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Free vibration of FGM cylindrical shells with holes under various boundary conditions ☆

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Abstract

Free vibration analysis of functionally graded material (FGM) cylindrical shells with holes was studied in this paper. The variational equation was founded firstly, and the unified displacement mode-shape function of the shells with various boundary conditions was put forward then. The general analytical expressions of natural frequency and mode-shape solutions given by functional variation and characteristic value analysis, which can be applied to FGM cylindrical shells with holes of arbitrary functionally graded distribution along the thickness direction and different boundary conditions. Non-dimensional frequencies of shell with holes of different shape, number, location were given in the end of this paper.

1. Introduction

The functionally graded material (FGM) is a new kind of composite material developed recently in which the material property graded distribution is formed by the composition of different material media in variant proportion in space, so as to meet the requirements of material properties for different parts of a member. In the meanwhile, as each part of the material of structure varies in continuous way, there is more advantage of the material properties than ordinary laminated and composed ones. Such kind of composite material composed non-uniformly and continuously in structure by different materials in desired way makes development of material stride forward towards a higher level [1].

The design idea of FGM, which was firstly proposed by Japanese scientists in the 1980s last century, was mainly used to meet the specific requirements of material properties in advanced science region of national defense. Though the original research of FGM was begun on releasing thermal stresses, the idea was gradually applied to the research and conception of other functional materials which are widely used and developed in some other important regions such as aerospace, energy sources, electron, chemical engineering, optical materials, biological engineering, etc. In recent years, more attention from international academic circles is paid to FGM and its structures [2]. And the development of FGM in various regions has been doing in succession by scientists of Japan, American, Russia, Germany, England, Switzerland, Finland, Ukraine, etc.

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The research work on FGM was started as a specific functional material of aerospace engineering. Its development period is not long. Therefore, more attention of study was paid to thermal stresses [3–8], crack [9], optimizing design [10], etc. Still much remains to be done, as many weak links exist in different fields of mechanics, infact, even in static and dynamic analyses of ordinary structural members [11]. Besides, the parameters of FGM are related to spatial coordinates, and the correspondent dominant equations are of variable coefficient, and therefore, it is very difficult to find analytic solution directly. Semi-analytical method [12] and numerical method [13–14] such as finite element method are used to solve these problems. In analytic research field of FGM, there are quite a lot of challenging research projects. Up till now, several methods, such as laminated model method [15], asymptotic approach [16], 3-D analysis method [17], simplified model technique [18], etc, are mainly applied to the analyses of FGM plate and shell structures. As the mathematics involved is difficult, the analytic solutions can only be obtained for some specific functional gradient functions and very limited kind of boundary conditions, usually for plate problems which are not concerned with shells with holes [19].

Free vibration analysis of FGM cylindrical shells with holes was studied in this paper. The variational equation was founded firstly, and the unified displacement mode-shape function of the shells with various boundary conditions was put forward then. The general analytical expressions of natural frequency and mode-shape solutions were given by functional variation and characteristic value analysis, which can be applied to FGM cylindrical shells with holes of arbitrary functionally graded distribution along the thickness direction and different boundary conditions.

2. Basic variational equations

2.1. Geometrical equations

According to the theory of shells [20,21], the strain–displacement relations of FGM cylindrical shell in cylindrical coordinates are the same as homogeneous one, namely (Fig. 1)

mid-plane strain:

$$\varepsilon_x^0 = \frac{\partial u}{\partial x}, \quad \varepsilon_\theta^0 = \frac{\partial v}{R\partial\theta} + \frac{w}{R}, \quad \varepsilon_{x\theta}^0 = \frac{\partial v}{\partial x} + \frac{\partial u}{R\partial\theta}, \tag{1a}$$

mid-plane curvature:

$$\chi_x = -\frac{\partial^2 w}{\partial x^2}, \quad \chi_\theta = \frac{1}{R^2}\left[\frac{\partial v}{\partial\theta} - \frac{\partial^2 w}{\partial\theta^2}\right], \quad \chi_{x\theta} = \frac{1}{R}\left[\frac{\partial v}{\partial x} - 2\frac{\partial^2 w}{\partial x\partial\theta}\right], \tag{1b}$$

shell strain:

$$\varepsilon_x = \varepsilon_x^0 + z\chi_x, \quad \varepsilon_\theta = \varepsilon_\theta^0 + z\chi_\theta, \quad \varepsilon_{x\theta} = \varepsilon_{x\theta}^0 + z\chi_{x\theta}, \tag{1c}$$

where u, v, w are the displacement components in the middle plane along the axial, circumferential and normal directions, respectively.

