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Performance and parametric study of infinite-multiple TMDs for structures under ground acceleration by H_∞ optimization

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Abstract

A new design method, i.e. the Infinite-Multiple tuned mass dampers (IMTMD) method, is proposed for the optimum configuration of classical MTMD. Firstly, the transfer function (TF) of IMTMD is obtained by integration, and then the dynamic magnification factor (DMF) of the single-degree-of-freedom (SDOF) main structure with IMTMD under base acceleration excitation is investigated as well as MTMD. Using IMTMD the best and critical performance of MTMD composed of many TMDs can be obtained together with optimum design parameters, which can only be conjectured based on a lot of data observation in previous studies. Moreover, the IMTMD method has excellent efficiency of iteration and it can give correct optimum parameters numerically for MTMD composed of more than 20 TMDs. For convenience, two types of MTMD hypotheses proposed in the previous literature are selected as the examples for demonstration and discussion.

1. Introduction

Tuned mass damper (TMD) consisting of a mass, a damper and a spring, is an effective and reliable structural vibration control device commonly attached to the vibrating structure to suppress its undesirable vibrations. In general, the natural frequency of TMD is tuned in resonance with the fundamental mode of the primary structure, so that a large amount of the structural vibrating energy is transferred to the TMD and then dissipated by the damping of the TMD as the primary structure is subjected to external disturbances. Consequently, the safety and habitability of the structure are greatly enhanced. Now the TMD devices have been successfully installed on many slender skyscrapers and towers to suppress their undesirable structural dynamic responses, such as the CN Tower (535 m) in Canada, John Hancock Building (sixty stories) in Boston, Center-Point Tower (305 m) in Sydney, and the tallest building in the world, Taipei 101 Tower (101 stories, 504 m) [1] in Taiwan. From the field vibration measurements, TMD has been proven to be an effective and feasible system for structural vibration control in most cases. Nevertheless, in the case of seismic excitation, the efficiency of these passive systems has not yet been established, because of their inability to respond to a variety of transient base excitations within a very short period of time. This has led to the

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development of active and hybrid mass damper systems that can accommodate these features. More information about these systems and their modeling and analysis may be found in other papers [2].

Additionally, it is well known that there exist some disadvantages of a single TMD, for example, the performance of single TMD is sensitive to the variation in the natural frequency of the structure and/or the damping ratio of the TMD. Either mistuning frequency or off-optimum damping can significantly reduce the performance of single TMD. To overcome these drawbacks, more than one tuned mass damper with different dynamic characteristics has been proposed. Iwanami and Seto [3] showed that two tuned mass dampers are more effective than a single TMD. However, the improvement on the performance was not significant. Recently, multiple tuned mass dampers (MTMD) with distributed natural frequencies were proposed by Xu and Igusa [4], Yamaguchi and Harnpornchai [5], Abe and Fujino [6], Jangid [7,8], Kareem and Kline [2], Joshi and Jangid [9], Bakre and Jangid [10], Li [11–13], Zuo [14] and Koc [15]. It was shown that the MTMD is more effective for vibration control compared to single TMD. In addition, the performance of the MTMD system is not sensitive to the change or estimation error in the natural frequency of the structure.

In the above-mentioned studies, the optimum parameters of MTMD are determined numerically through parametric studies or by their proposed optimal design methods. It is impossible to obtain the analytical optimum parameters because of the complexity of analytical equations, which does not happen for the single TMD implemented in single-degree-of-freedom (SDOF) structure. For the single TMD, many analytical solutions for optimum parameters have obtained until now [16]. As for MTMD, the introduction of multiple TMDs increases the order number of the state space or the length of Transfer Function (i.e. TF). Therefore, the more the number of TMDs is, the more difficult to obtain numerically the design parameters of MTMD. Moreover, it will take more CPU time and computer memory in every iteration step as MTMD are composed of a large number of TMDs. Because only the iteration method can be adopted to search optimum parameters, the difficulty increases with the increase of the number of TMDs. Sometimes the search process may be trapped in local minimum points. Some studies just investigated MTMD with 21 or less TMDs [2,11], some studies investigated more [14].

If the number of TMDs is large enough, how about the performance of the MTMD (actually infinite-multiple TMDs, i.e. IMTMD)? What are their optimum parameters? Can these optimum parameters take the place of that of MTMD with less TMDs? Especially important, does it take a great deal of time to solve optimum parameters of IMTMD?

This paper tries to study the IMTMD in a simple and effective way and answer these questions. Firstly, the TFs of IMTMD are given by integration method, which will be proved to be true by letting the number of TMDs of MTMD approach infinity in the following part. Secondly, based on the TFs obtained, the DMFs of the primary structure are derived, and then design parameters are given using H_∞ optimization. Thirdly, some comparisons are given for IMTMD and MTMD, and the differences between these two models are analyzed. In this paper all studies are based on two hypotheses selected from those proposed by Li [12].

