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Subharmonic resonance and transition to chaos of nonlinear oscillators with a combined softening and hardening nonlinearities

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Abstract

The concern of this work is the 1/3 sub-harmonic resonance response and transition to chaos in the response of harmonically driven single degree of freedom nonlinear oscillators with a combined static and inertia nonlinearities. Approximate analytical solutions to the 1/3 sub-harmonic resonance curves are obtained using the harmonic balance (HB) method and the multiple scales (MMS) perturbation method. Stability analyses of the obtained approximate solutions were used to determine zones of chaotic behavior in the primary frequency response curve. The obtained analytical results were verified for selected values of system parameters using computer simulations and with the aid of time histories, phase planes, Poincare map, FFT and Lyapunov exponents.
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1. Introduction

The interest of the this work is to study 1/3 subharmonic resonance in the class of nonlinear oscillators with static and inertia nonlinearities, given by

$$\ddot{u} + \delta\dot{u} + u + \varepsilon_1(u^2\ddot{u} + u\dot{u}^2) + \varepsilon_2 u^3 = P\cos(\Omega t), \tag{1}$$

where δ, ε_1, ε_2, P and Ω are constant positive parameters.

Examples of physical systems modeled by this oscillator include: a rotating flexible blade, an immersed beam and parametrically excited structures, e.g. [1–5].

For the immersed beam studied in Refs. [2,4], the parameters ε_1 and ε_2 in Eq. (1) depend on the physical parameters of the system, such as; fluid depth, fluid density, mass ratio and position, and the behavior of the nonlinear oscillator will change from softening to hardening depending on $\varepsilon_1/\varepsilon_2$ ratio [1].

It is well known that a harmonically driven single degree of freedom (sdof) oscillator with cubic hardening static nonlinearity, a subharmonic resonance, of amplitude larger than that of the fundamental response, may

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get excited for the right system parameters (e.g. forcing amplitude, system damping and initial conditions) when the forcing frequency Ω is in the range $\omega/3$, where ω is the linear oscillator natural frequency.

Stability analysis and numerical simulation studies of sub-harmonic resonances in various types of harmonically excited Duffing oscillators have shown that chaotic motions are associated with the loss of stability of a sub- or super-harmonic resonance, i.e. one characteristic precursor to chaotic motions is the appearance and then loss of stability of a sub- or super-harmonic resonance, e.g. Refs. [6–17].

Studies dealing with approximate analytic-numeric stability and bifurcation analyses of sub-harmonic resonance and its transition to chaos in sdof nonlinear oscillators have been limited, except in few cases, such as; first-order approximations statically stable and unstable oscillators with only static nonlinearity [12–17].

These studies have shown that the zone of a chaotic motion in the response, of various types of the classical Duffing oscillator, is found in a narrow zone just before entering the sub/super harmonic resonance region.

For a more general version of the oscillator considered in the present work, results were obtained for a cantilever beam when subjected to primary and principal parametric excitations using the method of multiple scales to determine the steady state responses and their stability. Amplitude and phase modulation are used to detect chaos and unbounded motions in the instability regions of the periodic solutions [18].

Consequently, analytical approaches have been used to seek a link between stability limits of sub-harmonic solutions and the onset of chaotic motions through a continuous sequence period doubling bifurcations [12–15].

The objective of this work is to seek an approximate analytical solution for the subharmonic resonance of order 1/3 and its stability in the nonlinear oscillator described in Eq. (1). Approximate analytical solutions for the primary and 1/3 sub-harmonic resonance curves are obtained using harmonic balance (HB) method and method of multiple scales (MMS). Stability analyses of the obtained approximate solutions are carried out to predict the regions on the primary resonance curves at which the subharmonic response may lose stability. Computer simulations using time histories, phase planes, Poincare map, FFT and Lyapunov exponents, were used to verify theoretical results for selected values of system parameters.

2. Analysis

In order to obtain an approximate solution to the oscillator described in Eq. (1), it is convenient to rewrite this equation in the form

$$\Omega^2\ddot{u} + \Omega\delta\dot{u} + u + \varepsilon_1\Omega^2u^2\ddot{u} + \varepsilon_1\Omega^2u\dot{u}^2 + \varepsilon_2u^3 = P\cos(T+\phi). \tag{2}$$

Here the phase shift ϕ, has been added such that one can obtain an approximate solution for the primary response using single term only, i.e. cosine term only. The authors in Refs. [1,2,5] studied the primary steady state response and bifurcation and transition to chaos of the nonlinear oscillators described in Eq. (2).