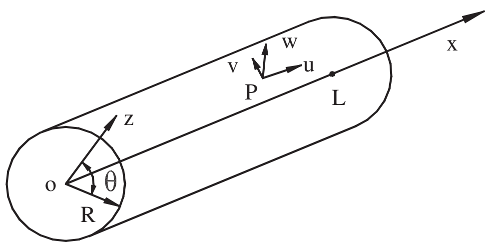

Fig. 1. FGM cylindrical shell.

2.2. Physical equations

According to the σ_z, ε_{xz} and $\varepsilon_{\theta z}$ assumptions of shell theory, stress–strain relations are given by considering the continuous change of material properties of FGM shell along the thickness direction:

$$\sigma_x = \frac{E(z)}{1-\mu^2(z)}[\varepsilon_x + \mu(z)\varepsilon_\theta], \quad \sigma_\theta = \frac{E(z)}{1-\mu^2(z)}[\varepsilon_\theta + \mu(z)\varepsilon_x], \quad \tau_{x\theta} = G(z)\varepsilon_{x\theta}. \tag{2}$$

2.3. Deformation energy

According to the assumption of shell theory, the specific energy is

$$\overline{W} = \tfrac{1}{2}(\sigma_x\varepsilon_x + \sigma_\theta\varepsilon_\theta + \tau_{x\theta}\varepsilon_{x\theta}).$$

Substituting Eqs. (1) and (2) into the above equation, $\overline{W}$ can be expressed as

$$\overline{W} = \frac{E(z)}{2[1-\mu^2(z)]}[(\varepsilon_x^0 + z\chi_x)^2 + (\varepsilon_\theta^0 + z\chi_\theta)^2 + 2\mu(z)(\varepsilon_x^0 + z\chi_x)(\varepsilon_\theta^0 + z\chi_\theta)] + \frac{G(z)}{2}(\varepsilon_{x\theta}^0 + z\chi_{x\theta})^2.$$

First, calculating integration

$$\int_{-(h/2)}^{h/2} \overline{W}\,\mathrm{d}z = \frac{K}{2}\left({\varepsilon_x^0}^2 + {\varepsilon_\theta^0}^2\right) + C(\varepsilon_x^0\chi_x + \varepsilon_\theta^0\chi_\theta) + \frac{D}{2}(\chi_x^2 + \chi_\theta^2) + K_{x\theta}(\varepsilon_x^0\varepsilon_\theta^0) + C_{x\theta}(\varepsilon_x^0\chi_\theta + \varepsilon_\theta^0\chi_x) + D_{x\theta}(\chi_x\chi_\theta) + \frac{K_G}{2}(\varepsilon_{x\theta}^0)^2 + C_G\varepsilon_{x\theta}^0\chi_{x\theta} + \frac{D_G}{2}\chi_{x\theta}^2,$$

where, membrane stiffness:

$$K = \int_{-(h/2)}^{h/2} \frac{E(z)}{1-\mu^2(z)}\,\mathrm{d}z, \quad K_{x\theta} = \int_{-(h/2)}^{h/2} \frac{E(z)\mu(z)}{1-\mu^2(z)}\,\mathrm{d}z, \quad K_G = \int_{-(h/2)}^{h/2} G(z)\,\mathrm{d}z, \tag{3a}$$

bending stiffness:

$$D = \int_{-(h/2)}^{h/2} \frac{E(z)z^2}{1-\mu^2(z)}\,\mathrm{d}z, \quad D_{x\theta} = \int_{-(h/2)}^{h/2} \frac{E(z)\mu(z)z^2}{1-\mu^2(z)}\,\mathrm{d}z, \quad D_G = \int_{-(h/2)}^{h/2} G(z)z^2\,\mathrm{d}z \tag{3b}$$

membrane–bending coupled stiffness:

$$C = \int_{-(h/2)}^{h/2} \frac{E(z)z}{1-\mu^2(z)}\,\mathrm{d}z, \quad C_{x\theta} = \int_{-(h/2)}^{h/2} \frac{E(z)\mu(z)z}{1-\mu^2(z)}\,\mathrm{d}z, \quad C_G = \int_{-(h/2)}^{h/2} G(z)z\,\mathrm{d}z \tag{3c}$$

in which, h is the thickness of shell. The deformation energy can be obtained as follows:

$$\begin{aligned} U &= \iiint_V \overline{W}\,\mathrm{d}V = \iint_{A_H}\left[\int_{-(h/2)}^{h/2}\overline{W}\,\mathrm{d}z\right]R\,\mathrm{d}x\,\mathrm{d}\theta \\ &= \iint_{A_H}\left[\frac{K}{2}\left({\varepsilon_x^0}^2 + {\varepsilon_\theta^0}^2\right) + K_{x\theta}(\varepsilon_x^0\varepsilon_\theta^0) + \frac{K_G}{2}(\varepsilon_{x\theta}^0)^2\right]R\,\mathrm{d}x\,\mathrm{d}\theta \\ &+ \iint_{A_H}\left[\frac{D}{2}(\chi_x^2 + \chi_\theta^2) + D_{x\theta}(\chi_x\chi_\theta) + \frac{D_G}{2}\chi_{x\theta}^2\right]R\,\mathrm{d}x\,\mathrm{d}\theta \\ &+ \iint_{A_H}\left[C(\varepsilon_x^0\chi_x + \varepsilon_\theta^0\chi_\theta + C_{x\theta}(\varepsilon_x^0\chi_\theta + \varepsilon_\theta^0\chi_x) + C_G\varepsilon_{x\theta}^0\chi_{x\theta})\right]R\,\mathrm{d}x\,\mathrm{d}\theta, \end{aligned} \tag{4a}$$

where the integration is calculated along the middle plane A_H of cylindrical shell with hole. Substituting the mid-plane strain expressions (1a) and (1b) into the above equation, the deformation energy can be expressed by the mid-plane displacements.

Deformation energy of the FGM cylindrical shell involves three portions, first portion and second portion of Eq. (4a) are the membrane strain energy and the bending strain energy, respectively; and the third portion is the membrane–bending coupled strain energy, which is special property of FGM shells.

2.4. Kinetic energy

The kinetic energy of FGM shell can be expressed as

$$T = \int\int\int_V \frac{\rho(z)}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] \mathrm{d}V$$
$$= \frac{1}{2}\overline{\rho h}\int\int_{A_H}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] R\,\mathrm{d}x\,\mathrm{d}\theta \tag{4b}$$

in which, $\partial u/\partial t$, $\partial v/\partial t$, $\partial w/\partial t$ are the velocities in three directions and the surface mass density is

$$\overline{\rho h} = \int_{-(h/2)}^{h/2} \rho(z)\,\mathrm{d}z. \tag{3d}$$

2.5. Variational equation

According to Hamilton's principle, when the transverse load $q(x, \theta, t)$ acts on the surface, the displacement variational equation can be obtained as

$$\delta\int_{t_0}^{t_1}(T - U)\,\mathrm{d}t + \int_{t_0}^{t_1}\int\int_{A_H} q\,\delta w R\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}t = 0. \tag{5a}$$

The free vibration variational equation is

$$\delta\int_{t_0}^{t_1}(T - U)\,\mathrm{d}t = 0. \tag{5b}$$

3. Free vibration of FGM cylindrical shells with holes under different boundaries

Assuming $u(x,\theta,t) = U(x,\theta)\sin(\omega t + \varphi)$, $v(x,\theta,t) = V(x,\theta)\sin(\omega t + \varphi)$, $w(x,\theta,t) = W(x,\theta)\sin(\omega t + \varphi)$, substituting the above expressions into Eqs. (1a) and (1b) firstly, then substituting them into Eqs. (4a), (4b), (5b), and integrating along $\omega t = 0\sim 2\pi$, the modal variational equation can be given as

$$\delta\int\int_{A_H}\left\{K\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{R\partial\theta}+\frac{W}{R}\right)^2\right] + 2K_{x\theta}\left(\frac{\partial U}{\partial x}\right)\left(\frac{\partial V}{R\partial\theta}+\frac{W}{R}\right) + K_G\left(\frac{\partial V}{\partial x}+\frac{\partial U}{R\partial\theta}\right)^2\right.$$
$$+ D\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \frac{1}{R^4}\left(\frac{\partial V}{\partial\theta}-\frac{\partial^2 W}{\partial\theta^2}\right)^2\right] - 2\frac{D_{x\theta}}{R^2}\left(\frac{\partial^2 W}{\partial x^2}\right)\left(\frac{\partial V}{\partial\theta}-\frac{\partial^2 W}{\partial\theta^2}\right) + \frac{D_G}{R^2}\left(\frac{\partial V}{\partial x}-2\frac{\partial^2 W}{\partial x\partial\theta}\right)^2$$
$$+ 2C\left[\frac{1}{R^2}\left(\frac{\partial V}{R\partial\theta}+\frac{W}{R}\right)\left(\frac{\partial V}{\partial\theta}-\frac{\partial^2 W}{\partial\theta^2}\right) - \left(\frac{\partial U}{\partial x}\right)\left(\frac{\partial^2 W}{\partial x^2}\right)\right] + 2C_{x\theta}\left[\frac{1}{R^2}\left(\frac{\partial U}{\partial x}\right)\left(\frac{\partial V}{\partial\theta}-\frac{\partial^2 W}{\partial\theta^2}\right)\right.$$
$$\left.\left. - \left(\frac{\partial V}{R\partial\theta}+\frac{W}{R}\right)\left(\frac{\partial^2 W}{\partial x^2}\right)\right] + 2\frac{C_G}{R}\left(\frac{\partial V}{\partial x}+\frac{\partial U}{R\partial\theta}\right)\left(\frac{\partial V}{\partial x}-2\frac{\partial^2 W}{\partial x\partial\theta}\right) - \omega^2\overline{\rho h}(U^2+V^2+W^2)\right\}R\,\mathrm{d}x\,\mathrm{d}\theta = 0. \tag{6}$$

