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Transverse vibrations of non-homogeneous orthotropic rectangular plates of variable thickness: A spline technique

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Abstract

An analysis and numerical results are presented for free transverse vibrations of non-homogeneous orthotropic rectangular plates of non-uniform thickness and resting on an elastic foundation of Winkler type on the basis of classical plate theory. The non-homogeneity of the plate material is assumed to arise due to the exponential variation in Young's moduli and density along one direction. Following Lévy approach i.e. the two parallel edges are simply supported, the fourth-order differential equation governing the motion of such plates of exponentially varying thickness in one direction, has been solved by using the quintic splines interpolation technique for three different combinations of clamped, simply supported and free boundary conditions at the other two edges. Effect of the non-homogeneity and elastic foundation together with other plate parameters such as orthotropy, aspect ratio and thickness variation on the natural frequencies of vibration is illustrated for the first three modes of vibration. Normalized displacements are presented for specified plates for all the three boundary conditions.
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1. Introduction

In the recent past, there has been a phenomenal increase in the development of fiber-reinforced materials due to the desirability of lightweight, high strength, corrosion resistance and high-temperature performance requirements in modern technology. Plates of composite materials are widely used in many engineering structures and machines. By appropriate variation of plate thickness, these plates have significantly greater efficiency for vibration as compared to the plates of uniform thickness and also provide the advantage of reduction in weight and size. Thus, their design requires an accurate determination of their natural frequencies and mode shapes. An extensive review of the work up to 1985 on linear vibration of isotropic/anisotropic plates of various geometries has been given by Leissa in his monograph [1] and in a series of review articles [2–5]. The studies of orthotropic rectangular plates with uniform/non-uniform thickness with various edge conditions after 1985 has been carried out by a number of researchers and are reported in Refs. [6–15] to mention the prominent ones.

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Non-homogeneous elastic plates find their applications in the design of space vehicles, modern missiles and aircraft wings [16]. Materials and structural components are often non-homogeneous, either by design or because of the physical composition and imperfections in the underlying materials. Thus non-homogeneous materials are of considerable interest to design engineers and have widespread application due to prevalence of these materials in the field of electronics and aerospace industry. Very few models representing the behavior of non-homogeneous materials have been reported in the literature. The earliest model was proposed by Bose [17], where Young's modulus and density are supposed to vary with radius vector i.e. $(E, \rho) = (E_0, \rho_0)r$. Biswas [18] in his model considered exponential variations for torsional rigidity $\mu = \mu_0 e^{-\mu_1 z}$ and the material density $\rho = \rho_0 e^{-\mu_1 z}$, where μ_0, μ_1 and ρ_0 are constants. Rao et al. [19] dealing with vibration of non-homogeneous isotropic thin plates have assumed linear variations for Young's modulus and density given by $E = E_0\ (1+\alpha x)$ and $\rho = \rho_0\ (1+\beta x)$. In a series of papers, Tomar et al. [20–23] have analyzed the dynamic behavior of non-homogeneous isotropic plates of variable thickness of different geometries. The non-homogeneity of the plate material is assumed to arise due to the variations of Young's modulus and the density exponentially along one direction as $(E, \rho) = (E_0, \rho_0)e^{\beta x}$, where E_0, ρ_0 are constants and β is the non-homogeneity parameter. In all the above studies, Poisson's ratio is assumed to remain constant. In a recent paper, Lal and Sharma [24] have analyzed the axisymmetric vibrations of non-homogeneous polar orthotropic annular plates of variable thickness assuming that Young's moduli and density of the plate material vary exponentially in the radial direction in the same manner i.e. $(E_r, E_\theta, \rho) = (E_1, E_2, \rho_0)e^{\beta r}$ where β is the non-homogeneity parameter and E_1, E_2, ρ_0 are constants. The similar type of variation in both E and ρ with non-homogeneity parameter β does not seem to have any justification as has also been pointed out by Rao et al. [19]. Very recently, Gupta et al. [25,26] have studied the axisymmetric vibrations of non-homogeneous circular plates of quadritically varying thickness where Young's modulus and density are assumed to vary exponentially in radial direction i.e. $E = E_0 e^{\mu x}$ and $\rho = \rho_0 e^{\eta x} (\mu \neq \eta)$. Poisson's ratio has been assumed to be constant.

In an up-to-date survey of literature, the authors have not come across any study dealing with vibration of non-homogeneous orthotropic rectangular plates with the exception of Fares and Zenkour [27], which deals with buckling, and free vibration of non-homogeneous composite laminated plates. Keeping this in view, a study dealing with transverse vibrations of non-homogeneous orthotropic rectangular plates of exponentially varying thickness along one direction and resting on a Winkler-type elastic foundation is presented employing classical plate theory. For non-homogeneity of the plate material it is assumed that Young's moduli and density vary exponentially along one direction. The governing differential equation for such plates with two opposite edges simply supported reduces to fourth-order differential equation with variable coefficients whose analytical solution is not feasible. Quintic splines interpolation technique has been employed to obtain the natural frequencies for three different combinations of clamped, simply supported and free boundary conditions at the other two edges. This method is preferred because a chain of lower-order approximations may yield a better accuracy than a global higher-order approximation [28] and natural boundary conditions can be considered easily. The effect of various plate parameters has been studied on the natural frequencies for the first three modes of vibration. A Huber-type orthotropic material 'ORTHO1' has been taken as an example of rectangular orthotropic material.

The consideration of non-homogeneity, orthotropy, thickness variation, elastic foundation and aspect ratio leads to a very complex problem involving several parameters. However, with the choice of Lévy approach and one-dimensional variations in thickness, Young's moduli and density one can find the approximate solution of the present problem. This type of orthotropy and non-homogeneity arises during the fiber-reinforced plastic structures, which use fibers with different moduli along two mutually perpendicular directions (i.e. along axes) and different strength properties. Further, this type of variation in thickness and non-homogeneity consideration is of interest since it provides reasonable approximation to linear variation. Thus, the present study of theoretically investigated vibrational characteristics will be of interest to design engineers.