2.1. Subharmonic response using harmonic balance method (HBM)

According to the HBM, an approximate solution of the subharmonic response of Eq. (2), takes the form

$$u(t) = A_1\cos(T) + A_{1/3}\cos(T/3) + B_{1/3}\sin(T/3). \tag{3}$$

Substituting Eq. (3) into Eq. (2), and comparing the coefficients of cos(*T*), sin(*T*), cos(*T*/3), sin(*T*/3), one obtains the following equations:

$$A_1\left\{1-\Omega^2+\left(\frac{3}{2}\varepsilon_2-\frac{5}{9}\varepsilon_1\Omega^2)(A_{1/3}^2+B_{1/3}^2\right)\right\}+\frac{A_1^3}{4}(3\varepsilon_2-2\varepsilon_1\Omega^2)$$
$$+A_{1/3}^3\left(\frac{\varepsilon_2}{4}-\frac{\varepsilon_1}{18}\Omega^2\right)+A_{1/3}B_{1/3}^2\left(\frac{\varepsilon_1}{6}\Omega^2-\frac{3\varepsilon_2}{4}\right)=P\cos\phi, \tag{4}$$

$$\Omega\delta A_1+B_{1/3}^2\left(\frac{\varepsilon_2}{4}-\frac{\varepsilon_1}{18}\Omega^2\right)+B_{1/3}A_{1/3}^2\left(\frac{\varepsilon_1}{6}\Omega^2-\frac{3\varepsilon_2}{4}\right)=P\sin\phi, \tag{5}$$

$$A_{1/3}\left\{1-\frac{\Omega^2}{9}+\frac{3\varepsilon_2}{4}(2A_1^2+((A_{1/3}^2+B_{1/3}^2))-\frac{5\varepsilon_1\Omega^2}{18}(2A_1^2+((A_{1/3}^2+B_{1/3}^2)\right\}$$
$$+A_1\left\{\left(\frac{3}{4}\varepsilon_2+\frac{\varepsilon_1\Omega^2}{6}\right)(A_{1/3}^2-B_{1/3}^2)\right\}+\frac{\delta\Omega}{3}B_{1/3}=0, \tag{6}$$

$$B_{1/3}\left\{1-\frac{\Omega^2}{9}+\frac{3\varepsilon_2}{4}(2A_1^2+((A_{1/3}^2+B_{1/3}^2))-\frac{\varepsilon_1\Omega^2}{18}(10A_1^2+(A_{1/3}^2+B_{1/3}^2)\right\}$$
$$+B_{1/3}A_{1/3}A_1\left(\frac{\varepsilon_1\Omega^2}{3}-\frac{3\varepsilon_2}{2}\right)-\frac{\delta\Omega}{3}A_{1/3}=0. \tag{7}$$

Eqs. (4)–(7), can be reduced to three nonlinear algebraic equations "by adding the squares of Eqs. (4) and (5), i.e. by eliminating the phase ϕ", and solved numerically for A_1, $A_{1/3}$ and $B_{1/3}$.

In this paper, the three Equations were solved by using an iterative numerical technique, i.e. by assuming an initial guess for the three unknowns A_1, $A_{1/3}$ and $B_{1/3}$, for a given system parameters Ω, δ, ε_1, ε_2 and P.

It is worth mentioning that for some points on the response curve, there were some numerical difficulties in obtaining solution "divergence" for the subharmonic resonance and no convergence was achieved.

Based on results of other researchers [13,19,20], and for different types of oscillators, the value of the A_1 in the assumed solution in Eq. (3), can be obtained from the linear solution of the nonlinear oscillator, i.e. $A_1 \approx (-9P/8\Omega^2)$, since at the region of interest near $\Omega \cong 3$ the value of A_1 is almost the same as that of the linear oscillator, as one can see from Figs. 1–4, regardless the values of ε_1 and ε_2 the primary response obtained using the HBM, is nearly the same as that obtained from $A_1 = (-9P/8\Omega^2)$.

In the present work the same approach was adopted to approximate the value of A_1, i.e. $A_1 = (-9P/8\Omega^2)$. Accordingly, Eqs. (6) and (7) are re-written in the following form:

$$A_{1/3}=-\frac{A_1\left\{\left(\frac{3}{4}\varepsilon_2+\frac{\varepsilon_1\Omega^2}{6}\right)(A_{1/3}^2-B_{1/3}^2)\right\}+\frac{\delta\Omega}{3}B_{1/3}}{\left\{1-\frac{\Omega^2}{9}+\frac{3\varepsilon_2}{4}(2A_1^2+((A_{1/3}^2+B_{1/3}^2))-\frac{5\varepsilon_1\Omega^2}{18}(2A_1^2+((A_{1/3}^2+B_{1/3}^2)\right\}}, \tag{8}$$