For the closed cylindrical shells, two end boundary conditions can be chosen in the following:

$$\text{fixed boundary}: \quad u = v = \theta_x = w = 0, \tag{7a}$$

$$\text{free boundary}: \quad N_x = T_{x\theta} = M_x = V_x = 0, \tag{7b}$$

$$\text{simple boundary}: \quad N_x = v = M_x = w = 0, \tag{7c}$$

where N_x, M_x, θ_x, $T_{x\theta}$, V_x is membrane force, moment, rotational angel, membrane shearing force, transverse shearing force, respectively [16]. Then mode-shape function can be taken as

$$U(x,\theta) = A_{mn}\frac{\mathrm{d}X_m(x)}{\mathrm{d}(x/L)}\cos n\theta, \tag{8a}$$

$$V(x,\theta) = B_{mn}X_m(x)\sin n\theta \quad (m,n = 1,2,3,\ldots), \tag{8b}$$

$$W(x,\theta) = C_{mn}X_m(x)\cos n\theta. \tag{8c}$$

In which, $X_m(x)$ is the mth order modal function of the beam with corresponding axial boundary conditions of shell [16]; A_{mn}, B_{mn}, C_{mn} are the unknown modal coefficients.

Substituting modal function (8) into the variational equation (6), getting through variational calculus, the homogeneous algebraic equations about A_{mn}, B_{mn}, C_{mn} can be given as

$$\begin{vmatrix} (\alpha_1\Omega^2 - S_{11}) & S_{12} & S_{13} \\ S_{21} & (\alpha_2\Omega^2 - S_{22}) & S_{23} \\ S_{31} & S_{32} & (\alpha_3\Omega^2 - S_{33}) \end{vmatrix} \begin{Bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \end{Bmatrix} = \{0\}. \tag{9}$$

In order to solve its non-zero solution, the determinant of coefficient must be zero, the frequency equation can be obtained as

$$\Omega^6 + a\Omega^4 + b\Omega^2 + c = 0, \tag{10a}$$

where

$$a = -\left(\frac{S_{11}}{\alpha_1} + \frac{S_{22}}{\alpha_2} + \frac{S_{33}}{\alpha_3}\right), \tag{10b}$$

$$b = \frac{S_{11}S_{22}}{\alpha_1\alpha_2} + \frac{S_{22}S_{33}}{\alpha_2\alpha_3} + \frac{S_{33}S_{11}}{\alpha_3\alpha_1} - \left(\frac{S_{12}^2}{\alpha_1\alpha_2} + \frac{S_{23}^2}{\alpha_2\alpha_3} + \frac{S_{31}^2}{\alpha_3\alpha_1}\right), \tag{10c}$$

$$c = [S_{11}S_{23}S_{32} + S_{22}S_{13}S_{31} + S_{33}S_{12}S_{21} + 2S_{12}S_{23}S_{31} - S_{11}S_{22}S_{33}]/\alpha_1\alpha_2\alpha_3 \tag{10d}$$

and non-dimensional frequency:

$$\Omega^2 = \frac{\overline{\rho h}}{K}R^2\omega^2 \tag{11}$$

coefficients of equations:

$$S_{11} = \frac{K_G}{K}n^2\alpha_4 + \left(\frac{R}{L}\right)^2\alpha_6, \tag{12a}$$

$$S_{12} = S_{21} = \frac{K_G R + C_G}{K}\left(\frac{n}{L}\right)\alpha_4 - \frac{K_{x\theta}R + C_{x\theta}}{K}\left(\frac{n}{L}\right)\alpha_5, \tag{12b}$$