2. Mathematical formulation

Consider a rectangular orthotropic plate of length a, width b, thickness $h = (x, y)$, density ρ, and resting on a Winkler-type elastic foundation of foundation modulus k_f. The plate is referred to a system of rectangular

Cartesian coordinates (x, y, z). The middle surface being $z=0$ and the origin is at one of the corners of the plate. The x- and y-axis are taken along the principle directions of orthotropy, the axis of z is perpendicular to the xy-plane. The differential equation governing the free transverse vibration of such plates is given by

$$D_x\frac{\partial^4 w}{\partial x^4}+D_y\frac{\partial^4 w}{\partial y^4}+2H\frac{\partial^4 w}{\partial x^2\partial y^2}+2\frac{\partial H}{\partial x}\frac{\partial^3 w}{\partial x\partial y^2}+2\frac{\partial H}{\partial y}\frac{\partial^3 w}{\partial y\partial x^2}+2\frac{\partial D_x}{\partial x}\frac{\partial^3 w}{\partial x^3}+2\frac{\partial D_y}{\partial y}\frac{\partial^3 w}{\partial y^3}$$
$$+\frac{\partial^2 D_x}{\partial x^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 D_y}{\partial y^2}\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 D_1}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 D_1}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+4\frac{\partial^2 D_{xy}}{\partial x\partial y}\frac{\partial^2 w}{\partial y\partial x}+\rho h\frac{\partial^2 w}{\partial t^2}+k_f w=0, \tag{1}$$

where $D_x=E_x^*h^3/12$, $D_y=E_y^*h^3/12$, $D_{xy}=G_{xy}h^3/12$, $D_1=E^*h^3/12$, $H=D_1+2D_{xy}$, $E^*=v_yE_x^*=v_xE_y^*$, $(E_x^*,E_y^*)=(E_x,E_y)/(1-v_xv_y)$, $w(x,y,t)$ is the transverse deflection, t the time, ρ the mass density and E_x, E_y, v_x, v_y and G_{xy} are material constants in proper directions defined by an orthotropic stress–strain law.

Let us assume that the thickness of the plate varies in the x-direction only i.e. $h=h(x)$ and the two opposite edges $y=0$ and b are simply supported. For a harmonic solution, the deflection w (Lévy approach) is assumed to be

$$w(x,y,t)=\bar{w}(x)\sin(p\pi y/b)e^{i\omega t}, \tag{2}$$

where p is the positive integer and ω the frequency in radians.

Further, for elastically non-homogeneous material, it is assumed that the Young's moduli E_x, E_y and density ρ are the functions of space variable x only and following Leikhnitskii [29], Panc [30], the shear modulus is $G_{xy}=\sqrt{E_xE_y}/2(1+\sqrt{v_xv_y})$.

Introducing the non-dimensional variables

$$X=x/a,\quad Y=y/b,\quad \bar{h}=h/a,\quad W=\bar{w}/a. \tag{3}$$

Eq. (1) reduces to

$$\begin{aligned}&\bar{h}^2E_xW^{iv}+[2(\bar{h}^3E_x'+\bar{h}^2\bar{h}'E_x)]W'''\\&+[(6\bar{h}\bar{h}'^2+3\bar{h}^2\bar{h}')E_x+6\bar{h}^2\bar{h}'E_x'+\bar{h}^3E_x''-2\lambda^2\bar{h}^3(E^*+2G_{xy})(1-v_xv_y)]W''\\&-[2\lambda^2\{3\bar{h}^2\bar{h}'(v_yE_x+2(1-v_xv_y)G_{xy})+\bar{h}^3(v_yE_x'+2(1-v_xv_y)G_{xy}')\}]W'\\&+[\lambda^4\bar{h}^3E_y-\lambda^2v_y\{\bar{h}^3E_x''+6\bar{h}^2\bar{h}'E_x'+(6\bar{h}\bar{h}'^2+3\bar{h}^2\bar{h}'')E_x\}\\&-12(1-v_xv_y)(\rho\bar{h}a^2\omega^2+ak_f)]W=0,\end{aligned} \tag{4}$$

where $\lambda^2=p^2a2\pi^2/\mathrm{b}^2$ and primes denote differentiation with respect to X.

For exponential variation in thickness [31,32], i.e., $\bar{h}=h_0e^{\alpha X}$ and following Refs. [20–24] for non-homogeneity of the plate material in X direction as follows:

$$E_x=E_1e^{\mu X},\quad E_y=E_2e^{\mu X},\quad \rho=\rho_0e^{\beta X}, \tag{5}$$

where h_0, ρ_0 are the thickness and density of the plate at $X=0$, μ the non-homogeneity parameter, α the taper parameter, β the density parameter and E_1, E_2 the Young's moduli in proper directions at $X=0$. The type of non-homogeneity considered here is mainly for illustration. The new present approach shows that how the non-homogeneity may be theoretically investigated in the study of vibrational behaviors of plate-type structures.

Eq. (4) now reduces to

$$A_0W^{iv}+A_1W'''+A_2W''+A_3W'+A_4W=0, \tag{6}$$

where

$$\begin{aligned}&A_0=1,\quad A_1=2(\mu+3\alpha),\quad A_2=(\mu+3\alpha)^2-2\sqrt{E_2/E_1}\lambda^2,\quad A_3=-2\lambda^2(\mu+3\alpha)\sqrt{E_2/E_1},\\&A_4=\lambda^4E_2/E_1-\lambda^2v_y(\mu+3\alpha)^2+12Ke^{(\mu+3\alpha)X}/h_0^3-\Omega^2e^{(-\mu-2\alpha+\beta)X},\\&K=ak_f(1-v_xv_y)/E_1,\quad \Omega^2=12\rho_0(1-v_xv_y)a^2\omega^2/E_1h_0^2.\end{aligned}$$

The solution of Eq. (6) together with the boundary conditions at the edges $X=0$ and $X=1$ constitutes a well-defined two-point boundary value problem. Owing to the presence of variable coefficients in Eq. (6), its closed form solution is not possible. Keeping this in view, an approximate solution is obtained by employing the quintic splines interpolation technique.

3. Solution by quintic splines

According to the spline technique [32], the interval [0, 1] is divided into n equal subintervals ΔX by means of points X_i, $i=0, 1, 2,\ldots, n$ so that $n\Delta X=1$. The quintic spline takes the form

$$W(X) = a_0 + \sum_{j=1}^{4} a_j (X - X_0)^j + \sum_{i=0}^{n-1} b_i (X - X_i)_+^5, \tag{7}$$

where

$$(X - X_i)_+ = \begin{cases} 0 & \text{if} \quad X \leq X_i, \\ X - X_i & \text{if} \quad X > X_i \end{cases}$$

and $a_0, \ldots, a_4, b_0, \ldots, b_{n-1}$ are $(n+5)$ unknown constants.