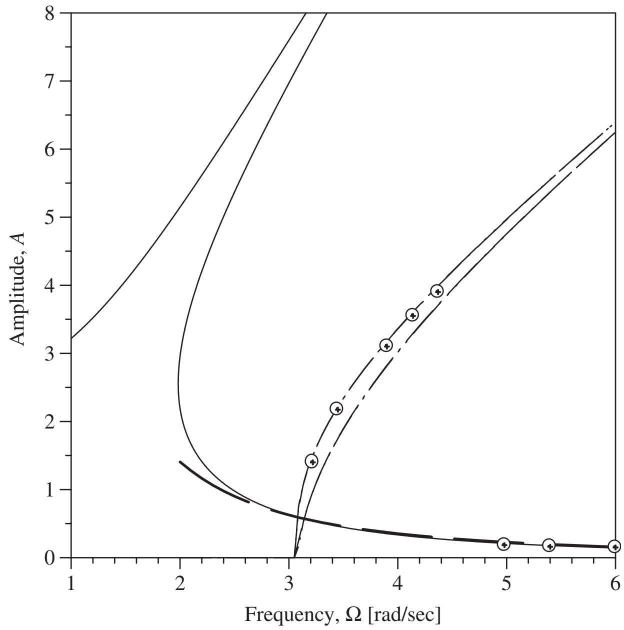

Fig. 1. Primary and subharmonic response. $P=5$, $\delta=0.01$, $\varepsilon_1=0$, $\varepsilon_2=0.1$, $\varepsilon=1$: ——— primary response, –·–·– subharmonic response using HB and MMS, —— linear amplitude, ⊕ numerical solution.

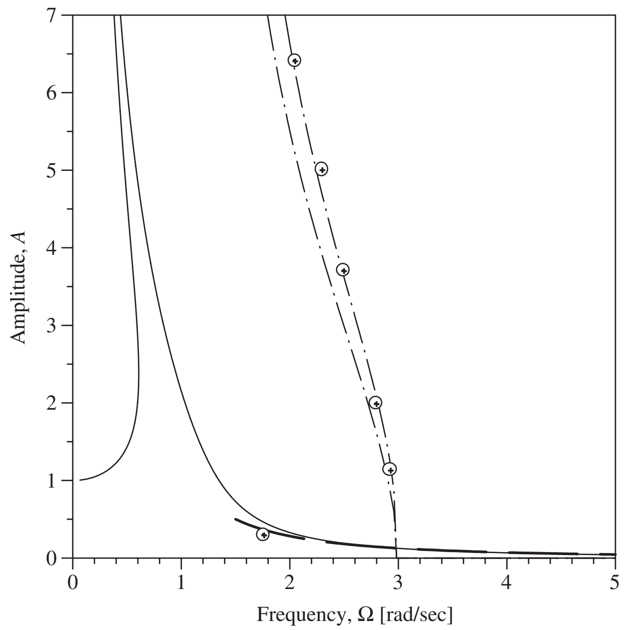

Fig. 2. Primary and subharmonic response. $P = 1$, $\delta = 0.01$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0$, $\varepsilon = 1$: ——— primary response, -·-·-·- subharmonic response using HB and MMS, —— linear amplitude, ⊕ numerical solution.

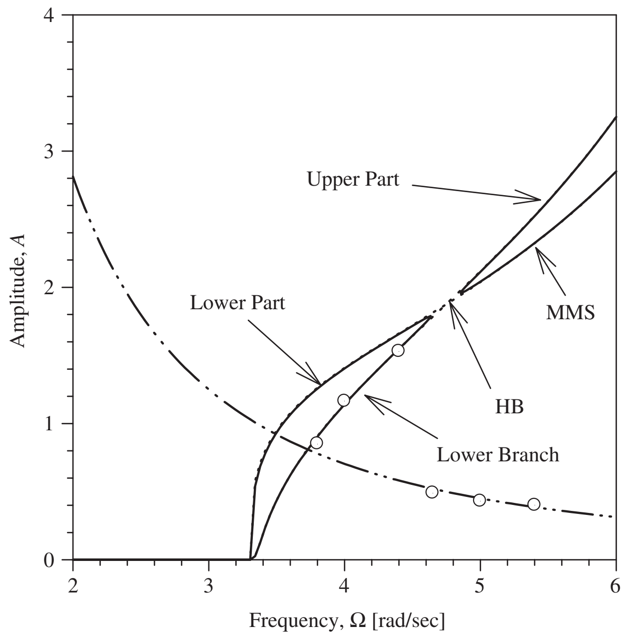

Fig. 3. Subharmonic response. $P = 10$, $\delta = 0.02$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 1$, $\varepsilon = 1$: ·············· subharmonic response using HB, ——— subharmonic response using MMS, –· ·–· · linear amplitude, ⊕ numerical solution.

$$B_{1/3} = \frac{\dfrac{\delta\Omega}{3}A_{1/3} - B_{1/3}A_{1/3}A_1\left(\dfrac{\varepsilon_1\Omega^2}{3} - \dfrac{3\varepsilon_2}{2}\right)}{\left\{1 - \dfrac{\Omega^2}{9} + \dfrac{3\varepsilon_2}{4}(2A_1^2 + ((A_{1/3}^2 + B_{1/3}^2)) - \dfrac{\varepsilon_1\Omega^2}{18}(10A_1^2 + (A_{1/3}^2 + B_{1/3}^2)\right\}}. \tag{9}$$

