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Short Communication

On a superposition method for the approximate determination of the eigenfrequencies of nonlinear conservative oscillators

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Abstract

Recently, Hu [A note on the frequency of nonlinear conservative oscillators, *Journal of Sound and Vibration* 286 (2005) 653–662] presented a superposition method for the approximate determination of frequencies of conservative oscillators when the nonlinear restoring force consists of a superposition of several individual characteristics. In this contribution, it is shown that the conjecture of Hu is not true in general, particularly for underlinear systems. Only for overlinear systems are there plausible reasons for the validity of the conjecture, but even for this case one has to use caution.
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1. Introduction

Recently in this journal, Hu [1] presented a superposition method for the approximate determination of frequencies of nonlinear conservative oscillators of the type:

$$\ddot{x}+f(x)=0,\quad x(0)=A,\quad \dot{x}(0)=0, \tag{1}$$

with

$$f(x)=\sum_{i=1}^{n} f_i(x), \tag{2}$$

where $f_i(-x)=-f_i(x)$ is assumed. Additionally, he considered the auxiliary equations:

$$\ddot{x}+f_i(x)=0,\quad x(0)=A,\quad \dot{x}(0)=0,\quad i=1,\dots,n. \tag{3}$$

Denoting the exact eigenfrequencies of the systems (1,3) by ω_e and ω_{ei}, $i=1,\dots,n$, and any approximation of them by ω_a and ω_{ai}, $i=1,\dots,n$, Hu stated the relation:

$$\omega_a^2=\sum_{i=1}^{n}\omega_{ai}^2. \tag{4}$$

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It should be mentioned that the relation (4) had been already introduced by Atkinson [2] more than 40 years ago. The formal difference that Atkinson considered, polynomial-type restoring forces instead of more general functions $f_i(x)$, is marginal. Both authors showed the existence of the relation (4) not in the general sense, but only for the approximation method of harmonic balance:

$$\omega_h^2 = \sum_{i=1}^{n} \omega_{hi}^2, \tag{5}$$

where ω_h and ω_{hi}, $i = 1,\ldots,n$, denote the approximate eigenfrequencies of the systems (1,3) according to the method of harmonic balance [3]. Additionally, they illustrated the approximation

$$\omega_e^2 \approx \omega_a^2 \;\text{ or rather }\; \omega_e^2 \approx \omega_h^2 \tag{6}$$

by certain examples which were essentially "overlinear" (see below).

In this contribution, it is shown that Eq. (4) does not hold in general, although Eq. (5) is valid, as has been proved in Refs. [1,2]. Additionally, approximation (6) may fail, especially in cases of "underlinear" restoring forces.

2. Qualitative analysis

A qualitative analysis of systems (1,3) can be simplified for special restoring forces which are either overlinear, linear, or underlinear. According to Ref. [4], a restoring force $f(x)$ with $f(0) = 0$ and $xf(x) > 0$, $x \neq 0$, is classified as follows:

$$f(x)\begin{cases}\text{overlinear}\\ \text{linear}\\ \text{underlinear}\end{cases} \quad \text{if} \quad \frac{f(x)}{x}\begin{cases}\text{increases monotonically,}\\ \text{remains constant,}\\ \text{decreases monotonically}\end{cases} \tag{7}$$

for increasing $|x|$. If $f(x)$ is one or two times differentiable, then following conditions hold:

$$f(x)\begin{cases}\text{overlinear}\\ \text{underlinear}\end{cases} \quad \text{if} \quad \begin{cases}xf'(x) - f(x) \geqslant 0 \;\text{ or }\; xf''(x) \geqslant 0,\\ xf'(x) - f(x) \leqslant 0 \;\text{ or }\; xf''(x) \leqslant 0.\end{cases} \tag{8}$$

Overlinear or underlinear restoring forces are also called hardening or softening spring forces, cf. Ref. [5]. A classical result for the periods of the oscillations of the system (1) is repeated here for the sake of completeness: if $f(x)$ is entirely overlinear then the period $T_e = 2\pi/\omega_e$ of the vibration decreases with increasing amplitude A; if $f(x)$ is linear, then T_e does not depend on A; and finally, if $f(x)$ is entirely underlinear, then T_e increases with increasing A.