Substitution for $W(X)$ and its derivatives into Eq. (6) gives, for satisfaction at the mth node

$$\begin{aligned} & A_4 a_0 + [A_4(X_m - X_0) + A_3]a_1 + [A_4(X_m - X_0)^2 + 2A_3(X_m - X_0) + 2A_2]a_2 \\ & \quad + [A_4(X_m - X_0)^3 + 3A_3(X_m - X_0)^2 + 6A_2(X_m - X_0) + 6A_1]a_3 \\ & \quad + [A_4(X_m - X_0)^4 + 4A_3(X_m - X_0)^3 + 12A_2(X_m - X_0)^2 + 24A_1(X_m - X_0) + 24A_0]a_4 \\ & \quad + \sum_{i=0}^{n-1} [A_4(X_m - X_i)_+^5 + 5A_3(X_m - X_i)_+^4 + 20A_2(X_m - X_i)_+^3 + 60A_1(X_m - X_i)_+^2 \\ & \quad + 120A_0(X_m - X_i)_+]b_i = 0. \end{aligned} \tag{8}$$

For $m=0(1)n$, one obtains a set of $(n+1)$ homogenous equations having $(n+5)$ unknowns a_i, $i=0(1)4$, b_j, $j=0, 1,\ldots, (n-1)$, which can be represented by the matrix equation

$$[A]\{B\} = \{0\}, \tag{9}$$

where A is a matrix of order $(n+1)\times(n+5)$ and, $\{B\}$ and $\{0\}$ are column vectors of order $(n+5)\times 1$.

4. Boundary conditions and frequency equations

The three sets of boundary conditions namely C–C, C–S, C–F have been considered in which the first symbol represents the condition at the edge $X=0$ and second symbol at the edge $X=1$ and C, S, F stand for clamped, simply supported and free edge, respectively. The relations that should be satisfied at clamped, simply supported and free edges are

$$W = \mathrm{d}W/\mathrm{d}X = 0; \quad W = (\mathrm{d}^2W/\mathrm{d}X^2) - (E^*/E_x^*)\lambda^2 W = 0 \quad \text{and}$$

$$(\mathrm{d}^2W/\mathrm{d}X^2) - (E^*/E_x^*)\lambda^2 W = \frac{\mathrm{d}}{\mathrm{d}X}\left(E_x\overline{h}^3\left\{\frac{\mathrm{d}^2W}{\mathrm{d}X^2} - \lambda^2 v_y W\right\}\right) - 4\lambda^2(1 - v_x v_y)G_{xy}\overline{h}^3(\mathrm{d}W/\mathrm{d}X) = 0,$$

respectively.

Applying the boundary condition C–C to the displacement function given by Eq. (7), one obtains a set of four homogenous equations in terms of unknown constants a_i, $i=0(1)4$, b_j, $j=0, 1,\ldots, (n-1)$, which can be written as

$$\left[B^{CC}\right]\{B\} = \{0\}, \tag{10}$$

where B^{CC} is a matrix of order $4\times(n+5)$.

Eq. (10) together with Eq. (9) gives a complete set of $(n+5)$ equations in $(n+5)$ unknowns that can be denoted as

$$\begin{bmatrix} A \\ B^{CC} \end{bmatrix} \{B\} = \{0\}. \tag{11}$$

For a non-trivial solution of Eq. (11), the frequency determinant must vanish and hence,

$$\begin{vmatrix} A \\ B^{CC} \end{vmatrix} = 0. \tag{12}$$

Similarly for C–S and C–F plates, the frequency determinants can be written as

$$\begin{vmatrix} A \\ B^{CS} \end{vmatrix} = 0, \quad \begin{vmatrix} A \\ B^{CF} \end{vmatrix} = 0, \tag{13,14}$$

respectively.

5. Numerical results and discussion

The frequency equations (12)–(14) provide the values of the frequency parameter Ω for various values of plate parameters. In the present work, numerical results have been computed for first three modes of vibration, for different values of foundation parameter $K = 0.00$, 0.01, 0.02, non-homogeneity parameter $\mu = -0.5, -0.3, -0.1, 0.0, 0.1, 0.3, 0.7, 1.0$ density parameter $\beta = -0.5, -0.3, -0.1, 0.0, 0.1, 0.3, 0.5, 0.7, 1.0$, taper parameter $\alpha = -0.5, -0.3, -0.1, 0.0, 0.1, 0.3, 0.5$, and aspect ratio $a/b = 0.3, 0.5, 1.0, 1.5, 2.0$, for three boundary conditions C–C, C–S and C–F. This is achieved by writing $p = 1$ in the frequency equations and determined the first three values of Ω. The elastic constants for the plate material are taken as $E_1 = 1 \times 10^{10}$ MPa, $E_2 = 5 \times 10^9$ MPa, $\upsilon_x = 0.2$, $\upsilon_y = 0.1$, given by Biancolini et al. [17] ('ORTHO1'). The thickness h_0 at the edge $X = 0$ has been taken as 0.1.

To choose the appropriate interval ΔX, the computer program developed for the evaluation of frequency parameter Ω was run for $n = 10(5)50$. The numerical values showed a consistent improvement with the increase in the number of nodes. In all the above computations, $n = 40$ has been fixed, since further increase in n does not improve the results except at the fourth decimal place even in the third mode (Fig. 1).

The results are presented in Figs. 2–7. It is observed that the frequencies for a C–S plate are greater than that for a C–F plate but less than that for a C–C plate for the same set of values of other plate parameters. Fig. 2(a) shows the behavior of frequency parameter Ω with the increasing values of non-homogeneity parameter μ for two different values of foundation parameter $K = 0.0$, 0.02, taper parameter $\alpha = \pm 0.5$ and $\beta = -0.5$, $a/b = 1.0$ for first mode of vibration. It is observed that the frequency parameter Ω increases with the increasing values of non-homogeneity parameter μ, taper parameter α and foundation parameter K for all the three plates. The rate of increase of Ω with μ is more pronounced in case of C–C plate as compared to C–S and C–F plates. This rate decreases with the increase in the value of foundation parameter K for all the three plates. A similar inference can be drawn from Figs. 2(b) and (c) when the plate is vibrating in the second mode as well as in the third mode of vibration except that the rate of increase of Ω with μ is much higher as compared to the first mode.

Fig. 3(a) gives the inference of density parameter β on frequency parameter Ω for two values of foundation parameter $K = 0.0$, 0.02, non-homogeneity parameter $\mu = \pm 0.5$ and $\alpha = 0.5$, $a/b = 1.0$ for the first mode of vibration. It is found that the frequency parameter Ω decreases with increasing values of density parameter β for all the three boundary conditions. The rate of decrease of frequency parameter Ω with β for C–C plate is higher than that for C–S and C–F plates keeping all other plate parameters fixed. This rate increases with the increase in the value of foundation parameter K. It also increases with the increase in the number of modes, as clear from Fig. 3(b) and (c) when the plate is vibrating in the second/third mode of vibration.