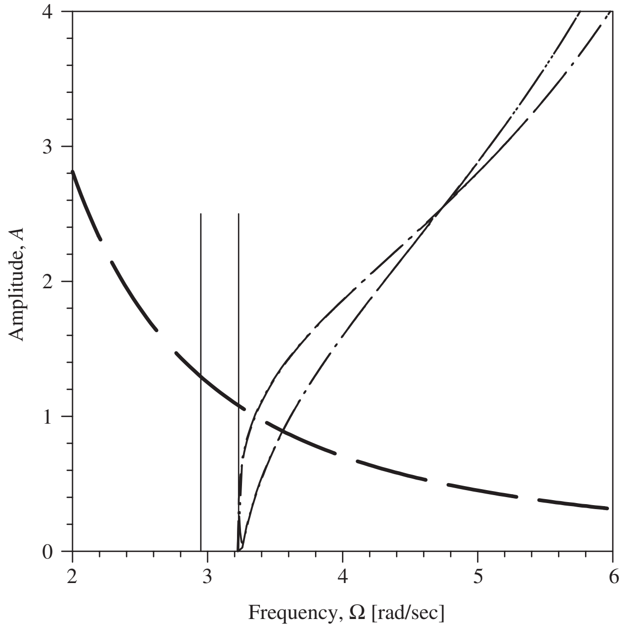

Fig. 4. Subharmonic response. $P = 10$, $\delta = 0.01$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.5$, $\varepsilon = 1$ ·············· subharmonic response using HB, -·-·-·- subharmonic response using MMS, ——- linear amplitude. For $\Omega = 2.95$, $\lambda_1 = 1.721 \times 10^{-3}$, $\lambda_2 = 0$ and $\lambda_3 = -1.733 \times 10^{-3}$, $d_f = 2.99$.

The solutions for $A_{1/3}$ and $B_{1/3}$ were obtained, as mentioned, by an iterative numerical technique, and for a given system parameters Ω, δ, ε_1, ε_2 and P.

2.2. Subharmonic response using method of multiple scales (MMS)

In order to apply the MMS perturbation technique, it is necessary to scale Eq. (2), i.e. by introducing a small gauge parameter ε. Accordingly, Eq. (2), can be written as

$$\ddot{u} + \varepsilon\delta\dot{u} + u + \varepsilon\varepsilon_1(u^2\ddot{u} + u\dot{u}^2) + \varepsilon\varepsilon_2 u^3 = P\cos\Omega t, \tag{10}$$

where ε is a small positive parameter ($0<\varepsilon\leqslant 1$). According to the MMS method [2], one defines a number of time scales $T_n = \varepsilon^n t$, $n = 0,1,2,\ldots$, where $T_0 = t$ is the fast time scale on which the main oscillatory behavior of the response occurs, and T_n, $n\geqslant 1$ are slow time scales on which the amplitude and phase modulations, caused by the nonlinearity, damping and resonance, take place. Then upon expressing the time derivatives in terms of the new time scales T_n become

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots, \tag{11}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(2D_0D_2 + D_1^2) + \ldots, \tag{12}$$

where

$$D_n = \frac{\partial}{\partial T_n}.$$

Also, the assumed series expansion for the dependent variables, take the form

$$u(t,\varepsilon) = u_0(T_0, T_1, T_2, \ldots) + \varepsilon u_1(T_0, T_1, T_2, \ldots) + \varepsilon^2 u_2(T_0, T_1, T_2, \ldots) + O(\varepsilon^3). \tag{13}$$

As mentioned above, the interest here is in the subharmonic resonance of order 1/3, and accordingly a detuning parameter σ, which measure the nearness of Ω to 3, such that

$$\Omega^2 = 9 + \varepsilon\sigma. \tag{14}$$

Substituting Eqs. (11)–(14) into Eq. (10) and equating the equal powers for ε, one obtains

$$D_0^2 u_0 + \frac{1}{9} u_0 = \frac{P}{2}(e^{iT_0} + e^{-iT_0}), \tag{15}$$

$$D_0^2 u_1 + \frac{1}{9} u_1 = -2D_0 D_1 u_0 - \delta D_0 u_0 - \sigma u_0 - (D_0 u_0)^2 \varepsilon_1 u_0 - D_0^2 u_0 \varepsilon_1 u_0^2 - \varepsilon_2 u_0^3, \tag{16}$$

$$\begin{aligned} D_0^2 u_2 + \frac{1}{9} u_2 = & -2D_0 D_2 u_0 - 2D_0 D_1 u_1 - D_1^2 u_0 - \delta(D_0 u_1 + D_1 u_0) \\ & - 2\varepsilon_1 u_0 (D_0 u_0 D_0 u_1 + D_0 u_0 D_1 u_0)\varepsilon_1 u_0 - \varepsilon_1 u_0^2 (2D_0 D_1 u_0 + D_0^2 u_1) \\ & - \sigma u_1 - 2\varepsilon_1 u_0 u_1 D_0^2 u_0 - 3\varepsilon_2 u_0^2 u_1. \end{aligned} \tag{17}$$