2. Modeling of structure-IMTMD system

The model of classical MTMD is shown in Fig. 1(a). Let ω_S, ζ_S, c_S, m_S, k_S denote the natural frequency, damping ratio, damping coefficient, mass and spring stiffness of the main structure respectively. Clearly there

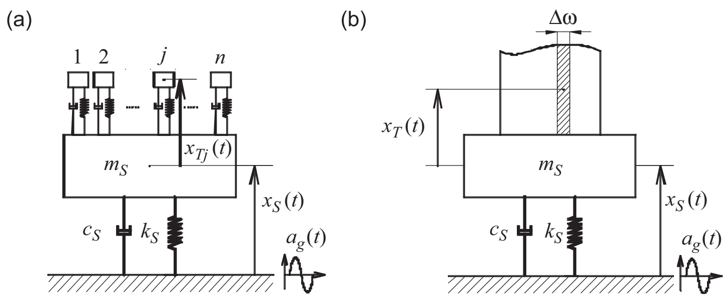

Fig. 1. Analytical model of MTMD and IMTMD under ground acceleration.

exist the following relationships:

$$m_S\omega_S^2 = k_S, \quad c_S = 2m_S\omega_S\zeta_S. \tag{1}$$

The total mass ratio of MTMD system is μ, and then the total mass of MTMD can be expressed as

$$m_T = m_S\mu. \tag{2}$$

Individual natural frequencies of MTMD made up of n TMDs are a set of discrete values listed as $\omega_1, \omega_2, \ldots, \omega_n$. Assuming that these frequencies are distributed uniformly, we define λ and σ as the tuning frequency ratio and frequency half-spacing coefficient of MTMD, then

$$\begin{aligned}
&\sigma = \frac{\omega_n - \omega_1}{2\omega_S}, \quad \lambda = \frac{\omega_n + \omega_1}{2\omega_S},\\
&\omega_j = \omega_1 + (j-1)\frac{2\sigma\omega_S}{n-1}(\lambda-\sigma)\omega_S \leqslant \omega_j \leqslant (\lambda+\sigma)\omega_S\\
&\forall j = 1, 2, \ldots, n.
\end{aligned} \tag{3}$$

As the number of TMDs increases infinitely, MTMD become IMTMD, which is shown in Fig. 1(b). Individual natural frequencies of IMTMD cover continuously the interval $[(\lambda-\sigma)\omega_S, (\lambda+\sigma)\omega_S]$. Select an infinitesimal $\Delta\omega$ from this interval, which stands for an infinitesimal TMD with the single natural frequency ω. Its mass, damping coefficient and spring stiffness are denoted by Δm, Δc and Δk respectively. Then the density function of mass, damping coefficient and spring stiffness can be defined as the following:

$$\begin{aligned}
m(\omega) &= \frac{\mathrm{d}m}{\mathrm{d}\omega} = \lim_{\Delta\omega\to 0}\frac{\Delta m}{\Delta\omega},\\
c(\omega) &= \frac{\mathrm{d}c}{\mathrm{d}\omega} = \lim_{\Delta\omega\to 0}\frac{\Delta c}{\Delta\omega},\\
k(\omega) &= \frac{\mathrm{d}k}{\mathrm{d}\omega} = \lim_{\Delta\omega\to 0}\frac{\Delta k}{\Delta\omega}.
\end{aligned} \tag{4}$$

It is evident that the following equations hold:

$$\int_{(\lambda-\sigma)\omega_S}^{(\lambda+\sigma)\omega_S} m(\omega)\,\mathrm{d}\omega = m_S\mu, \tag{5}$$

$$k(\omega) = m(\omega)\omega^2, \tag{6}$$

$$\theta = \frac{\omega}{\omega_S} \in [\lambda-\sigma, \lambda+\sigma]. \tag{7}$$

Similarly, we may define the damping ratio of the infinitesimal TMD

$$\zeta(\omega) = \lim_{\Delta\omega\to 0}\frac{\Delta c}{2\Delta m\omega} = \frac{c(\omega)}{2m(\omega)\omega}. \tag{8}$$

As for the infinitesimal $\Delta\omega$, which is an infinitesimal TMD denoted by Δm, Δc and Δk, there exists only the interaction between it and the primary structure attaching. The dynamical equation can be expressed as

$$m(\omega)\Delta\omega(\ddot{x}_T + \ddot{x}_S + a_g) + c(\omega)\Delta\omega\dot{x}_T + k(\omega)\Delta\omega x_T = 0, \tag{9}$$

where x_S denotes the displacement of the main structure relative to the ground, x_T the displacement of the infinitesimal TMD relative to the main structure and a_g denotes the ground acceleration.