Fig. 4(a) shows the graphs of frequency parameter Ω versus taper parameter α for two different values of foundation parameter $K = 0.0$, 0.02, non-homogeneity parameter $\mu = \pm 0.5$, and $\beta = -0.5$, $a/b = 1.0$, for the first mode of vibration. It is found that the frequency parameter Ω increases continuously with the increasing

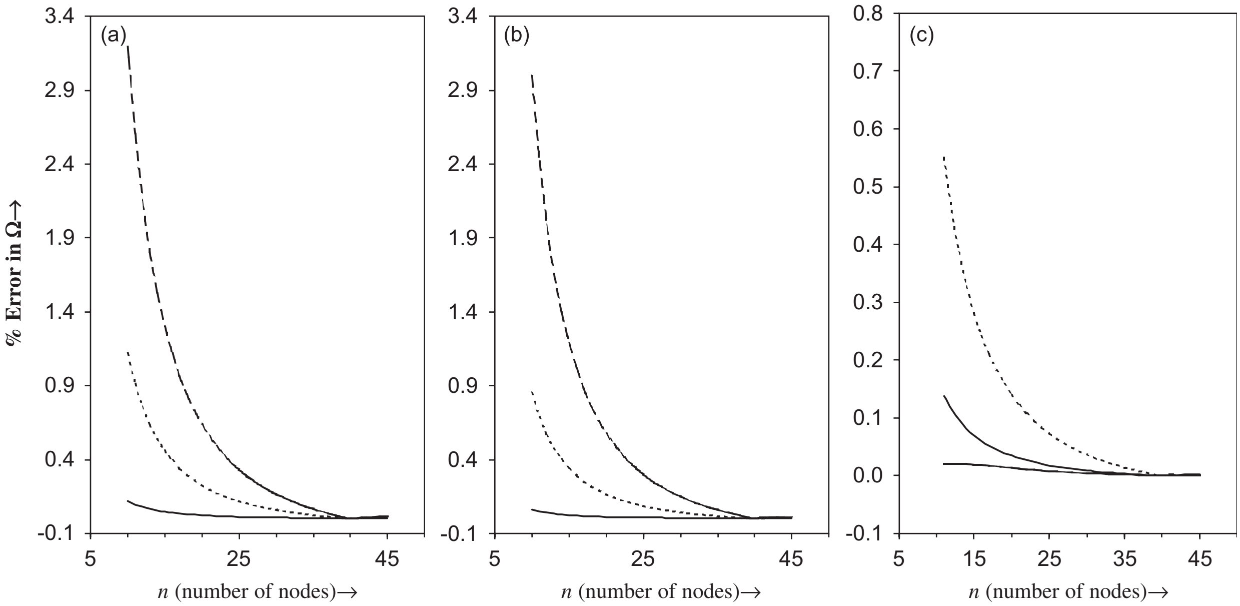

Fig. 1. Percentage error in frequency parameter Ω: (a) C–C plate, (b) C–S plate and (c) C–F plate, for $a/b=1.0$, $K=0.02$, $\mu=0.5$, $\beta=0.5$ and $\alpha=-0.5$. ——, First mode; - - - - - -, second mode; – – – – –, third mode. Percentage error $=[(\Omega_n-\Omega_{40})/\Omega_{40}]\times 100$; $n=10(5)45$.

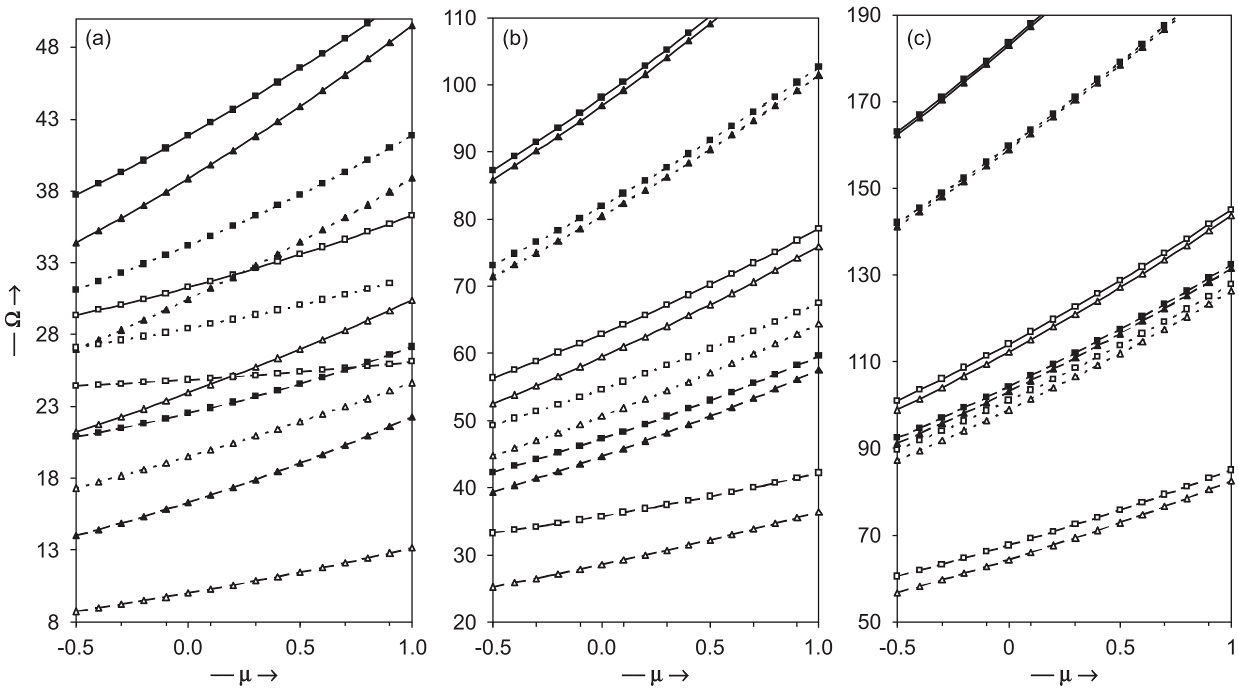

Fig. 2. Natural frequencies of C–C, C–S and C–F plates: (a) first mode (b) second mode and (c) third mode, for $\beta=-0.5$ $a/b=1.0$. ——, C–C; - - - - - -, C–S; – – – – –, C–F; △, $\alpha=-0.5$, $K=0.0$; ▲,$\alpha=0.5$, $K=0.0$; □, $\alpha=-0.5$, $K=0.02$; ■, $\alpha=0.5$, $K=0.02$.

