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Classical Jacobi polynomials, closed-form solutions for transverse vibrations

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Abstract

This paper deals with transverse vibrations of nonuniform homogeneous beams and plates. Classes of beams and axisymmetrical circular plates whose boundary value problems of free transverse vibrations and free transverse axisymmetrical vibrations, respectively, can be reduced to an eigenvalue singular problem (singularities occur at both ends) of orthogonal polynomials, are reported. Exact natural frequencies and Jacobi polynomials as exact mode shapes, which result directly from eigenvalues and eigenfunctions of eigenvalue singular problems of classical orthogonal polynomials, are reported for these classes. The above classes of beams and plates hereafter called Jacobi classes are given by geometry and boundary conditions. The geometry consists of parabolic thickness variation, with respect to the axial coordinate for beams, and with respect to the radius for plates. Beams belonging to this class have either one or two sharp ends (singularities) along with certain boundary conditions. Plates have zero thickness at zero and outer radii. The boundary value problems associated with plates, and beams of two sharp ends, are free boundary problems. Two other boundary value problems, hinged-free and sliding-free, are reported for beams with one sharp end. Also, exact natural frequencies and mode shapes for uniformly rotating beams with hinged-free boundary are reported.

1. Introduction

During the past few decades, a significant amount of literature reporting either analytical or numerical solutions has been devoted to transverse vibrations of nonuniform beams, uniformly rotating beams, and circular plates. The characteristic of the governing differential equations of transverse vibrations of nonuniform Euler–Bernoulli beams, and classical circular plates, is that they are fourth-order linear equations with variable coefficients. Closed-form solutions of transverse vibrations have been found for a limited class of nonuniform beams and plates. New findings in the area of orthogonal polynomials theory [1–3] allow extending this class.

Transverse vibrations of nonuniform beams have been studied by numerous investigators due to their relevance to aeronautical, mechanical, and civil engineering. These studies reported either analytical [1,4–22]

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or approximate [23–37] solutions. Analytical solutions, either orthogonal polynomials [1], solutions in terms of Bessel functions [4–12], hypergeometric series [13–16] or power series by Frobenius method [17–21], were reported. A method in which the equation of motion of a class of nonuniform beams is transformed into one of a uniform beam is presented in Ref. [22]. Approximate methods such as Rayleigh–Ritz, employing either orthogonal polynomials [22,25,26] or Fourier series [27] as trial functions, Ritz [28], Galerkin [29], finite difference [30], or finite element [31–37] have been used to obtain approximate natural frequencies of nonuniform beams.

Most of the nonuniform beams whose transverse vibrations have been investigated were linearly tapered. They were (1) beams of circular cross-section either truncated [6–9,17], having one sharp end [11,18] or two sharp ends [1,11], and (2) beams of rectangular cross-section of constant width [4–7,18,27–30,34–36], constant thickness [11,20,21] or pyramids [5–12]. Among the papers studying transverse vibration of nonuniform beams of rectangular cross-section, only a few were dedicated to beams with one sharp end [11,17–20] and only two to the case of two sharp ends [1,11]; all others being dedicated to truncated beams. In Ref. [11], symmetric compound beams of rectangular cross-section of linear thickness variation and width varying with any positive power of the longitudinal coordinate are also considered. A few papers presented a more general case of beams of rectangular cross-section in which the width varied with any positive power of the longitudinal coordinate and the thickness was either constant [19], linear [6], or varied with any positive power of the longitudinal coordinate [13,24]. Cases of parabolic thickness variation along with constant width [34] or singular width at both ends [14] have also been reported. The case of constant thickness and exponential width has been presented in Ref. [11].

Transverse vibrations of uniformly rotating beams have been considered by several investigators. Power series solutions were obtained for uniform [38,39], and nonuniform [20] beams. Solutions in terms of hypergeometric functions were obtained for an entire class of nonuniform beams [14,40] either truncated or with one end sharp. The finite element method was also used to investigate rotating beams [31].

Free transverse axisymmetric vibrations of circular plates were reported in the literature as well. Power series solutions [41–44], solutions in terms of Bessel functions [8], and hypergeometric functions [45–47] were found for nonuniform plates. Approximate methods such as Rayleigh–Ritz [48–53], generalized differential quadrature [54], and finite element [55] have been also used to study different circular plates of variable thickness.

One can see that beams and plates are elements of continuous research interest. Recent applications in the area of MicroElectroMechanical Systems (MEMS), using either beams or plates, have been reported as follows: analytical models for support loss in clamped-free and clamped–clamped micromachined beam resonators with in-plane flexural vibrations [56], resonant techniques to characterize the Poisson's ratio of film materials [57], exact solutions for the coupled thickness-shear and flexural vibrations of quartz strips with linearly varying thickness [58], and analysis of a complete hydrophone system, including effects of the nonuniform plate resonator [59].