The force applied to the main structure by the infinitesimal TMD is expressed as

$$\Delta f_T = c(\omega)\Delta\omega\dot{x}_T + k(\omega)\Delta\omega x_T. \tag{10}$$

The Laplace transformation of the force can be obtained according to Eqs. (9) and (10):

$$\begin{aligned}\Delta F_T &= \frac{-m(\omega)s^2(c(\omega)s+k(\omega))}{m(\omega)s^2+c(\omega)s+k(\omega)}X_S\Delta\omega + \frac{-m(\omega)(c(\omega)s+k(\omega))}{m(\omega)s^2+c(\omega)s+k(\omega)}A_g\Delta\omega \\ &\triangleq \Delta G_T X_S + \Delta\bar{G}_T A_g,\end{aligned} \tag{11}$$

where

$$\begin{aligned}\Delta G_T &= \frac{-m(\omega)s^2(c(\omega)s+k(\omega))}{m(\omega)s^2+c(\omega)s+k(\omega)}\Delta\omega, \\ \Delta\bar{G}_T &= \frac{-m(\omega)(c(\omega)s+k(\omega))}{m(\omega)s^2+c(\omega)s+k(\omega)}\Delta\omega.\end{aligned} \tag{12}$$

Clearly

$$\Delta\bar{G}_T = \frac{\Delta G_T}{s^2}. \tag{13}$$

The total force applied to the primary structure by the whole IMTMD system is

$$\begin{aligned}F_T &= \int_\omega \mathrm{d}F_T \\ &= X_S\int_\omega \mathrm{d}G_T + A_g\int_\omega \mathrm{d}\bar{G}_T \\ &\triangleq X_S G_T + A_g\bar{G}_T,\end{aligned} \tag{14}$$

where

$$G_T = \int_\omega \mathrm{d}G_T,\ \bar{G}_T = \int_\omega \mathrm{d}\bar{G}_T. \tag{15}$$

According to Eq. (13), there also exists

$$\bar{G}_T = \frac{G_T}{s^2}. \tag{16}$$

Substituting Eq. (16) into Eq. (14), we can obtain

$$F_T = X_S G_T(s) + A_g\frac{G_T(s)}{s^2}. \tag{17}$$

As for $G_T(s)$, it can be derived further as following:

$$G_T(s) = \int_{(\lambda-\sigma)\omega_S}^{(\lambda+\sigma)\omega_S} \frac{-m(\omega)s^2(c(\omega)s+k(\omega))}{m(\omega)s^2+c(\omega)s+k(\omega)}\,\mathrm{d}\omega \tag{18}$$

or

$$G_T(s) = \int_{(\lambda-\sigma)\omega_S}^{(\lambda+\sigma)\omega_S} \frac{-m(\omega)s^2(2\omega\zeta(\omega)s+\omega^2)}{s^2+2\omega\zeta(\omega)s+\omega^2}\,\mathrm{d}\omega. \tag{19}$$

Based on classical control theories, the secondary parts (i.e. MTMD or IMTMD) can be considered as feedback components, which receive the signals of the primary structure's displacement X_S as well as the base acceleration excitation A_g, at the same time, output the corresponding control force F_T onto the primary structure (Ref. to Eq. (17)). To obtain the TF of the whole system, that of the primary structure under the external force should be solved in advance. We define P as the force applied on the primary structure, which actually is the feedback force F_T exerted by the secondary parts added by the inertial force $-m_S A_g$ caused by the base acceleration. That is

$$P = F_T - m_S A_g \triangleq F_T + KA_g, \tag{20}$$

where

$$K = -m_S. \tag{21}$$

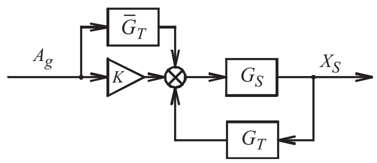

Fig. 2. The block diagram of the whole system.

The primary structure is a SDOF spring–damper–mass system, and then the TF from P to its displacement X_S can be easily expressed as

$$G_S(s) = \frac{X_S}{P} = \frac{1}{m_S s^2 + 2c_S s + k_S} = \frac{1}{(s^2 + 2\zeta_S\omega_S s + \omega_S^2)m_S}. \quad (22)$$

Based on the above description the block diagram of the whole IMTMD–Structure system can be plotted as shown in Fig. 2.