values of taper parameter α for C–C and C–S plates whatever be the values of other plate parameters. However, in case of C–F plate the frequency parameter Ω increases with the increasing value of α in the absence of foundation i.e. $K=0.0$ while in the presence of K the values of frequency parameter first decreases

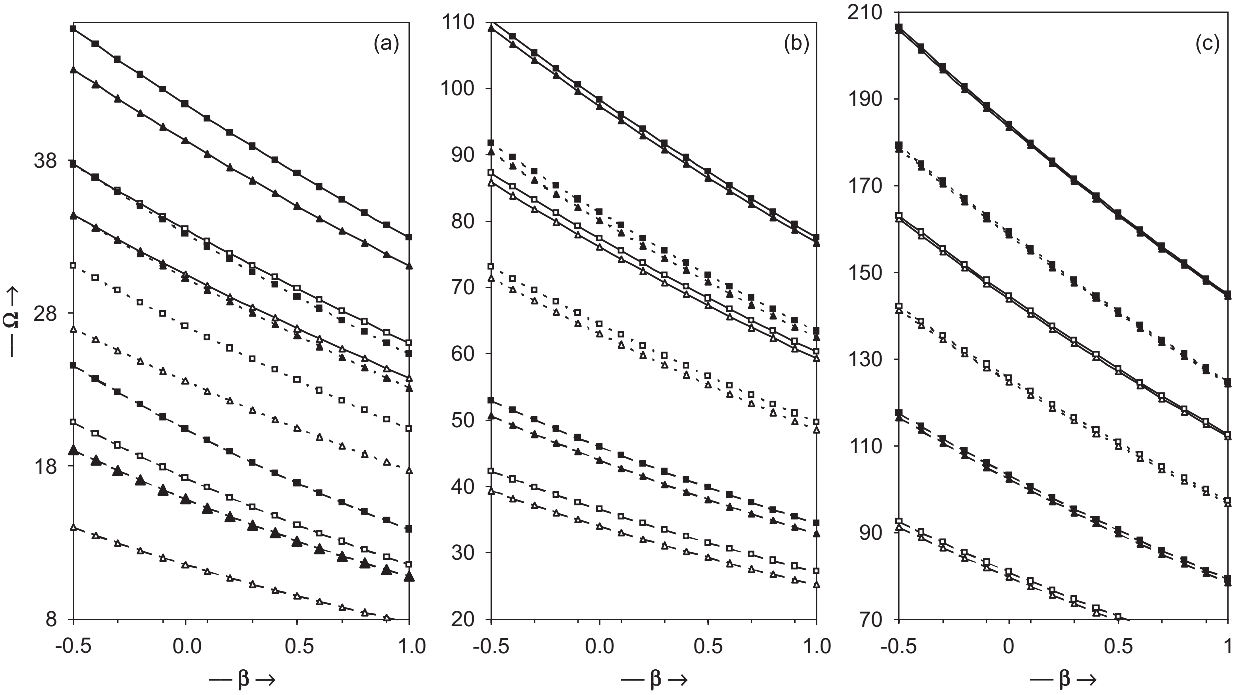

Fig. 3. Natural frequencies of C–C, C–S and C–F plates: (a) first mode, (b) second mode and (c) third mode, for $\alpha = 0.5$, $a/b = 1.0$. ——, C–C; - - - - - -, C–S; – – – – –, C–F; △, $\mu = -0.5$, $K = 0.0$; ▲, $\mu = 0.5$, $K = 0.0$; □, $\mu = -0.5$, $K = 0.02$; ■, $\mu = 0.5$, K = 0.02.

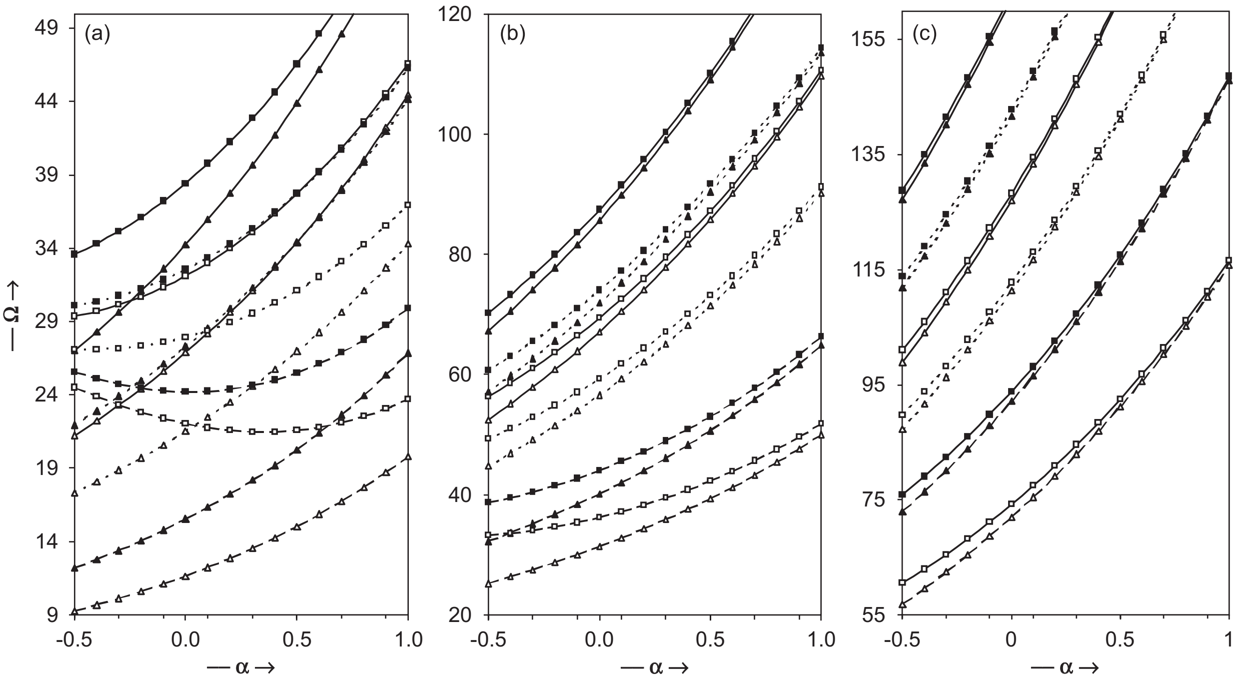

Fig. 4. Natural frequencies of C–C, C–S and C–F plates: (a) first mode, (b) second mode and (c) third mode, for $\beta = -0.5$, $a/b = -1.0$. ——, C–C; - - - - - -, C–S; – – – – –, C–F; △, $\mu = -0.5$, $K = 0.0$; ▲, $\mu = 0.5$, $K = 0.0$; □, $\mu = -0.5$, $K = 0.02$; ■, $\mu = 0.5$, $K = 0.02$.