As a consequence, if all $f_i(x)$, $i = 1,\ldots,n$, are overlinear restoring forces, then the superposition of $f_i(x)$ in Eq. (2) gives a "more" overlinear function and the period T_e is smaller than the single periods $T_{ei} = 2\pi/\omega_{ei}$. In other words, the eigenfrequency ω_e becomes progressively larger by adding the various restoring forces $f_i(x)$. Therefore, a result in the sense of Eq. (4) looks plausible. In the opposite case, for underlinear restoring functions $f_i(x)$, $i = 1,\ldots,n$, the problem is more complicated. The superposition of underlinear restoring forces leads to an overall characteristic which is still underlinear but "less" underlinear than each single $f_i(x)$. Therefore, even though ω_e decreases with increasing amplitude A, it increases with the addition of a single $f_i(x)$. Given these two opposite trends, the total behavior is not predetermined, and a result like Eq. (4) is not certain.

This qualitative analysis is strengthened by the fact that the relation between the exact eigenfrequency ω_e and the eigenfrequency ω_a, obtained with an approximation method, depends strongly on the type of approximation method used. In Ref. [6], it has been shown that for all types of restoring forces $f(x)$ with $f(0) = 0$, $xf(x) > 0$, $x \neq 0$, the method of harmonic balance always leads to

$$\omega_e(A) \leqslant \omega_h(A) \tag{9}$$

resulting in:

$$\omega_e^2(A) \leqslant \sum_{i=1}^{n} \omega_{hi}^2(A) \tag{10}$$

according to Eq. (5). But for other averaging methods (energy-averaging, various force averaging methods), a relation like Eq. (9) does not exist in general. Depending on the type of restoring force and on the approximation method, one can find

$$\omega_e(A) \underset{>}{\leqslant} \omega_a(A). \tag{11}$$

3. Counterexample

By a counterexample, it is shown that conjectures (4) and (6) by Hu [1] and Atkinson [2] do not hold in the general case. The exact calculation of the eigenfrequencies is used as the approximation method here.

Consider the wobble vibrations of a vertically placed block (rectangular parallelepiped) on a horizontal plane. The equation of motion is given by

$$\ddot{x} + f_w(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{12}$$

$$f_w(x) = f_0\left(1 - \frac{x}{x_0}\right), \quad x > 0; \quad f_w(-x) = -f_w(x), \tag{13}$$

where x corresponds to the angular displacement of the block, [7]. The characteristic $f_w(x)$ is shown in Fig. 1. This restoring force can be considered as a limit of a differentiable underlinear force function, but its piecewise linear behavior allows explicit calculation of all values of interest. The vibration amplitude A is bounded by x_0, and the limiting case of $A \to x_0$ will be discussed for simplicity.

The period T_w is determined by

$$T_w(A) = 4\sqrt{\frac{x_0}{f_0}} \ln \frac{1 - \frac{A}{x_0}}{1 - \sqrt{1 - \left(1 - \frac{A}{x_0}\right)^2}}. \tag{14}$$

According to the underlinear character of the restoring force $f_w(x)$, the period (14) increases with increasing amplitude. In the limit as $A \to x_0$, it is

$$\lim_{A \to x_0} T_w(A) = \infty. \tag{15}$$

Based on the wobble oscillator, the system (1,2) is considered with

$$f(x) = f_1(x) + f_2(x), \quad f_1(x) = f_w(x), \quad f_2(x) = mx. \tag{16}$$

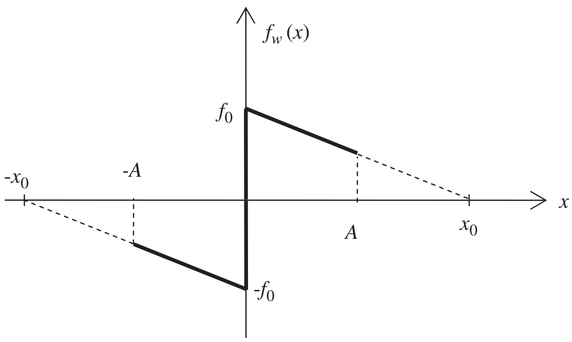

Fig. 1. Restoring force of the wobble oscillator.

