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Short Communication

Short-term flutter-type instability of undamped tdof system with randomly varying bifurcation parameter

M.F. Dimentberg[a,*], A. Naess[b]

[a]Department of Mechanical Engineering, Worcester Polytechnic Institute, Worcester, MA 01609, USA
[b]Centre for Ships and Ocean Structures and Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491, Trondheim, Norway

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Abstract

A model problem of a two-degrees-of-freedom (tdof) flutter is considered for an undamped system with random temporal variations of its bifurcation parameter. The nominal system, i.e. one with the mean value of the bifurcation parameter is assumed to be stable; however the above variations may occasionally bring the system temporarily into its domain of dynamic instability. A procedure for predicting probability density function (PDF) of the peaks of the corresponding intermittent response is outlined as based on parabolic approximation for the parameter variation in the vicinity of its peaks. For the case of relatively slow parameter variations the equations of motion are reduced by Krylov–Bogoliubov averaging to those describing static instability of the response amplitude. The basic relation between peak values of the bifurcation parameter and of the corresponding response outbreak for the reduced system is therefore available from previous studies of short-term static instability in a sdof system.

1. Introduction

Common practice for most designs of machines and structures is to completely preclude static and/or dynamic instability during long-term service, so that only stable designs are qualified as being acceptable. In some cases however, this practice may result in too conservative impractical designs. This may happen, for example, in cases of "temporary" or short-term instability due to very high short-term fluid loads such as those produced by wind gusts during hurricanes or ocean waves in severe storms or in cases of aeroelasticity of flight vehicles operating close to their instability boundary. In such cases the system may be designed to operate within its stability domain as long as "nominal" design parameters are considered. However, if the parameters may experience random temporal variations around their nominal or expected values, the system may become "temporary unstable" occasionally whenever its instability boundary is crossed. As long as complete elimination of such brief excursions into the instability domain may lead to impossible or impractical design the corresponding short-time outbreaks in response should be analyzed to evaluate the system's

*Corresponding author. Tel.: +1 508 831 5368; fax: +1 508 831 5680.
E-mail address: diment@wpi.edu (M.F. Dimentberg).

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reliability. In particular, predicting the probability density function (PDF) of the response peaks may be important for estimating corresponding damage accumulation in low-cycle fatigue due to the outbreaks.

The basic approach to such prediction has been outlined in [1,2] for the cases of static and dynamic instability respectively in a single-degree-of-freedom (sdof) system. The approach relies on the following approximation of a stationary zero-mean random process $g(t)$ with unit standard deviation in the vicinity of its peak which exceeds a given level u [3,4], that is after upcrossing level u at time instant $t = 0$:

$$g(t/u) \cong u + (1/u)(\varsigma t - \lambda^2 t^2/2) \text{ so that } g(t) \cong u + \varsigma t - (u/2)(\lambda t)^2$$
$$\text{for } t \in [0, 2\varsigma/\lambda^2 u] \text{ and } \max_t g(t) = g(\varsigma/\lambda^2 u) = g_p = u + \varsigma^2/2\lambda^2 u. \quad (1)$$

Here subscript "p" is used for peak values of random processes, ζ is the random slope of $g(t)$ at the instant of upcrossing and $\lambda^2 = \sigma_{\dot{g}}^2 = \int_{-\infty}^{\infty} \omega^2 \Phi_{gg}(\omega) d\omega / \int_{-\infty}^{\infty} \Phi_{gg}(\omega)\, \mathrm{d}\omega$ where $\Phi_{gg}(\omega)$ is power spectral density (PSD) of $g(t)$ so that λ is a mean frequency of $g(t)$. Thus the parabolic approximation (1) implies that the random process $g(t)$ is regarded as deterministic within the high-level excursion of duration $\tau_f = \lambda t_f = 2\varsigma/\lambda u$ above level u; during this time interval it depends just on its initial slope ζ at upcrossing which is regarded as a random *variable* for the excursion. Furthermore, the instant of downcrossing τ_f is clearly obtained as a second root of equation $g(t) = u$, the first one being $t = 0$. This probabilistic description may be used together with the solution for the transient response within the instability domain.

A sdof system with randomly varying stiffness as described by the approximation (1) has been considered in Ref. [1] accordingly; the PDF of peaks in the intermittent response was predicted using a numerical solution for the transient response during outbreak. The case of temporary dynamic instability in a sdof system with randomly varying apparent damping factor has been studied in Ref. [2]; using Krylov–Bogoliubov (KB) averaging [5] the problem has been reduced to a first-order equation for the response amplitude which permitted to obtain an explicit analytical solution for the response (and thus for the PDF of its peaks). This analytical solution has been extended later to a certain "axisymmetric" two-degrees-of-freedom (tdof) system through the use of a single complex generalized displacement [6].

In this Short Communication a model problem is considered of a "classical" flutter in a linear undamped tdof system with potential dynamic instability due to coalescing or merging of its natural frequencies [7]. Through the use of KB-averaging the problem of transient motion due to short-term *dynamic* instability is reduced to that due to short-term *static* instability for slowly varying amplitude—that is to the problem solved in Ref. [1] by numerical integration for the transient response.