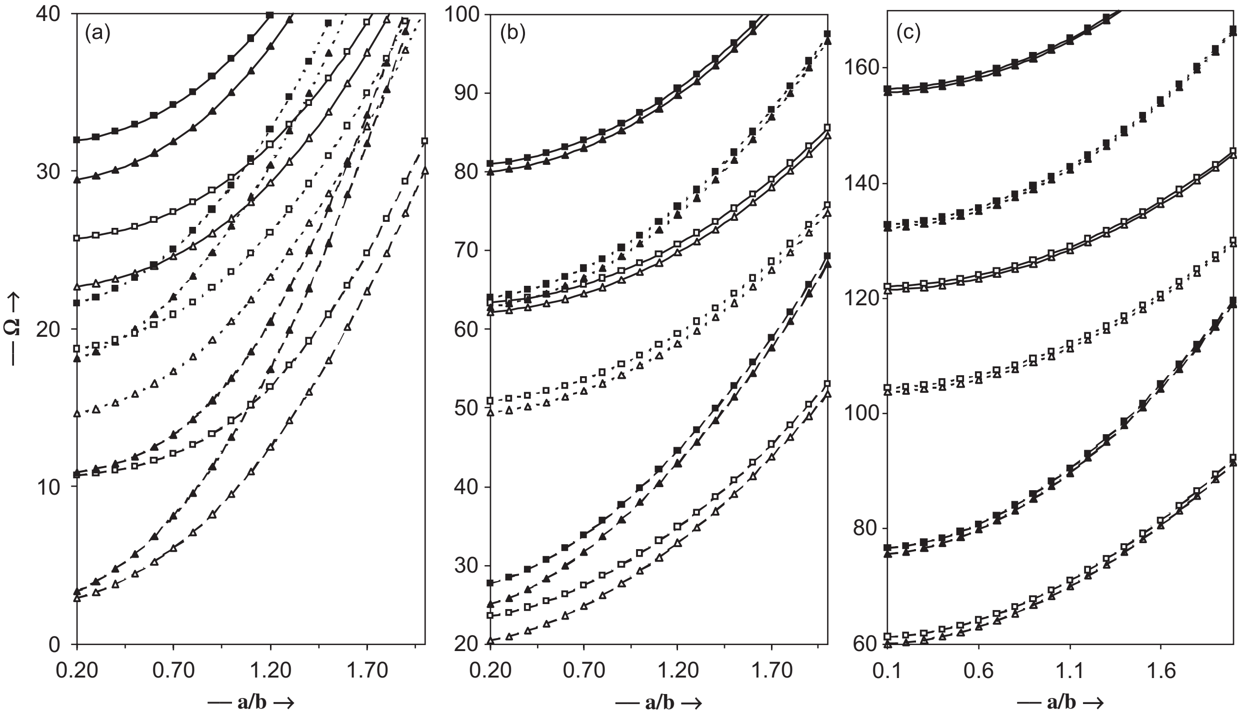

Fig. 5. Natural frequencies of C–C, C–S and C–F plates: (a) first mode, (b) second mode and (c) third mode, for $\alpha = 0.5$, $\beta = 0.5$. ——, C–C, - - - - - -, C–S, – – – – –, C–F; △, $\mu = -0.5$, $K = 0.0$; ▲, $\mu = 0.5$, $K = 0.0$; □, $\mu = -0.5$, $K = 0.02$; ■, $\mu = 0.5$, $K = 0.02$.

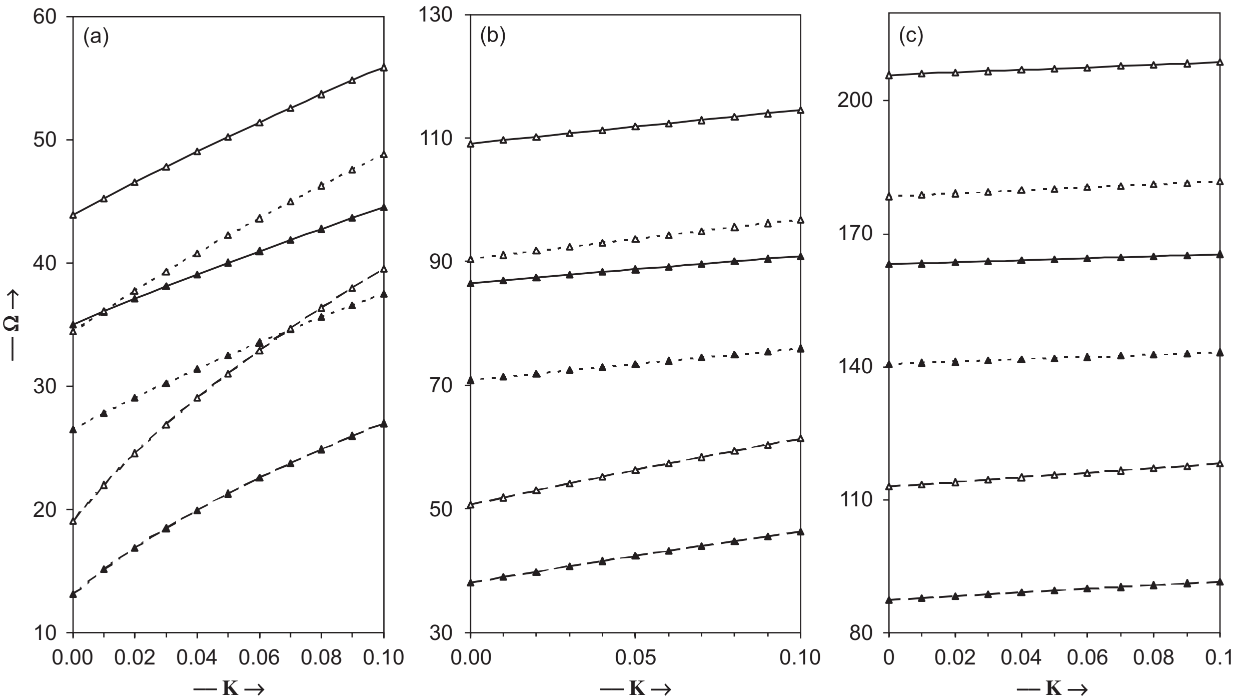

Fig. 6. Natural frequencies of C–C, C–S and C–F plates: (a) first mode, (b) second mode and (c) third mode, for $\alpha = 0.5$, $\mu = 0.5$, $a/b = 1$. ——, C-C; - - - - - -, C-S; – – – – –, C-F; △, $\beta = -0.5$; ▲, $\beta = 0.5$.