According to Fig. 2 the TF of the IMTMD-structure system can be obtained with the help of Eqs. (21) and (22):

$$G(s) = \frac{X_S}{A_g} = \frac{(K + \bar{G}_T(s))G_S(s)}{1 - G_S(s)G_T(s)} = \frac{-m_S + \dfrac{G_T(s)}{s^2}}{(s^2 + 2\zeta_S\omega_S s + \omega_S^2)m_S - G_T(s)}. \quad (23)$$

The DMF is defined as [12]

$$\text{DMF} = \left|\frac{X_S}{A_g m_S / k_S}\right| = |G(s)|\omega_S^2. \quad (24)$$

3. The first hypothesis

To further simplify the above model, assume that

$$m(\omega) \equiv \tilde{m}_T, \quad \zeta(\omega) \equiv \zeta_T. \quad (25)$$

Here ζ_T is an undetermined value. According to Eq. (5), $\tilde{m}_T$ must satisfy the following equation:

$$2\tilde{m}_T\sigma\omega_S = m_S\mu \quad (26)$$

or

$$\tilde{m}_T = \frac{m_S\mu}{2\sigma\omega_S}. \quad (27)$$

Actually Eq. (25) is in accordance with the 2nd MTMD hypothesis proposed by Li [12] expressed as

$$\begin{aligned} &m_1 = m_2 = \cdots = m_n, \\ &\zeta_1 = \zeta_2 = \cdots = \zeta_n = \zeta_T, \\ &k_j = m_j\omega_j^2, \quad c_j = 2\zeta_j m_j \omega_j, \quad \forall j = 1, \ldots, n. \end{aligned} \quad (28)$$

Substituting Eqs. (25), (27) into (19), the TF of IMTMD system can be obtained as

$$G_T(s) = -\frac{m_S\mu s^2}{2\sigma\omega_S}\int_{(\lambda-\sigma)\omega_S}^{(\lambda+\sigma)\omega_S} \frac{2\omega\zeta_T s + \omega^2}{s^2 + 2\omega\zeta_T s + \omega^2}\,d\omega. \quad (29)$$

Let ω_I represents the input frequency of ground acceleration, and the ratio of input frequency can be defined as

$$\eta = \frac{\omega_I}{\omega_S}. \quad (30)$$

Substituting

$$s = \omega_I \mathrm{i} = \eta\omega_S \mathrm{i} \tag{31}$$

and Eq. (7) into Eq. (29)

$$G_T(\eta\omega_S i) = \frac{m_S\omega_S^2\mu\eta^2}{2\sigma}\int_{\lambda-\sigma}^{\lambda+\sigma}\frac{2\theta\zeta_T\eta\mathrm{i}+\theta^2}{-\eta^2+2\theta\zeta_T\eta\mathrm{i}+\theta^2}\,\mathrm{d}\theta, \tag{32}$$

where i denotes the imaginary unit $\sqrt{-1}$.

Let

$$\begin{aligned} A(\eta,\lambda,\sigma,\zeta_T) &= \frac{\mu\eta^2}{2\sigma}\int_{\lambda-\sigma}^{\lambda+\sigma}\frac{2\theta\zeta_T\eta\mathrm{i}+\theta^2}{-\eta^2+2\theta\zeta_T\eta\mathrm{i}+\theta^2}\,\mathrm{d}\theta \\ &= \frac{\mu\eta^2}{2\sigma}\left(2\sigma+\frac{\eta}{\mathrm{i}\sqrt{1-\zeta_T^2}}\mathrm{Arctan}\frac{\lambda+\sigma+\zeta_T\eta\mathrm{i}}{\eta\mathrm{i}\sqrt{1-\zeta_T^2}}-\frac{\eta}{\mathrm{i}\sqrt{1-\zeta_T^2}}\mathrm{Arctan}\frac{\lambda-\sigma+\zeta_T\eta\mathrm{i}}{\eta\mathrm{i}\sqrt{1-\zeta_T^2}}\right). \end{aligned} \tag{33}$$

The function Arctan is a complex function, which is defined as follows:

$$\mathrm{Arctan}z = -\frac{\mathrm{i}}{2}\mathrm{Ln}\frac{1+z\mathrm{i}}{1-z\mathrm{i}}, \quad z \in \mathbb{C}. \tag{34}$$

Apparently $A(\eta, \lambda, \sigma, \zeta_T)$ is a dimensionless variable, which is a function of η and undetermined values λ, σ, ζ_T. It can be seen from Eqs. (32) and (33) that

$$G_T(\eta\omega_S\mathrm{i}) = m_S\omega_S^2 A(\eta,\lambda,\sigma,\zeta_T) \tag{35}$$

Substituting Eqs. (31) and (35) into Eq. (23), the TF of IMTMD-structure system is

$$G(\eta\omega_S\mathrm{i}) = \frac{1}{\omega_S^2}\left(\frac{-1-\dfrac{A(\eta,\lambda,\sigma,\zeta_T)}{\eta^2}}{-\eta^2+2\zeta_S\eta\mathrm{i}+1-A(\eta,\lambda,\sigma,\zeta_T)}\right). \tag{36}$$