$$S_{13} = S_{31} = 2\frac{C_G n^2}{KL}\alpha_4 - \frac{K_{x\theta}R + C_{x\theta}n^2}{KL}\alpha_5 + \frac{CR^2}{KL^3}\alpha_6, \tag{12c}$$

$$S_{22} = \left(1 + 2\frac{C}{KR} + \frac{D}{KR^2}\right)n^2\alpha_3 + \frac{K_G R^2 + 2C_G R + D_G}{KL^2}\alpha_4, \tag{12d}$$

$$S_{23} = S_{32} = -\left(1 + \frac{C}{KR} + \frac{Cn^2}{KR} + \frac{Dn^2}{KR^2}\right)n\alpha_3 - \frac{2n(C_G R + D_G)}{KL^2}\alpha_4 + \frac{n(C_{x\theta}R + D_{x\theta})}{KL^2}\alpha_5, \tag{12e}$$

$$S_{33} = \left(1 + 2\frac{Cn^2}{KR} + \frac{Dn^4}{KR^2}\right)\alpha_3 + 4\frac{D_G n^2}{KL^2}\alpha_4 - 2\frac{C_{x\theta}R + D_{x\theta}n^2}{KL^2}\alpha_5 + \frac{DR^2}{KL^4}\alpha_6 \tag{12f}$$

in which, the modal parameters are:

$$\alpha_1 = L\int\int_{A_H} (\mathrm{X}'_m)^2(\cos n\theta)^2\,\mathrm{d}x\,\mathrm{d}\theta, \tag{13a}$$

$$\alpha_2 = \frac{1}{L}\int\int_{A_H} (\mathrm{X}_m)^2(\sin n\theta)^2\,\mathrm{d}x\,\mathrm{d}\theta, \tag{13b}$$

$$\alpha_3 = \frac{1}{L}\int\int_{A_H} (\mathrm{X}_m)^2(\cos n\theta)^2\,\mathrm{d}x\,\mathrm{d}\theta, \tag{13c}$$

$$\alpha_4 = L\int\int_{A_H} (\mathrm{X}'_m)^2(\sin n\theta)^2\,\mathrm{d}x\,\mathrm{d}\theta, \tag{13d}$$

$$\alpha_5 = L\int\int_{A_H} (\mathrm{X}_m\mathrm{X}''_m)(\cos n\theta)^2\,\mathrm{d}x\,\mathrm{d}\theta, \tag{13e}$$

$$\alpha_6 = L^3\int\int_{A_H} (\mathrm{X}''_m)^2(\cos n\theta)^2\,\mathrm{d}x\,\mathrm{d}\theta \tag{13f}$$

and $X'_m(x)$, $X''_m(x)$ are the first- and second-order derivate, respectively. Eq. (10a) is a cubic algebraic equation about Ω^2, its three roots can be solved as

$$\Omega^2_{imn} = -\frac{1}{3}\left\{2\sqrt{a^2 - 3b}\cos\left[\gamma + \frac{2\pi}{3}(i-1)\right] + a\right\} \quad (i = 1, 2, 3) \tag{14a}$$

in which

$$\gamma = \frac{1}{3}\cos^{-1}\left[\frac{a^3 - 4.5ab + 13.5c}{(a^2 - 3b)^{3/2}}\right]. \tag{14b}$$

Substituting Ω_{imn} into Eq. (9), the corresponding ratio of A^i_{mn}, B^i_{mn}, C^i_{mn} can be solved, then substituting them back into Eqs. (8), and the corresponding natural mode-shapes can be obtained. The natural frequency solutions of FGM cylindrical shells with holes under different boundaries can be uniformly expressed as

$$\omega^i_{mn} = \frac{\Omega_{imn}}{R}\sqrt{\frac{K}{\overline{\rho h}}}\left(\begin{array}{c} i = 1, 2, 3 \\ m, n = 1, 2, 3\ldots \end{array}\right). \tag{15}$$

Here, K is given in Eq. (3a); $\overline{\rho h}$ is given in Eq. (3d); R is the radius of cylindrical shell; the solution of Ω_{imn} is given in Eqs. (14), in which, a, b and c is given in Eqs. (10); S_{ij} and $\alpha_k (k = 1{\sim}6)$ of Eqs. (10) are given in Eqs. (12) and (13), respectively. According to different graded distribution of the materials, the stiffness parameters K, $K_{x\theta}$, K_G, C, $C_{x\theta}$, C_G, D, $D_{x\theta}$ and D_G of Eqs. (12) can be solved by Eqs. (3a) and (3b).