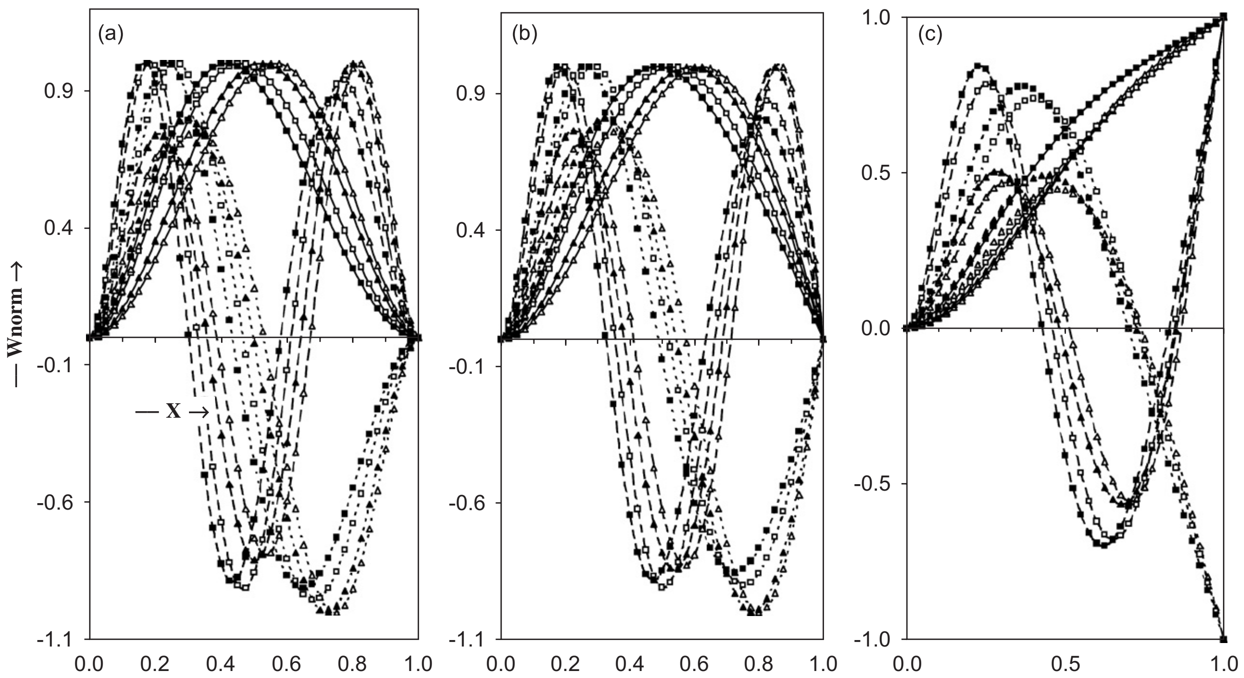

Fig. 7. Normalized displacements: (a) C–C plate, (b) C–S plate and (c) C–F plate, for $a/b = 1.0$, $\beta = -0.5$, $K = 0.02$. ——, First mode; - - - - - -, second mode; – – – – –, third mode; △, $\mu = -0.5$, $\alpha = -0.5$; ▲, $\mu = 0.5$, $\alpha = -0.5$; □, $\mu = -0.5$, $\alpha = 0.5$; ■, $\mu = 0.5$, $\alpha = 0.5$.

slightly and than increases gradually with increasing values of taper parameter α. In particular, for $K = 0.02$, $\beta = -0.5$, $a/b = 1.0$, there is a local minima in the vicinity of $\alpha = 0.3$ for $\mu = -0.5$, which shifts towards $\alpha = 0.0$ for $\mu = 0.5$ i.e. local minima shifts towards the decreasing values of α as plate becomes stiffer and stiffer towards the edge $X = 1$. The behavior of frequency parameter Ω with α in case of second mode of vibration is shown in Fig. 4(b). It is found that the frequency parameter Ω increases with the increasing values of α for all the three boundary conditions whatever be the values of other plate parameters. In case of third mode of vibration the frequency parameter also increases with the increasing values of α as clear from Fig. 4(c). The rate of increase is much higher as compared to the second mode for all the three plates.

The effect of aspect ratio a/b on the frequency parameter Ω for two values of foundation parameter $K = 0.0$, 0.02, non-homogeneity parameter $\mu = \pm 0.5$ and $\alpha = 0.5$, $\beta = 0.5$, has been shown in Fig. 5(a) for the first mode of vibration. It is found that the frequency parameter Ω increases with increasing values of aspect ratio a/b for all the three plates. The effect of elastic foundation is more pronounced for $a/b < 1$ as compared to $a/b > 1$ for all the three boundary conditions. This effect decreases with the order of boundary conditions C–F, C–S and C–C. It is also noticed that the frequency parameter Ω increases more rapidly for $a/b > 1$ when μ changes from negative to positive i.e. plate becomes more and more stiff towards the edge $X = 1$. From Fig. 5(b) and (c) when the plate is vibrating in second/third mode of vibration it is found that the behavior of aspect ratio a/b on the frequency parameter Ω is almost same as that of the first mode except that the rate of increase of Ω with a/b increases with the increase in the number of modes for all the three boundary conditions.

The graphs of frequency parameter Ω versus foundation parameter K for two values of density parameter $\beta = \pm 0.5$ and $\mu = 0.5$, $\alpha = 0.5$, $a/b = 1.0$ have been plotted in Fig. 6(a) when the plate is vibrating in the first mode of vibration. It is found that the frequency parameter increases with the increasing values of foundation parameter K for all the three boundary conditions. The rate of increase in frequency parameter Ω with foundation parameter K is higher in case of C–F plate as compared to C–S and C–C plates for the same set of values of other plate parameters. From Fig. 6(b) when the plate is vibrating in the second mode of vibration, the effect of foundation parameter is found to increase the frequency parameter Ω, however the rate of increase gets reduced to almost half of the first mode for all the three boundary conditions. In case of third mode, this rate of increase further decreases and becomes nearly half of the second mode as is evident from Fig. 6(c).

