}{2}. \end{cases} \tag{34}$$

We, therefore, obtain the following results:

$$u_1(t) = \begin{cases} \frac{1}{2\Omega^2}\left(4A\Omega^2 \cos \Omega t - A\Omega^2 \cos 2\Omega t - A\Omega^2 + 2 \cos \Omega t - 2\right), & -\frac{\pi}{2} \leqslant \Omega t \leqslant \frac{\pi}{2}, \\ \frac{1}{2\Omega^2}\left(4A\Omega^2 \cos \Omega t - A\Omega^2 \cos 2\Omega t - A\Omega^2 - 2 \cos \Omega t + 2\right), & \frac{\pi}{2} \leqslant \Omega t \leqslant \frac{3\pi}{2}. \end{cases} \tag{35}$$

The above results are in good agreement with the results obtained by modified Lindstedt–Poincare method reported in Ref. [24] and homotopy perturbation method reported in Ref. [19].

4. Conclusions

He's variational iteration method (VIM), for the first time, was applied to nonlinear oscillators with discontinuities. We demonstrated the accuracy and efficiency of the method by solving some examples. Moreover, we showed that the obtained solutions are valid for the whole domain. Furthermore, we concluded that discontinuous function had no tangible effect on the efficiency of the method.

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