4. Examples

Following examples are the FGM shells with simply supported ends, namely the modal function is $X_m(x) = \sin(m\pi/L)x$. Gradient functions are as follows:

$$E(z) = E_f \mathrm{e}^{-(\lambda_E/R)(z+(h/2))};\ \mu(z) = \mu_f \mathrm{e}^{-(\lambda_\mu/R)(z+(h/2))};\ G(z) = G_f \mathrm{e}^{-(\lambda_G/R)(z+(h/2))}. \tag{16}$$

where E_f, μ_f, G_f are material parameters in shell surface; λ_E, λ_μ, λ_G are corresponding gradients of parameters. A gradient group is assumed in following examples ($\mu_f = 0.3$), gradient values and corresponding stiffness are shown in Table 1. Substituting modal function into Eqs. (13a)–(13f), α_1–α_6 can be obtained, and non-dimensional frequency are given by solving Eq. (10a).

Example 1. Fundamental frequency coefficient versus aperture ratio of FGM shell with middle square hole.

Hole and shell sizes are shown in Fig. 2. In this example: $d_1 = d_2$; $e = 0$, aperture ratio is defined as: $\beta = d_2/R$. Fig. 3 are curves of fundamental frequency coefficient versus aperture ratio of different shells. It shows that the curve pattern is different for different shells. For slender shell (a), the shell with bigger hole has the less frequency; and the tubby shell (c) are reverse. For middle case (b), frequency is first reduced and increased then.

Example 2. Fundamental frequency coefficient versus radius–span ratio of FGM shell with middle square hole.

In this example, $h{:}R = 1{:}20$; $\beta = 0.5$. Curves of Ω_{111} versus $\eta = R/L$ are shown in Fig. 4, which shows that Ω_{111} is larger when η is larger.

Table 1
Gradients and corresponding stiffness

Gradient group			Membrane stiffness			Bending stiffness			Membrane–bending coupled stiffness		
$\lambda_E \frac{h}{R}$	$\lambda_\mu \frac{h}{R}$	$\lambda_G \frac{h}{R}$	$K/E_f h$	$K_{x\theta}/E_f h$	$K_G/G_f h$	$D/E_f h^3$	$D_{x\theta}/E_f h^3$	$D_G/G_f h^3$	$C/E_f h^2$	$C_{x\theta}/E_f h^2$	$C_G/G_f h^2$
2.0	0.5	1.0	0.463	0.118	0.608	0.0436	0.0117	0.0543	−0.0745	−0.0229	−0.0520

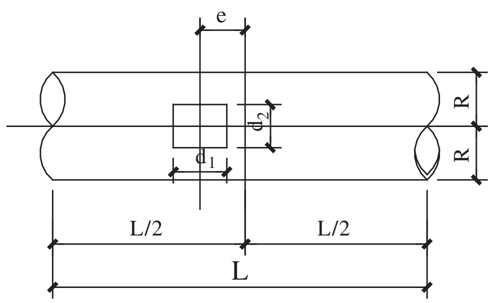

Fig. 2. Cylindrical shell with hole.

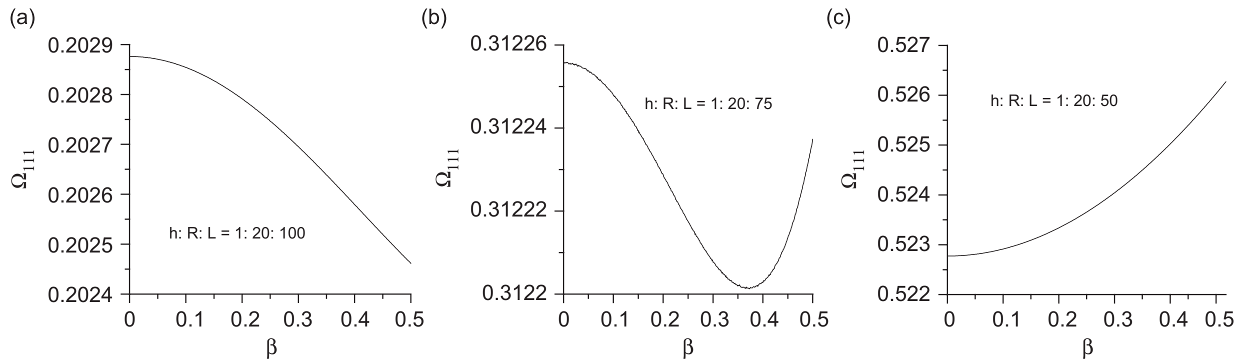

Fig. 3. Fundamental frequency coefficient versus aperture ratio.