The TF of the classical MTMD system under the Hypothesis of Eq. (28) [12] can be expressed as

$$G_T(s) = m_S\omega_S^2 A_D(\eta,\lambda,\sigma,\zeta_T), \tag{37}$$

where

$$A_D(\eta,\lambda,\sigma,\zeta_T) = \frac{\eta^2\mu}{n}\sum_{j=1}^{n}\frac{2\theta_j(\lambda,\sigma)\zeta_T\eta\mathrm{i}+(\theta_j(\lambda,\sigma))^2}{-\eta^2+2\theta_j(\lambda,\sigma)\zeta_T\eta\mathrm{i}+(\theta_j(\lambda,\sigma))^2}, \tag{38}$$

where $\theta_j(\lambda, \sigma)$ denotes the ratio of the natural frequencies of the jth TMD versus ω_S. As for the uniform distribution of frequency, we can obtain

$$\theta_j(\lambda,\sigma) = \frac{\omega_j}{\omega_S} = \lambda-\sigma+\frac{2\sigma(j-1)}{n-1}. \tag{39}$$

The TF of MTMD-structure system is

$$G_D(\eta\omega_S\mathrm{i}) = \frac{1}{\omega_S^2}\left(\frac{-1-\dfrac{A_D(\eta,\lambda,\sigma,\zeta_T)}{\eta^2}}{-\eta^2+2\zeta_S\eta\mathrm{i}+1-A_D(\eta,\lambda,\sigma,\zeta_T)}\right). \tag{40}$$

That is to say, when the MTMD is composed of a large enough number of TMDs, i.e. IMTMD, the TF of the whole system can be expressed in a more simple way compared to that of classical MTMD, while the previous methods [11,12] will lead to lengthy series expression.

According to Eq. (24), the DMF for the primary structure equipped with IMTMD and MTMD can be expressed as the following equations respectively:

$$\mathrm{DMF} = \left| \frac{1 + \dfrac{A(\eta, \lambda, \sigma, \zeta_T)}{\eta^2}}{-\eta^2 + 2\zeta_S \eta \mathrm{i} + 1 - A(\eta, \lambda, \sigma, \zeta_T)} \right| \tag{41}$$

and

$$\mathrm{DMF} = \left| \frac{1 + \dfrac{A_D(\eta, \lambda, \sigma, \zeta_T)}{\eta^2}}{-\eta^2 + 2\zeta_S \eta \mathrm{i} + 1 - A_D(\eta, \lambda, \sigma, \zeta_T)} \right|. \tag{42}$$

Let $n \to \infty$, then

$$\begin{aligned} \lim_{n\to\infty} A_D(\eta, \lambda, \sigma, \zeta_T) &= \lim_{n\to\infty} \frac{\eta^2 \mu}{2\sigma} \frac{2\sigma}{n} \sum_{j=1}^{n} \frac{2\theta_j(\lambda, \sigma)\zeta_T \eta \mathrm{i} + (\theta_j(\lambda, \sigma))^2}{-\eta^2 + 2\theta_j(\lambda, \sigma)\zeta_T \eta \mathrm{i} + (\theta_j(\lambda, \sigma))^2} \\ &= \frac{\eta^2 \mu}{2\sigma} \int_{\lambda-\sigma}^{\lambda+\sigma} \frac{2\theta \zeta_T \eta \mathrm{i} + \theta^2}{-\eta^2 + 2\theta\zeta_T \eta \mathrm{i} + \theta^2} \, \mathrm{d}\theta. \end{aligned} \tag{43}$$

Hence

$$\lim_{n\to\infty} A_D(\eta, \lambda, \sigma, \zeta_T) = A(\eta, \lambda, \sigma, \zeta_T). \tag{44}$$

That is to say, Eqs. (41) and (42) are in accordance actually if $n \to \infty$.

4. The second hypothesis

Another assumption is adopted in the following:

$$\begin{aligned} k(\omega) &\equiv \tilde{k}_T, \\ m(\omega) &= \tilde{k}_T / \omega^2, \\ c(\omega) &\equiv \tilde{c}_T, \end{aligned} \tag{45}$$

where $\tilde{k}_T$ and $\tilde{c}_T$ are constant values independent of ω. The above hypothesis (45) just matches the 1st MTMD hypothesis of those proposed by Li [12], which is most commonly employed by researchers because of its well performance and convenience for manufacture and application in practice. The hypothesis can be expressed as

$$\begin{aligned} & m_1 \neq m_2 \neq \cdots \neq m_n, \\ & c_1 = c_2 = \cdots = c_n, \\ & k_1 = k_2 = \cdots = k_n, \\ & k_j = m_j \omega_j^2, c_j = 2\zeta_j m_j \omega_j, \quad \forall j = 1, \ldots, n. \end{aligned} \tag{46}$$