The solution of Eq. (15), i.e. the homogeneous and particular solution can be expressed as

$$u_0 = A(T_1, T_2)e^{1/3\Omega T_0} + \Lambda e^{-i\Omega T_0} + \text{cc}, \tag{18}$$

where cc stands for complex conjugate, $A = (a/2)e^{i\phi}$ and $\Lambda = -(9P/16\Omega^2)$. Upon substituting Eq. (18) into Eq. (16) and eliminating the secular terms, one obtains

$$\begin{aligned} & -\frac{2}{3} iD_1 A - \frac{1}{3} i\delta\Omega A - \sigma A + \left(\frac{2}{9}\varepsilon_1 \Omega^2 - 3\varepsilon_2\right) A^2 \overline{A} + \left(\frac{2}{3}\varepsilon_1 \Omega^2 - 3\varepsilon_2\right) \overline{A}^2 \Lambda \\ & + \left(\frac{20}{9}\varepsilon_1 \Omega^2 - 6\varepsilon_2\right) \Lambda^2 A = 0. \end{aligned} \tag{19}$$

Substituting $A = (a/2)e^{i\beta}$, $\bar{A} = (a/2)e^{-i\beta}$, $D_1 A = (a'/2)e^{i\beta} + i(a/2)\beta' e^{i\beta}$ into Eq. (19), and separating real and imaginary parts, one obtains

$$a' = \frac{3}{4}\Lambda a^2 \left(3\varepsilon_2 - \frac{2}{3}\varepsilon_1 \Omega^2\right) \sin(3\beta) - \frac{1}{2}\delta\Omega a, \tag{20}$$

$$\begin{aligned} a\beta' = & \frac{3}{2}\sigma a + \frac{3}{8}\left(3\varepsilon_2 - \frac{2}{9}\varepsilon_1 \Omega^2\right) a^3 + \frac{3}{4}\left(3\varepsilon_2 - \frac{2}{3}\varepsilon_1 \Omega^2\right) \Lambda a^2 \cos(3\beta) \\ & + \left(9\varepsilon_2 - \frac{10\varepsilon_1}{3}\Omega^2\right) \Lambda^2 a. \end{aligned} \tag{21}$$

Equating a' and β' to zero, Eqs. (20) and (21) take the form

$$\delta\Omega = \frac{3}{2}\Lambda a^2 \left(3\varepsilon_2 - \frac{2}{3}\varepsilon_1 \Omega^2\right) \sin(3\beta), \tag{22}$$

$$\frac{3}{2}\sigma + \frac{3}{8}\left(3\varepsilon_2 - \frac{2}{9}\varepsilon_1 \Omega^2\right) a^2 + \left(9\varepsilon_2 - \frac{10}{3}\varepsilon_1 \Omega^2\right) \Lambda^2 = \frac{3}{4}\left(3\varepsilon_2 - \frac{2}{3}\varepsilon_1 \Omega^2\right) \Lambda a \cos(3\beta). \tag{23}$$

The steady state subharmonic response of order 1/3 can be obtained from Eqs. (22) and (23) by making use of the trigonometric identity $\sin^2(3\beta) + \cos^2(3\beta) = 1$.

Results obtained for the subharmonic response from using HB and MMS were similar and are of the same qualitative nature.

2.3. Stability analysis

The stability of the subharmonic response can be examined by introducing a small perturbation to solutions obtained from Eqs. (20) and (21), [21], i.e. by substituting

$$a = a_0 + a_1, \tag{24}$$

$$\beta = \beta_0 + \beta_1, \tag{25}$$

where a_0 and β_0 represent the steady state solution and a_1 and β_1 represent the perturbation. Substituting (24) and (25) into (20) and (21) and keeping linear terms, one obtains

$$a_1' = \left[\frac{9}{4}\Lambda(3\varepsilon_2 - \frac{2}{3}\varepsilon_1\Omega^2)a_0 \sin\beta_0 - \frac{\delta\Omega}{2}\right]a_1 + \left(\frac{9}{4}\Lambda(3\varepsilon_2 - \frac{2}{3}\varepsilon_1\Omega^2\right)_1 a_0^2 \cos 3\beta_0)\beta_1, \tag{26}$$

$$\beta_1' = \left\{2\left(9\varepsilon_2 - \frac{10}{3}\varepsilon_1\Omega^2\right)\Lambda^2 a_0\right\}a_1 - \left\{\frac{9\Lambda}{4}\left(3\varepsilon_2 - \frac{2}{3}\varepsilon_1\Omega^2\right)a_0 \sin 3\beta_0\right\}\beta_1, \tag{27}$$

$\sin(3\beta_0)$ and $\cos(3\beta_0)$ can be obtained from steady state solution, i.e. (22) and (23). Substituting $a_1 = a_{10}e^{\lambda T_1}$ and $\beta_1 = \beta_{10}e^{\lambda T_1}$ into Eqs. (26) and (27).