Example 3. High-frequency coefficients versus radius–span ratio of FGM shell with middle square hole.

In this example, h:$R = 1$:20; $\beta = 0.4$. Curves of Ω_{1mn} versus η are shown in Fig. 5, where $m = 1$ and $n = 1,2,3,4$. Table 2 gives the value of Ω_{1mn} and n of the lowest frequency (same η). For the shorter shell (larger η) and the same m, Ω_{1mn} is not always increasing with n.

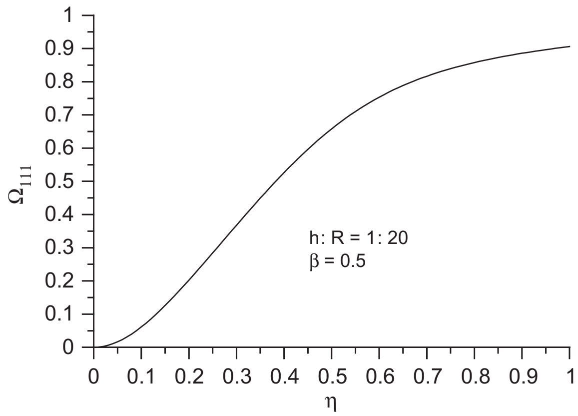

Fig. 4. Cuver of $\Omega_{111}-\eta$.

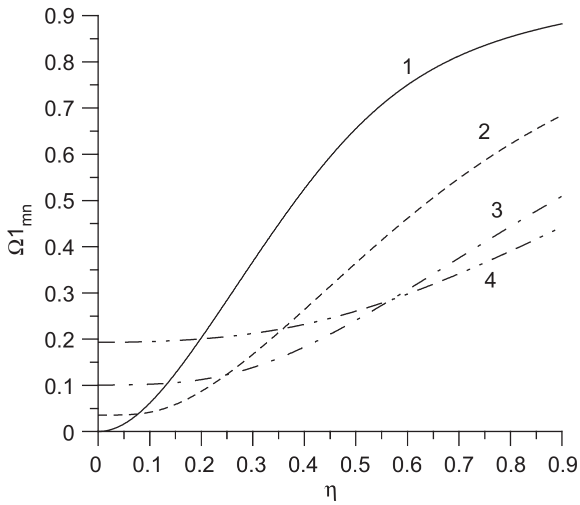

Fig. 5. Cuver of $\Omega_{1\text{mn}}-\eta$.

Table 2
High-frequency coefficient with ratio of radius and span of middle square hole

Ω_{1mn}		η						
m	n	0.05	0.1	0.2	0.3	0.4	0.5	0.6
1	1	0.0164	0.0614	0.2030	0.3680	0.5250	0.6550	0.7500
	2	0.0361	0.0417	0.0870	0.1670	0.2640	0.3650	0.4610
	3	0.1010	0.1020	0.1120	0.1390	0.1830	0.2410	0.3070
	4	0.1930	0.1950	0.2000	0.2120	0.2320	0.2600	0.2980
n of lowest frequency		$n = 1$	$n = 2$	$n = 2$	$n = 3$	$n = 3$	$n = 3$	$n = 4$

Example 4. Fundamental frequency coefficient versus offsetting span ratio of FGM shell with offset square hole.

In this example, $h:R:L = 1:20:60$; $\beta = 0.4$, curves of Ω_{111} versus $\gamma = e/L$ are shown in Fig. 6. the largest Ω_{111} is obtained when hole is in the middle.

Example 5. Fundamental frequency coefficient versus length–width ratio of FGM shell with middle rectangular hole.

In this example, $h:R:L = 1:20:100$; $\beta = 0.4$, length–width ratio of hole is defined as: $\phi_p = d_2/d_1$, which d_2 and d_1 are circumferential and axial length of hole, respectively. Fig. 7 shows the relation of $\Omega_{111}-\phi_p$. Ω_{111} is increasing with ϕ_p.

Example 6. Fundamental frequency coefficient of FGM shell with multi-holes.

In this example, $h:R:L = 1:20:30$, $d_2/R = 0.06$, $d_1/L = 0.1$. When the holes distributed along circumferential or axial directions, the values of Ω_{111} are shown in Table 3. Ω_{111} is increasing with axial hole number.