Based on Eqs. (5) and (45), we can obtain

$$\tilde{k}_T \int_{(\lambda-\sigma)\omega_S}^{(\lambda+\sigma)\omega_S} \frac{1}{\omega^2} \, \mathrm{d}\omega = m_S \mu \tag{47}$$

or

$$\tilde{k}_T = \frac{m_S \mu \omega_S (\lambda^2 - \sigma^2)}{2\sigma} \tag{48}$$

and

$$m(\omega) = \frac{m_S \mu \omega_S (\lambda^2 - \sigma^2)}{2\sigma\omega^2}. \tag{49}$$

According to Eq. (8),

$$\zeta(\omega) = \frac{\tilde{c}_T}{2\omega m(\omega)}. \tag{50}$$

Then we can define the average of damping ratio

$$\zeta_T = \frac{\int_{(\lambda-\sigma)\omega_S}^{(\lambda+\sigma)\omega_S} \zeta(\omega)\,\mathrm{d}\omega}{2\sigma\omega_S}. \tag{51}$$

Substituting Eqs. (49) and (50) into Eq. (51), we can obtain

$$\tilde{c}_T = \frac{m_S \mu (\lambda^2 - \sigma^2)}{\sigma\lambda} \zeta_T \tag{52}$$

and

$$\zeta(\omega) = \frac{\zeta_T \omega}{\lambda \omega_S}. \tag{53}$$

With the help of Eqs. (49) and (53), Eq. (19) can be simplified further as

$$G_T(s) = \frac{-s^2 \omega_S m_S \mu (\lambda^2 - \sigma^2)}{2\sigma} \int_{(\lambda-\sigma)\omega_S}^{(\lambda+\sigma)\omega_S} \frac{1}{\omega^2 + \dfrac{\lambda \omega_S s^2}{2\zeta_T s + \lambda \omega_S}} \mathrm{d}\omega. \tag{54}$$

Substituting Eqs. (7) and (31) into Eq. (54), then

$$G_T(\eta\omega_S \mathrm{i}) = \frac{m_S \omega_S^2 \mu \eta^2 (\lambda^2 - \sigma^2)}{2\sigma} \int_{\lambda-\sigma}^{\lambda+\sigma} \frac{1}{\theta^2 - \dfrac{\lambda\eta^2}{2\zeta_T \eta i + \lambda}} \mathrm{d}\theta$$
$$\triangleq m_S \omega_S^2 B(\eta, \lambda, \sigma, \zeta_T), \tag{55}$$

where

$$B(\eta, \lambda, \sigma, \zeta_T) = \frac{\mu\eta^2(\lambda^2 - \sigma^2)}{2\sigma} \int_{\lambda-\sigma}^{\lambda+\sigma} \frac{1}{\theta^2 - \dfrac{\lambda\eta^2}{2\zeta_T \eta \mathrm{i} + \lambda}} \mathrm{d}\theta$$
$$= \frac{\eta^2 \mu (\lambda^2 - \sigma^2)}{2\sigma \mathrm{i} \sqrt{\dfrac{\lambda\eta^2}{2\zeta_T \eta \mathrm{i} + \lambda}}} \left(\mathrm{Arctan} \frac{\lambda + \sigma}{\mathrm{i}\sqrt{\dfrac{\lambda\eta^2}{2\zeta_T \eta \mathrm{i} + \lambda}}} - \mathrm{Arctan} \frac{\lambda - \sigma}{\mathrm{i}\sqrt{\dfrac{\lambda\eta^2}{2\zeta_T \eta \mathrm{i} + \lambda}}} \right). \tag{56}$$

Apparently, B is a dimensionless variable dependent on η and undetermined parameters λ, σ, ζ_T as well as A. Similar to Eq. (36), the TF of the whole IMTMD-structure system can be expressed as

$$G(\eta\omega_S \mathrm{i}) = \frac{1}{\omega_S^2} \left(\frac{-1 - \dfrac{B(\eta, \lambda, \sigma, \zeta_T)}{\eta^2}}{-\eta^2 + 2\zeta_S \eta \mathrm{i} + 1 - B(\eta, \lambda, \sigma, \zeta_T)} \right). \tag{57}$$

The DMF can be obtained based on Eq. (24)

$$\mathrm{DMF} = \left| \frac{1 + \dfrac{B(\eta, \lambda, \sigma, \zeta_T)}{\eta^2}}{-\eta^2 + 2\zeta_S \eta \mathrm{i} + 1 - B(\eta, \lambda, \sigma, \zeta_T)} \right|. \tag{58}$$