For nontrivial solution the determinant of the coefficient matrix for a_{10} and β_{10} must vanish, which leads to a quadratic equation for the eigen value λ. The stability of the subharmonic solution can be examined by evaluating the sign of the real part of the eigen value, unstable (positive real part) and stable (negative real part), [4,5].

3. Results and discussion

Examples of the 1/3 subharmonic response of Eq. (1), for a selected range of system parameters, were obtained using the analytical solutions in Section 2 as well as those obtained numerically are displayed in Figs. 1–7. The numerical solutions were obtained by integrating Eq. (1) up to $t = 5000$ s and time step of 0.005 s, using the fourth-order Runge–Kutta method. The FFT analysis, was carried out using the last 2048 points of time history of the displacement using MATLAB, taking into consideration the effect of integration step size on accuracy.

It is to be noted that in some cases a total integration time of less than $t = 5000$ s and an integration step of 0.1 was sufficient for the response to reach the steady state, especially at frequencies away from the subharmonic resonance region.

In Fig. 1, results were obtained for a purely hardening case, $\varepsilon_1 = 0$ and $\varepsilon_2 = 0.1$, and verified numerically in a zone near a point on the frequency axis at which 1/3 subharmonic resonance emanates.

These results show that the subharmonic response: (a) consist of two non-terminating branches, (b) the resonance starts at a frequency of $\Omega = 3.05$, (c) the upper branch is stable while the lower branch is unstable, (d) the HBM method and the MMS produce identical results, (e) numerical solutions show good agreement with the analytical results up to $\Omega \approx 4.8$ after which numerical solution show a jump down to the primary resonance response, i.e. subharmonic response of order 1/3 were obtained only in the range $3.05 < \Omega < 4.8$.

Fig. 2 shows typical results obtained for the case were the nonlinearity in the oscillator is purely softening. These results show that: (a) the subharmonic resonance consists of two branches which, unlike the hardening case, bent to the left rather than to the right, (b) the upper branch is stable and the lower branch is unstable, (c) the subharmonic resonance emanates at a frequency slightly less than 3, (d) the HBM and the MMS produce the same, nearly identical, results which show good agreement with those obtained numerically, (e) the numerical solutions show that the subharmonic resonance is possible to a frequency down to about $\Omega \approx 2$, below which the response of the oscillator becomes dominated by the primary resonance response.

Fig. 3 displays the results obtained for the case where the oscillator is dominated by the hardening nonlinearity but with a small softening nonlinearity.

These results show: (a) the HB solution consist of two branches which tend to coalesce as the frequency is increased and re-separate again as the frequency is increased further, (b) the MMS solution is composed of