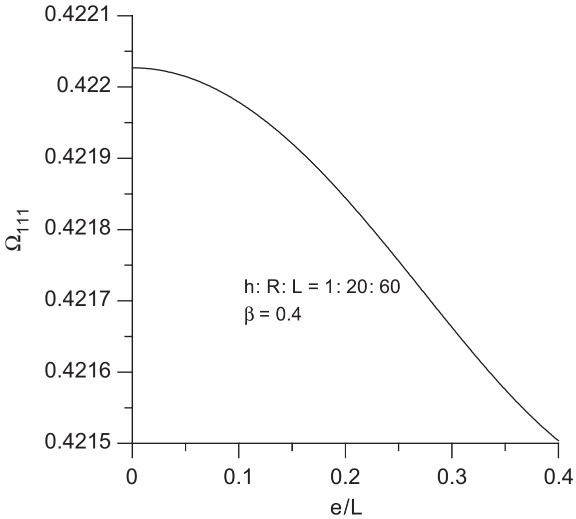

Fig. 6. Curves of $\Omega_{111}-\gamma$.

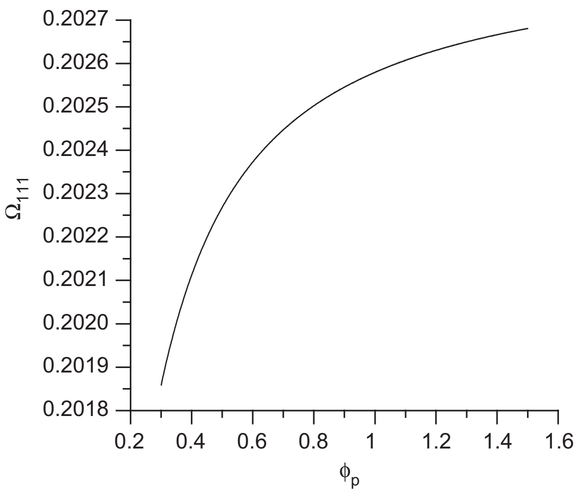

Fig. 7. Curves of $\Omega_{111}-\phi_p$.

Table 3
Fundamental frequency coefficient of the shell with multi-holes

Hole number	Circumferential	1	1	1	1	1	1	2	3	4	5
	Axial	1	2	3	4	5	1	1	1	1	1
Ω_{111}		0.32466	0.32473	0.32481	0.32489	0.32498	0.32466	0.32484	0.32479	0.32521	0.32501

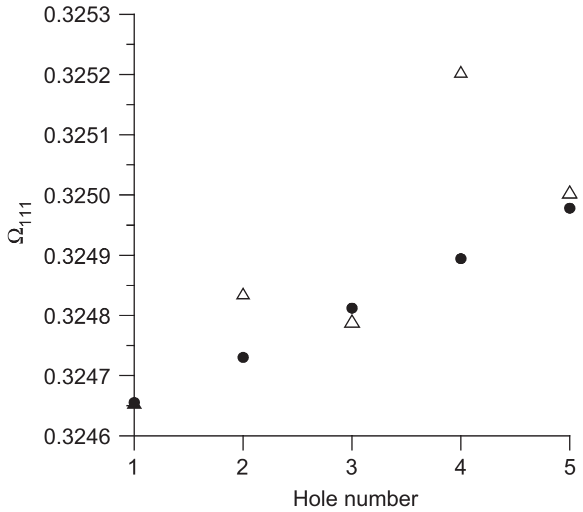

Fig. 8. Fundamental frequency coefficient of the shell with multi-holes.

Fig. 8 shows the corresponding point 1 (black point in Fig. 8), but when hole number is even and holes distributed along circumferential, Ω_{111} is larger than odd number holes (triangle point in Fig. 8).

5. Conclusions

(1) The FGMs shells were used in the aerospace engineering mostly. Holes with various shapes and numbers must be opened in order to satisfy the demand of engineering. In addition, the dynamic characters of structures are the important and necessary data for the design of spacecraft. The study of dynamic characters of functionally graded cylindrical shells with holes is very important and necessary, but few research is developed in this field.

(2) The general expression (15) and (14) of natural frequencies of functionally graded cylindrical shells with hole can be applied in the structures with arbitrary holes (A_n is the middle plane of shell with arbitrary holes in Eqs. (13)), various boundary conditions (different boundary of $X_m(x)$ in Eqs. (8)), various orders (i, m, n in Eq. (15)), and arbitrary graded distributions along the thickness direction (arbitrary $E(z)$, $\mu(z)$, $G(z)$ in Eqs. (3)).

(3) The influence of radius–span ratio, aperture ratio, offsetting, length–width ratio of hole, and numbers of holes on the natural frequency of functionally graded cylindrical shells is given by the numerical analyses, the conclusion can be supplied to the engineering design.

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