Similarly, the DMF for the primary structure equipped with MTMD is expressed as [12]

$$\mathrm{DMF} = \left| \frac{1 + \dfrac{B_D(\eta, \lambda, \sigma, \zeta_T)}{\eta^2}}{-\eta^2 + 2\zeta_S \eta \mathrm{i} + 1 - B_D(\eta, \lambda, \sigma, \zeta_T)} \right|, \tag{59}$$

where $B_D(\eta,\ \lambda,\ \sigma,\ \zeta_T)$ is defined as

$$B_D(\eta, \lambda, \sigma, \zeta_T) = \frac{\mu \eta^2}{\sum_{k=1}^{n} \dfrac{1}{(\theta_k(\lambda, \sigma))^2}} \sum_{j=1}^{n} \frac{1}{\left((\theta_j(\lambda, \sigma))^2 + \dfrac{-\lambda \eta^2}{2\zeta_T \eta \mathrm{i} + \lambda} \right)}. \tag{60}$$

Let $n \to \infty$, then

$$\begin{aligned} \lim_{n\to\infty} B_D(\eta, \lambda, \sigma, \zeta_T) &= \lim_{n\to\infty} \frac{\mu \eta^2}{\dfrac{2\sigma}{n} \sum_{k=1}^{n} \dfrac{1}{(\theta_k(\lambda, \sigma))^2}} \frac{2\sigma}{n} \sum_{j=1}^{n} \frac{1}{\left((\theta_j(\lambda, \sigma))^2 + \dfrac{-\lambda \eta^2}{2\zeta_T \eta \mathrm{i} + \lambda} \right)} \\ &= \frac{\mu \eta^2}{\int_{\lambda-\sigma}^{\lambda+\sigma} \dfrac{1}{\theta^2} \mathrm{d}\theta} \int_{\lambda-\sigma}^{\lambda+\sigma} \frac{1}{\theta^2 - \dfrac{\lambda \eta^2}{2\zeta_T \eta \mathrm{i} + \lambda}} \mathrm{d}\theta \\ &= \frac{\mu \eta^2 (\lambda^2 - \sigma^2)}{2\sigma} \int_{\lambda-\sigma}^{\lambda+\sigma} \frac{1}{\theta^2 - \dfrac{\lambda \eta^2}{2\zeta_T \eta \mathrm{i} + \lambda}} \mathrm{d}\theta. \end{aligned} \tag{61}$$

Hence

$$\lim_{n\to\infty} B_D(\eta, \lambda, \sigma, \zeta_T) = B(\eta, \lambda, \sigma, \zeta_T). \tag{62}$$

That is, Eqs. (58) and (59) are in accordance if $n \to \infty$.

Now we can know that IMTMD is not a real physical system, but a approximate mathematical model close to MTMD made up of many TMDs.

5. Optimization method

If H_∞ optimization is adopted, which often appeared in previous studies [11–13], and then the objective function can be expressed as

$$\min_{\lambda, \sigma, \zeta_T} \max_{\eta} \mathrm{DMF}. \tag{63}$$

There are just three undetermined design parameters, i.e. λ, σ and ζ_T, so it is possible to obtain the numerical optimum solutions by the gradient search method as well as some previous studies. It is well known that during each iteration step the gradient search method will give a set of tentative parameters and find the steepest decrease direction, then adjust the parameters towards this direction. The performance, i.e. the objective function, must be evaluated in each iteration step. Consequently, the less the CPU time spent on each evaluation is, the better the efficiency of the method is.

6. Numerical result and comparison

In this paper, for the convenience of the engineering design and application of the MTMD/IMTMD, the damping ratio $\zeta_S = 0.02$ is selected for the steel-structure buildings according to the Chinese code-*Technical specification for steel structure of tall buildings*. For three different total mass ratio μ, the performances of MTMD made up of 2–200 TMDs and IMTMD are listed below. At the same time, in order to demonstrate the superior efficiency of IMTMD versus classical MTMD, the approximate time for each iteration step in gradient search are given, too. The MTMD model is calculated using TF by series forms Eqs. (42) and (59), while IMTMD by Eqs. (41) and (58). All these computations are carried out on the same PC (512 MB, Pentium 2.8 GHz, Matlab 7.1 by Mathworks). In Table 1, the row in which n is assigned to Inf denotes the IMTMD.