2. Analysis

A model tdof system loaded by a nonconservative force is governed by equations of motion

$$\ddot{X}_1 + \Omega_1^2 X_1 + \gamma X_2 = 0, \quad \ddot{X}_2 + \Omega_2^2 X_2 - \gamma X_1 = 0, \quad (2a,b)$$

where dots denote differentiation over time t. This system is known to become dynamically unstable at sufficiently high value of the bifurcation parameter γ. The condition for this flutter-type instability in the undamped system (2) is that of coalescing or merging of natural frequencies of the tdofs. It will be assumed that the mean value of the bifurcation parameter belongs to the stability domain whereas its random temporal variations are sufficiently slow, thereby permitting transformations that should facilitate analysis of the transient response. Denoting

$$X_\pm = X_1 \pm X_2, \quad \Lambda^2 = \frac{1}{2}(\Omega_2^2 + \Omega_1^2), \quad \sigma = \frac{\Omega_2^2 - \Omega_1^2}{\Omega_2^2 + \Omega_1^2} \quad (3a-c)$$

(it will be assumed for definiteness that $\Omega_2 > \Omega_1$) Eqs. (2) may be rewritten for the transformed generalized displacements as

$$\ddot{X}_+ + \Lambda^2 X_+ = (\sigma\Lambda^2 + \gamma)X_-, \ddot{X}_- + \Lambda^2 X_- = (\sigma\Lambda^2 - \gamma)X_+. \quad (4a,b)$$

Assume now that the coefficients in RHSs of both Eqs. (4) are proportional to a small parameter so that the asymptotic method of KB-averaging can be applied [5]. As long as all actual parameters have finite albeit small values it may be assumed that $\sigma \ll 1, \gamma \ll \Lambda^2$ along with $\lambda \ll \Omega$, the latter condition being that of slow variations of the bifurcation parameter as mentioned before (zero-mean part of the variations will be eventually represented via process $g(t)$ as present in Eq. (1)).

A solution to the equation set (Eq. (4)) may be sought in the form

$$X_+(t) = X_{+c}(t)\cos\Lambda t, \quad \dot{X}_+(t) = -\Lambda X_{+c}(t)\sin\Lambda t$$
$$X_-(t) = X_{-s}(t)\sin\Lambda t, \quad \dot{X}_-(t) = \Lambda X_{-s}(t)\cos\Lambda t. \tag{5a−d}$$

From Eqs. (5a,b) we may single out X_{+c} and differentiate. Then $X_{+c}\dot{X}_{+c} = (\mathrm{d}/\mathrm{d}t)\left(X_+^2 + \dot{X}_+^2\big/\Lambda^2\right) = \left(\dot{X}_+/\Lambda^2\right)\left(\ddot{X}_+ + \Lambda^2 X_+\right)$ and substituting RHS of Eq. (4a) yields $\dot{X}_{+c} = -\Lambda^{-1}\sin\Lambda t\left[\left(\sigma\Lambda^2+\gamma\right)X_-\right]$.

The RHS of the resulting equation may be approximated by its average over "rapid" time within the response period $2\pi/\Lambda$. The above operations are then repeated for the second response amplitude X_{-s} as defined by the lower row of relations (5). Thus the following pair of first-order ODEs is obtained for slowly varying response amplitudes:

$$\dot{X}_{+c} = -\left(\sigma\Lambda/2 + \gamma/2\Lambda\right)X_{-s} \text{ and } \dot{X}_{-s} = \left(\sigma\Lambda/2 - \gamma/2\Lambda\right)X_{+c}. \tag{6a,b}$$

Two ODEs (6a,b) may be transformed to an equivalent single second-order ODE for any one of the two amplitudes. Thus

$$\ddot{X}_{+c} + \left[\left(\sigma\Lambda/2\right)^2 - \left(\gamma/2\Lambda\right)^2\right]X_{+c} = 0. \tag{7}$$

It is clearly seen that the equilibrium solution $X_{+c} \equiv 0$ to Eq. (7) for the response amplitude is unstable *statically* if $\gamma > \gamma_* = \sigma\Lambda^2$. This critical value of the bifurcation parameter as obtained by asymptotic analysis clearly coincides with the exact condition for *dynamic* instability of the original system (2) which corresponds to coalescing or merging of the system's natural frequencies as obtained from the corresponding characteristic equation [7].

Upon arriving to system (7) with "slow" time it is necessary now to "remember" that the bifurcation parameter γ is actually time-variant. Separating mean value and zero-mean part of its square denoted by angular brackets and subscript "zero", respectively, we may rewrite the ODE as

$$\ddot{X}_{+c} + \left[\Delta^2 - q(t)\right]X_{+c} = 0 \text{ where } \Delta^2 = \left(\sigma\Lambda/2\right)^2 - \left(\gamma/2\Lambda\right)^2, q(t) = \gamma_0^2(t)\big/(2\Lambda)^2. \tag{8}$$

It is assumed that $\Delta^2 > 0$ so that the nominal or mean system is stable. The zero-mean process $q(t)$ may now be scaled to its standard deviation σ_q by introducing process $g(t) = q(t)\big/\sigma_q$ with unit standard deviation. The parabolic approximation (1) may be used for this process, with scaled instability threshold defined as $u = \Delta^2\big/\sigma_q$. Upon introducing transformed local time $\tau = \lambda(t - t_u)$ with origin at the instant of upcrossing t_u Eq. (8) is transformed to

$$X''_{+c} + (\Delta/\lambda)^2\left[-\frac{\varsigma\tau}{\lambda u} + \tau^2/2\right]X_{+c} = 0, \tag{9}$$

where primes denote differentiation over τ.