Mode shapes for a square plate i.e. $a/b = 1$, have been computed for two values of non-homogeneity parameter $\mu = \pm 0.5$, taper parameter $\alpha = \pm 0.5$and $\beta = -0.5$, $K = 0.02$, for all the three boundary conditions. Normalized displacements are shown in Fig. 7(a)–(c) for the first three modes of vibration. The nodal lines are seen to shift towards the edge $X = 1$, as the edge $X = 0$ increase in thickness and stiffness for all the three plates. No special change was seen in the pattern of nodal lines by taking different values of β and K as normalized displacements were differing only at the third or fourth place after decimal for all the three boundary conditions.

A comparison of results with those available in the literature obtained by other methods has been presented in Tables 1 and 2. Table 1 shows a comparison of results for homogeneous isotropic plates of uniform thickness ($\alpha = 0.0$) with exact solutions [1] and non-uniform thickness ($\alpha = \pm 0.5$) with those obtained by Chebyshev collocation technique [12], Frobènius method [33], differential quadrature method [34], finite element method [31] and optimized Kantorovich method [31], for two values of aspect ratio $a/b = 0.5$, 1.0. In Table 2, the natural frequencies for C–C and C–S plates of uniform thickness have been presented for $K = 0.0$

Table 1
Comparison of frequency parameter Ω for isotropic C–C, C–S and C–F plates for $v = 0.3$

Boundary condition	Mode/α	$K = 0.0$						$K = 0.01$					
		$a/b = 0.5$			$a/b = 1$			$a/b = 0.5$			$a/b = 1$		
		−0.5	0.0	0.5	−0.5	0.0	0.5	−0.5	0.0	0.5	−0.5	0.0	0.5
C–C	I	18.655	23.820	30.757	22.604	28.950	37.268	22.467	26.219	32.266	25.844	30.953	38.524
		18.657[a]	23.816[a]	30.759[a]	22.609[a]	28.951[a]	37.276[a]	22.468[a]	26.214[a]	32.268[a]	25.848[a]	30.954[a]	38.532[a]
		–	–	–	–	28.946[b]	–						
		–	23.816[c]	–	–	28.951[c]	–						
		–	23.820[d]	–	–	28.960[d]	–						
	II	49.539	63.603	81.676	54.005	69.380	89.039	51.100	64.539	82.257	55.440	70.239	89.572
		49.499[a]	63.635[a]	81.611[a]	53.577[a]	69.327[a]	88.992[a]	51.061[a]	64.472[a]	82.192[a]	55.412[a]	70.187[a]	89.526[a]
		–	–	–	–	69.320[b]	–						
		–	63.535[c]	–	–	69.327[c]	–						
C–S	I	14.161	17.335	21.260	18.665	23.647	30.128	19.025	20.506	23.316	22.577	26.061	31.623
		14.161[a]	17.332[a]	21.262[a]	18.667[a]	23.646[a]	30.135[a]	19.025[a]	20.503[a]	23.318[a]	22.578[a]	26.060[a]	31.699[a]
		–	–	–	–	23.646[b]	–						
		–	17.332[c]	–	–	23.646[c]	–						
	II	41.264	52.150	65.742	46.119	58.688	74.538	43.146	53.288	66.455	47.811	59.702	75.167
		41.232[a]	52.098[a]	65.700[a]	46.094[a]	58.646[a]	74.510[a]	43.115[a]	53.237[a]	66.414[a]	47.787[a]	59.661[a]	75.151[a]
		–	–	–	–	58.641[b]	–						
		–	52.097[c]	–	–	58.646[c]	–						
C–F	I	4.999	5.703	6.868	9.575	12.684	17.255	14.297	12.350	11.322	16.456	16.760	19.496
		5.002[a]	5.704[a]	6.870[a]	9.812[a]	12.687[a]	17.258[a]	14.296[a]	12.351[a]	11.323[a]	16.458[a]	16.762[a]	19.499[a]
		–	–	–	–	12.680[b]	–						
		–	–	–	–	12.680[e]	–						
		–	–	–	–	12.830[f]	–						
	II	20.633	24.949	30.194	26.218	33.064	42.045	24.304	27.248	31.660	29.187	34.831	43.110
		20.639[a]	24.944[a]	30.192[a]	26.230[a]	33.065[a]	42.057[a]	24.308[a]	27.243[a]	31.658[a]	29.197[a]	34.839[a]	43.121[a]

[a]Values from Chebyshev collocation technique [14].
[b]Exact values from Liessa [1].
[c]Values by Frobènius method [33].
[d]Values by differential quadrature method [34].
[e]Values from finite element method [31].
[f]Values by optimized Kantorovich method [31].

Table 2
Comparison of frequency parameter Ω for isotropic C–C and C–S plates for $p = 2$, and $v = 0.3$

Mode	C–C		C–S	
	Values of a/b		Values of a/b	
	0.5	1.0	0.5	1.0
I	28.9499	54.7312	23.6468	51.6700
	28.9508[a]	54.7431[a]	23.6464[a]	51.6742[a]
	28.9508[b]	54.7430[b]	23.6463[b]	51.6742[b]
II	69.3796	94.5927	58.6880	86.1493
	69.3270[a]	94.5853[a]	58.6464[a]	86.1350[a]
	69.3270[b]	94.5852[b]	58.6463[b]	86.1344[b]
III	129.3675	154.9509	113.4541	141.0035
	129.0951[a]	154.7754[a]	113.2377[a]	140.8484[a]
	129.0956[b]	154.7757[b]	113.2281[b]	140.8455[b]

[a]Values by Chebyshev technique [14].
[b]Values taken from Ref. [35].

and $p = 2$ obtained by Frobènius method [35]. A close agreement of the results shows the versatility of the present technique.

6. Conclusion

The effect of non-homogeneity, which is assumed to arise due to the unidirectional variation in Young's moduli and density of the plate material on the natural frequencies of rectangular orthotropic plates of exponentially varying thickness along one direction and resting on a Winkler elastic foundation has been studied on the basis of classical plate theory. It is observed that frequency parameter Ω increases with the increasing values of non-homogeneity parameter μ, aspect ratio a/b, foundation parameter K, while it decreases with increasing values of density parameter β keeping all other plate parameter fixed for all the three boundary conditions. The behavior of frequency parameter Ω is found to increase with increasing values of taper parameter α with and without elastic foundation for all the three plates except in the first mode of C–F plate. In this case in the presence of elastic foundation ($K = 0.02$) there is a local minima in the vicinity of $\alpha = 0.3$ which shifts towards the decreasing (negative) values of α with increasing values of non-homogeneity parameter μ as well as aspect ratio a/b. Thus a change in the natural frequencies of a plate can be achieved by a proper choice of various plate parameters considered here, which will be of great practical interest to design engineers.

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