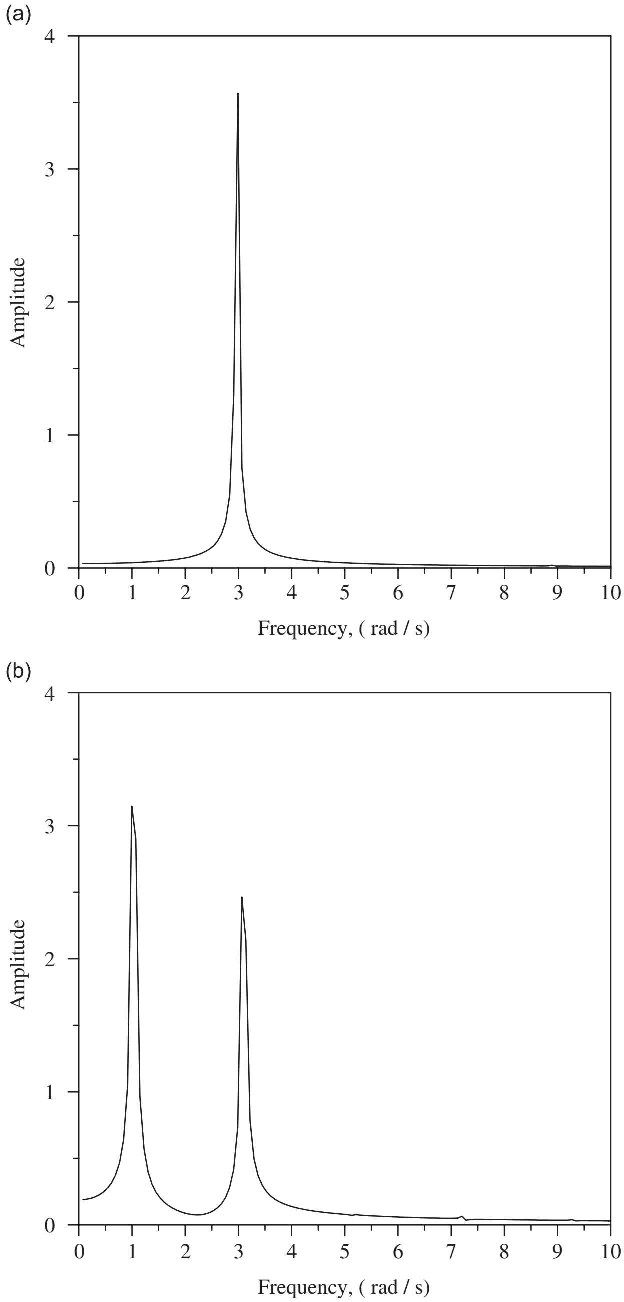

Fig. 5. Fast Fourier transform FFT of the response for $P = 2$, $\delta = 0$, $\varepsilon_1 = 1$, $\varepsilon_2 = 3$: (a) $\Omega = 2.97$, (b) $\Omega = 3.1$.

two parts, each consist of two branches, separated by a small island where no subharmonic resonance response exists, (c) unlike the cases where the oscillator is purely hardening or softening type, the stability analysis show that only the lower branch of the lower part is stable, and (d) the numerical solutions were only possible for points on the lower branch on the lower part for both the HB and MMS solutions.

Numerical simulations have shown that, for the hardening type oscillator, the chaotic behavior may occur just before entering the region at which the subharmonic resonance curve intersects with the primary response curve. Fig. 4 shows that the zone of chaotic behavior on the frequency response curve occurs in the shaded region, i.e. $2.9 < \Omega < 3.25$, just before the subharmonic resonance emanates, which agrees with the findings

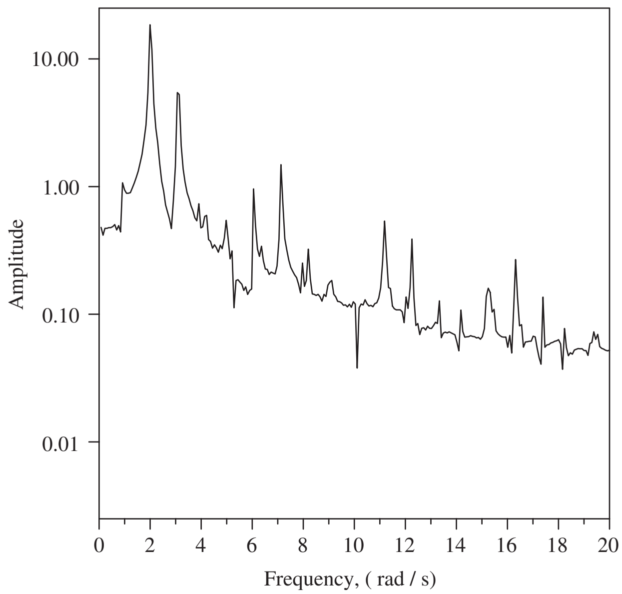

Fig. 6. Fast Fourier transform FFT of the response for $P = 20$, $\delta = 0.005$, $\varepsilon_1 = 1$, $\varepsilon_2 = 3$ and $\Omega = 3.1$, $\lambda_1 = 1.765 \times 10^{-3}$, $\lambda_2 = 0$ and $\lambda_3 = -3.20 \times 10^{-3}$, $d_f = 2.55$.

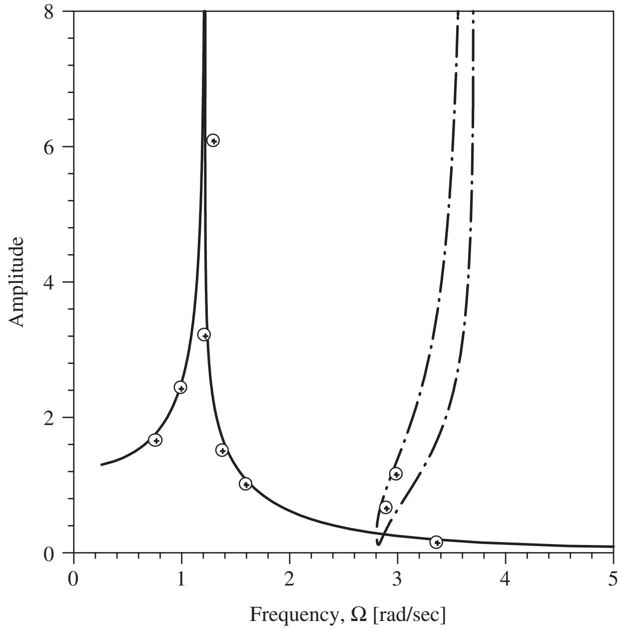

Fig. 7. Primary and subharmonic response. $P = 2$, $\delta = 0.01$, $\varepsilon_1 = \varepsilon_2 = 0.5$, ——— primary response, -·-·-·- subharmonic response using HB, ⊕ numerical solution.

presented in Ref. [13]. And this was detected with the aid of the simulation tools; phase planes, Poincare maps and Lyaponuv exponents (λ_i) and fractal dimension (d_f).