Eq. (9) coincides (with just slight differences in notation) with the equation for static instability of a sdof system—one that has been solved numerically in Ref. [1] for its transient response during short-term outbreak. The results provided the relation between the peak value of the response (it would be amplitude in the present case) and that of the process $g(t)$. Thus a reliability study of system (2) which is prone to short-term or temporary instability—e.g. analysis of the PDF of response peaks for evaluating low-cycle fatigue, or predicting first-passage failure—is reduced to analysis of relevant statistics of temporal variations in the bifurcation parameter. One favorable difference from the case of a damped system as considered in Eq. (1) should be mentioned here. Namely, no extrapolation is required for approximation (1) beyond final instant $\tau_f = 2\varsigma/\lambda u$ of downcrossing zero level by $g(t)$. At this instant peak of first derivative of X'_{+c} is clearly attained but not of X_{+c} itself in general; however for the present case it does not matter as long as $X'_{+c} = X_{-s}$ and the latter variable satisfies the same ODE (8).

3. Conclusions

A model problem of a tdof flutter due to coalescing or merging of natural frequencies at high level of nonconservative loading has been considered for an undamped system with random temporal variations of its bifurcation parameter. Whilst the nominal system, i.e. one with mean value of the bifurcation parameter is assumed to be stable, the above variations may occasionally bring the system temporarily into its domain of dynamic instability. Procedure for predicting PDF of the peaks in the resulting intermittent response is outlined as based on parabolic approximation of the parameter variation in the vicinities of its peaks together with solution for transient amplitude response of the system during short-term instability. Solution to the latter problem of transient response is reduced using KB-averaging, to previously obtained numerical solution for transient response of a sdof system during short-term *static* instability. The results may be used for predicting low-cycle fatigue in marginally unstable structures, i.e. those with relatively rare and brief potential excursions into domain of dynamic instability.

A certain comment on potential extension of the basic model seems appropriate here. Namely, in some applications nonconservative terms in Eqs. (2) may contain additional different constant coefficients so that coefficient γ in RHSs should be replaced by γ_1 and γ_2 in the first and second equations (2), respectively; see example of this case in Chapter 5 of the book [8]—tdof flutter of a row of cylinders in a cross flow of fluid. Direct stability analysis of the corresponding characteristic equation shows that only product of these coefficients does enter the condition for neutral stability derived as that for merging of natural frequencies; this (exact) condition may be written as $\tilde{\gamma}_* = \sigma\Lambda^2$ where $\tilde{\gamma} = \sqrt{\gamma_1\gamma_2}$ and star subscript corresponds to critical value at the neutral stability boundary. On the other hand, repeating the above KB-averaging analysis we obtain the same (approximate) results as before if γ is replaced by the arithmetic mean $\bar{\gamma} = \frac{1}{2}(\gamma_1 + \gamma_2)$ so that $\bar{\gamma}_* = \sigma\Lambda^2$. It is clearly seen that the exact and approximate stability conditions coincide when $\gamma_1 = \gamma_2$; otherwise the approximate condition provides the conservative estimate for stability as long as $\bar{\gamma} > \tilde{\gamma}$ for $\gamma_1 \neq \gamma_2$. Thus the simple procedure for transient response analysis for the case of short-term instability should also be conservative if both actual coefficients γ_1 and γ_2 are replaced by their arithmetic mean values in Eqs. (2) (and both are proportional to the same function of time as is the case with the above potential application described in Ref. [8]). The above considerations are also important for the case where the only available data on γ_1 and γ_2 are obtained from stability tests where just onset of instability is observed and thus only their product would be known.

A final comment should be made here regarding an important potential limitation in applicability of the present analysis to linearized model of *nonlinear* systems. A thorough study of influence of nonlinearities on aeroelastic behavior of flight vehicles is presented in Refs. [9,10]. First of all, quantitative results of the present analysis may not be directly applicable to systems with nonsmooth nonlinearities, such as those due to gaps, pre-stress, etc. Furthermore, the linearized model may not be applicable in case of smooth "softening" nonlinearity(ies) which may lead to a stepwise "jump" of the system from equilibrium state to a limit cycle (such nonlinearities were called "evil" in Ref. [10]). On the other hand, smooth "stiffening" nonlinearity (which is called "good" in Ref. [10]) may by itself restrict growth of response amplitude in case of "long-term" instability whereas in case of the "short-term" instability it may be of minor importance provided that response remains sufficiently small during its transient outbreaks; it is the latter case for which the present linearized model is quantitatively adequate.

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