For $i = 1{,}2$ the squared eigenfrequencies are determined as

$$\omega_{e1}^2 = \frac{4\pi^2}{T_{e1}^2}, \quad T_{e1} = T_w(A), \quad \omega_{e2}^2 = m. \tag{17}$$

To determine ω_e, the results of the wobble oscillator can be transferred. For $0 < m < f_0/x_0$ the restoring force $f(x)$ looks similar to that of $f_w(x)$, but only the zero crossing $f(x_1) = 0$ is shifted from x_0 to

$$x_1 = \frac{f_0 x_0}{f_0 - m x_0}. \tag{18}$$

Therefore, the period $T_e(A)$ runs as

$$T_e(A) = 4\sqrt{\frac{x_1}{f_0}} \ln \frac{1 - \frac{A}{x_1}}{1 - \sqrt{1 - \left(1 - \frac{A}{x_1}\right)^2}}. \tag{19}$$

Particularly, evaluating Eq. (19) for $A \to x_0$:

$$T_e(x_0) = 4\sqrt{\frac{x_0}{f_0}} \frac{1}{\sqrt{1-r}} \ln \frac{r}{1 - \sqrt{1 - r^2}}, \quad r = \frac{m x_0}{f_0} \tag{20}$$

is obtained. With respect to conjecture (4) for $A = x_0$, the squared eigenfrequencies $\omega_{e1}^2 = 0$ according to Eq. (15), $\omega_{e2}^2 = m$ according to Eq. (17), and

$$\omega_e^2 = \frac{\pi^2}{4} \frac{f_0}{x_0} (1 - r) \frac{1}{\left[\ln \frac{r}{1 - \sqrt{1 - r^2}}\right]^2} \tag{21}$$

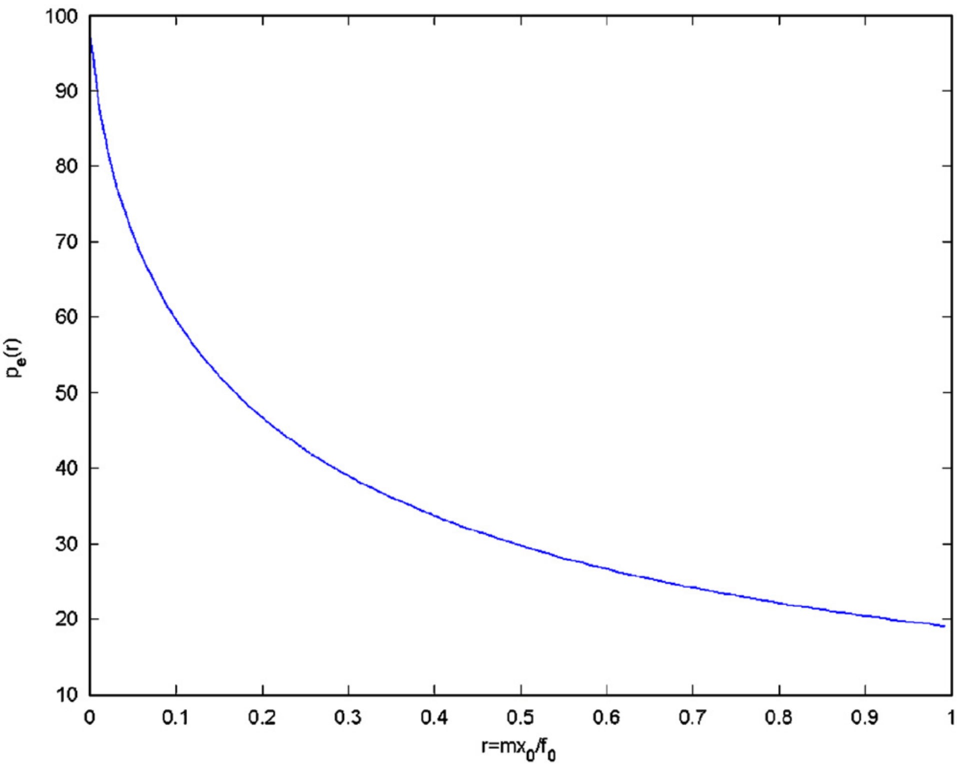

Fig. 2. Percentage of approximation error $p_e(r) = 100(1 - (m/\omega_e^2))$.

according to Eq. (20), should satisfy the following equation:

$$\omega_e^2 \approx \omega_{e1}^2 + \omega_{e2}^2 = m. \tag{22}$$

Fig. 2 shows the percentage of the approximation error $p_e(r) = 100(1 - (m/\omega_e^2))$. Estimate (22) completely fails for small values of $r = mx_0/f_0$; for $r = 0$ the error is 100%. For increasing r the error becomes smaller but even for $r = 1$ (which corresponds to a signum function of the restoring force $f(x)$), a failure of 19% still remains. Therefore, the conjectured relation (4) cannot be reasonably applied.