Some conclusions can be drawn from Table 1 as follows:

1. It seems that the 2nd hypothesis is better than the 1st one, which has been pointed out in the previous study [12].
2. The IMTMD can reflect the best or critical performance of MTMD composed of a large number of TMDs. In other words, for the optimum IMTMD and MTMD under the H_∞ optimization, there is a relationship between the TF of the primary structures equipped with them expressed as

$$\sup_{\eta}(|G_{\mathrm{IMTMD}}|) = \inf_{n}\left(\sup_{\eta}(|G_{\mathrm{MTMD}}|)\right). \tag{64}$$

Table 1

μ	The 1st model						The 2nd model					
	ζ_T	σ	λ	n	Max. DMF	Time per itr. step (s)	ζ_T	σ	λ	n	Max. DMF	Time per itr. step (s)
0.03	0.076	0.050	0.958	2	5.983	0.0021	0.071	0.054	0.963	2	5.975	0.015
	0.039	0.112	0.959	10	5.690	0.0035	0.034	0.117	0.971	10	5.440	0.0153
	0.027	0.125	0.965	20	5.689	0.0049	0.032	0.124	0.971	20	5.425	0.0161
	0.031	0.124	0.962	30	5.668	0.0057	0.028	0.129	0.971	30	5.422	0.0167
	0.031	0.126	0.962	50	5.667	0.0099	0.028	0.131	0.971	50	5.420	0.0166
	0.031	0.127	0.962	100	5.667	0.0221	0.028	0.133	0.971	100	5.419	0.0181
	0.031	0.128	0.962	200	5.667	0.0226	0.028	0.133	0.971	200	5.419	0.0248
	0.031	**0.129**	**0.962**	**Inf**	**5.667**	**0.0022**	**0.028**	**0.134**	**0.971**	**Inf**	**5.419**	**0.0023**
0.1	0.128	0.081	0.882	2	3.899	0.0028	0.127	0.094	0.899	2	3.915	0.0151
	0.074	0.189	0.879	10	3.795	0.0038	0.047	0.215	0.915	10	3.545	0.0154
	0.085	0.190	0.878	20	3.792	0.0054	0.053	0.219	0.920	20	3.495	0.0158
	0.084	0.198	0.876	30	3.789	0.0077	0.052	0.223	0.920	30	3.490	0.0166
	0.085	0.200	0.875	50	3.789	0.0101	0.051	0.227	0.920	50	3.488	0.0179
	0.074	0.208	0.879	100	3.790	0.0159	0.051	0.229	0.920	100	3.488	0.0197
	0.074	0.209	0.879	200	3.790	0.0225	0.051	0.231	0.920	200	3.488	0.0213
	0.084	**0.205**	**0.876**	**Inf**	**3.788**	**0.0024**	**0.051**	**0.232**	**0.920**	**Inf**	**3.487**	**0.0028**
0.2	0.176	0.099	0.788	2	3.167	0.0021	0.198	0.115	0.824	2	3.251	0.0164
	0.127	0.245	0.772	10	3.104	0.0037	0.067	0.289	0.851	10	2.989	0.0153
	0.130	0.249	0.775	20	3.099	0.0052	0.061	0.311	0.867	20	2.916	0.0157
	0.129	0.253	0.776	30	3.099	0.0071	0.062	0.308	0.860	30	2.874	0.0159
	0.130	0.256	0.775	50	3.099	0.0094	0.061	0.313	0.861	50	2.867	0.0165
	0.130	0.258	0.775	100	3.098	0.016	0.060	0.317	0.861	100	2.865	0.0201
	0.130	0.260	0.775	200	3.098	0.0224	0.060	0.318	0.861	200	2.864	0.0208
	0.130	**0.261**	**0.775**	**Inf**	**3.098**	**0.0023**	**0.060**	**0.320**	**0.861**	**Inf**	**2.864**	**0.0029**

3. The IMTMD have excellent efficiency of iteration. Its time spent on each iteration step is about $\frac{1}{10}$ of MTMD composed of 200 TMDs, however it still can give correct design parameters. Actually, from the above table it can be demonstrated that the MTMD can fully adopt design parameters from IMTMD when MTMD have more than 20 TMDs.
4. The objective function of IMTMD is smoother than that of MTMD, so it has less chance to be trapped in local minimum points.

7. Conclusion

A new optimization method is proposed for the MTMD, i.e. the IMTMD method. Some optimum parameters under various total mass ratios for the damped primary structure, whose damping ratio is selected as 0.02 according to the Chinese code—*Technical specification for steel structure of tall buildings* for the convenience of the engineering design and use, are given. At the same time, the approximate time spent on each iteration step of MTMD and IMTMD are listed for comparison. It is demonstrated that (1) the TF or DMF of the main structure with IMTMD can be expressed with a more simple way compared to that of the main structure with MTMD, (2) using the IMTMD method, we can obtain the critical performance of MTMD made up of many TMDs, (3) the IMTMD method have excellent iteration efficiency compared to the existing method, (4) if MTMD is composed of more than 20 TMDs, the IMTMD method can be used for the optimization design of MTMD. Hence, the IMTMD method is a simple and effective alternative; moreover, it is a very potential one for application in practice.

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