Other results but not shown, indicated that, for a softening type oscillator, the chaos may occur at lower values of Ω by decreasing the excitation amplitude P and for high values of Ω but slightly less than 3 when

Table 1
Summary of some numerical simulation for different values of; Excitation amplitude P, Damping δ and initial conditions $u(0)$ and $\dot{u}(0)$ at $\Omega = 3.1$

P	δ	$u(0)$	$\dot{u}(0)$	Behavior
2	0	0	0	3 T
10	0.01	0	0	3 T
10	0.01	2	0	3 T
10	0.01	5	0	3 T
10	0.01	10	0	Chaos
12	0.01	0	0	3 T
13	0.01	0	0	7 T
14	0.01	0	0	7 T
15	0.01	0	0	PD
15	0.01	5	2	3 T
10	0.005	0	0	3 T
15	0.005	0	0	PD
20	0.005	0	0	Chaos

increasing the excitation level. To summarize, for the same oscillator shown in Fig. 2, the chaos was observed at $\Omega = 1$ and for $P = 1$ and at $\Omega = 2.8$ with excitation level of $P = 5$.

To study the behavior of the nonlinear oscillator inside the region of intersection of the subharmonic and the primary resonance for $P = 2$, $\delta = 0$, $\varepsilon_1 = 1$, $\varepsilon_2 = 3$, results were obtained and the FFT were presented in Fig. 5 for some frequencies inside the range $2.95 < \Omega < 3.15$. As one can see, the subharmonic resonance can occur inside the frequency range when the primary resonance immerges in the subharmonic instability boundary.

It was found that by increasing the frequency, the behavior of the nonlinear oscillator is periodic for a frequency $2.95 < \Omega < 2.97$, as shown in Fig. 5a, and the subharmonic resonance of order 1/3 "3 T attractor" was first observed at a frequency equals to $\Omega = 2.98$. Numerical investigations have shown that the subharmonic resonance can occur inside the frequency range $2.98 < \Omega < 3.10$, as shown in Figs. 5b.

Other results, but not shown, have indicated that increasing the excitation level P and the initial conditions $u(0)$, $\dot{u}(0)$ may change the response of the system from period (1, 3 or 7 T) to a chaotic response through period doublings, depending on the amount of damping δ present in the system. Those results are summarized in Table 1.

From the results obtained, it was noticed that the response of the nonlinear oscillator might culminate in chaos through period doublings when increasing the excitation level (P) and the amount of damping δ, as shown in Fig. 6 for the case; $P = 20$, $\delta = 0.005$, $\varepsilon_1 = 1$ and $\varepsilon_2 = 3$.

Finally, in Fig. 7, results were obtained and for a case at which the nonlinearities are of the same order. The primary response resembles a linear behavior and the results obtained from the numerical simulations indicated that the response is periodic for all values of Ω and the excitation level P.

4. Conclusions

The results presented in this paper have shown that the subharmonic response of order 1/3 appear, for given system parameters; ε_1, ε_2, δ provided that the excitation level P is above certain critical value and the initial conditions are within the basin of attraction, whenever Ω is in the range of $3\omega_0$.

From the results obtained, the subharmonic resonance is a bifurcation phenomenon, and it might culminate in chaos, depending on the critical values of the system parameters.

In light of the results presented the following conclusions can be drawn:

- Increasing the value of ε_1, i.e. increasing the softening effect of the nonlinearity, will decrease the critical value of Ω at which the subharmonic resonance may initiated. This means that the subharmonic resonance may starts at value; less than 3 if the value of ε_1 is greater than ε_2 and value greater than 3 if the value of ε_1 is less than ε_2.

- For a hardening type nonlinear oscillator, the behavior may culminate to chaos through period doublings, just before entering the subharmonic region depending on the critical values of the excitation level, damping and the initial conditions.
- For a softening type nonlinear oscillator, decreasing the excitation level P tends to lower the excitation frequency at which chaotic motion may appear.
- When the oscillator is dominated by the hardening nonlinearity but with a small softening nonlineraity, the HB solution consist of two branches which tend to coalesce and the MMS solution is composed of two parts, each consist of two branches, separated by a small island where no subharmonic resonance response exists.
- When the values of ε_1 and ε_2 are of the same order, the frequency response resembles that of a linear oscillator and chaotic behavior was not observed even for relatively high excitation levels.

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