This example can also be used to show the quality of the method of harmonic balance. Applying this method to the restoring forces in Eq. (16), the approximate squared eigenfrequencies

$$\omega_{h1}^2 = \frac{4}{\pi A}\int_0^{\pi/2} f_w(A\cos\tau)\cos\tau\,\mathrm{d}\tau = \frac{4f_0}{\pi A} - \frac{f_0}{x_0}, \tag{23}$$

$$\omega_{h2}^2 = m, \tag{24}$$

$$\omega_h^2 = \frac{4f_0}{\pi A} - \frac{f_0}{x_1} = \frac{4f_0}{\pi A} - \frac{f_0}{x_0} + m \tag{25}$$

are obtained. Hence, the equality (5) is confirmed. But how good are the approximations? To answer that question, the wobble oscillator is considered again. The percentage of the approximation error $p_h(\lambda) = 100(1 - (\omega_{h1}^2/\omega_{e1}^2))$ is represented in Fig. 3 for $0<\lambda<1$, $\lambda = A/x_0$. For small amplitudes the result is very satisfactory, but even for $A = 0.75x_0$ an error of about 20% appears, and for $A \to x_0$ the error increases to infinity. The negative sign of the error corresponds to the inequality in Eq. (9). Although the method of harmonic balance leads to good approximations in many applications, there is no guarantee that it works well in all cases.

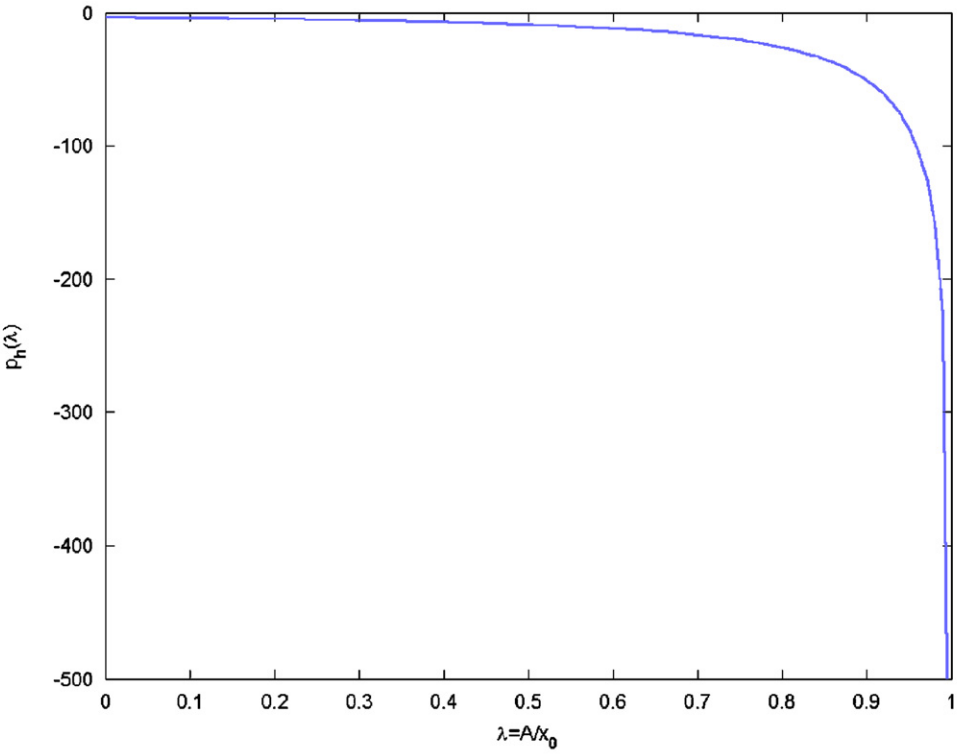

Fig. 3. Percentage of approximation error $p_h(\lambda) = 100(1 - (\omega_{h1}^2/\omega_{e1}^2))$.

4. Conclusions

Conjectures (4) and (6) of Hu [1] and Atkinson [2] are not true in general. Although the result (5) is valid, its application has to be considered carefully because the method of harmonic balance may fail.

The conjecture (6) and the relation (5) may be applied for overlinear systems. Although there is not a proof for the validity of Eq. (6), there are plausible reasons that for this case the error remains small. But for underlinear systems, the conjecture (6) is not trustworthy. The counterexample showed its failure. For mixed-type characteristics (where the restoring force is not exclusively overlinear, linear, or underlinear), it is also not recommended to apply the method of Hu [1] and Atkinson [2].

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