Among the results reported in the literature, exact closed-form solutions have a special place due to the fact that "they are not only of intrinsic interest, but they also serve as benchmarks against which the accuracy of various approximate solutions (the Rayleigh–Ritz, Boobnov–Galerkin, finite differences, finite elements, differential quadrature and others) can be ascertained," [42]. Also, they serve as testing packages for numerical solvers. A testing package consisting of exact solutions of several second-order Sturm–Liouville boundary value problems, "which offer a realistic performance test of the currently available automatic codes for eigenvalues of the classical Sturm–Liouville problems," can be found in Ref. [60].

Numerical methods, software packages, and commercial software packages have been developed for solving differential equations, and Boundary Value Problems (BVP). Yet they are limited. A review of numerical methods for self-adjoint and nonself-adjoint nonsingular boundary eigenvalue Sturm–Liouville problems can be found in Ref. [61]. A software package, SLEIGN [62], for computing eigenvalues and eigenfunctions of either regular or singular second-order Sturm–Liouville boundary value problems has been reported in the literature. In the singular case, SLEIGN "has no serious competitor," [63]. The only code available dealing with fourth-order Sturm–Liouville boundary value problems is SLEUTH [64]; yet, limited to regular problems. Another numerical BVP solver, COLSYS, which implements a method of which the differential equation is not evaluated at the ends of the interval, so it can be applied to singular problems, "returned numerical solutions which had no resemblance to the true solution for a range of tolerances'' [65]. Among the commercial software packages available for solving differential equations, MATHEMATICA is one of the most successful packages. Yet, its symbolic differential equation solver DSolve, can only ''solve linear ordinary differential equations of any order with constant coefficients. It can solve also many linear equations up to second-order with non-constant coefficients'' [66].

The survey by the author shows that there is a gap in the literature. Nonuniform beams, uniformly rotating beams, and circular plates, whose exact mode shapes of transverse vibrations are classical orthogonal polynomials, have not been reported yet, except [1]. It appears that [1] is the only paper reducing the boundary value problem of transverse vibrations of nonuniform Euler–Bernoulli beams to an eigenvalue singular problem of a fourth-order differential equation of classical orthogonal polynomials. Yet only beams of circular cross-section have been considered. Also, only two papers reporting vibrations of beams with two sharp ends [1,11] can be found in the literature. The survey also shows that there is a continuous effort for developing numerical methods, numerical solvers, and symbolic solvers, but so far, to the best of our knowledge, a general solver for boundary singular value problems of fourth-order differential equations has not been reported in the literature, yet.

The purpose of this paper is to fill this gap finding the class of nonuniform beams and circular plates along with the necessary boundary conditions in order to have classical orthogonal polynomials as exact mode shapes of transverse vibrations. This paper reports the classes of nonuniform (1) Euler–Bernoulli beams of rectangular and/or elliptical cross-section, (2) uniformly rotating Euler–Bernoulli beams, and (3) circular plates whose exact mode shapes are Jacobi polynomials. These classes hereafter called Jacobi classes consist of beams and plates of convex parabolic thickness variation with the axial coordinate and radius, respectively. Specifically, natural frequencies and mode shapes of transverse vibrations are reported in three boundary cases for nonuniform beams, one boundary case for uniformly rotating beams, and one boundary case for axisymmetric vibrations of circular plates. These cases can be summarized as follows: (1) Jacobi beams (sharp at either end) with free–free boundary conditions, (2) Jacobi half-beams, halves of symmetric complete-beams, with boundary conditions of large end sliding and sharp end free (SF), (3) Jacobi half-beams with large end hinged and sharp end free (HF), (4) uniformly rotating Jacobi half-beam with HF boundary, and (5) Jacobi plates, circular plates with zero thickness at zero and outer radii and free boundary. In all the above boundary value problems, the mode shape equation and boundary value problem reduce to a fourth-order differential equation of orthogonal polynomials and its eigenvalue singular problem [1,2]. Results presented in this paper cannot be compared to other papers since no results regarding these classes of beams and plates can be found in the literature. Even so, the present results are compared to those of uniform beams and plates, and where possible to beams of near rigidity variation with the axial coordinate.

Boundary conditions such as those presented in this paper are encountered in civil, micromechanical, and aeronautical applications. For example, free–free beam boundary conditions have been considered for new beam-type dynamic absorbers [67], layered piezoelectric beam micromechanical resonators [68,69], and viscoelastic measurement resonance methods for low-loss materials [70]. Hinged-free beam boundary conditions have been considered for multibody dynamic systems with flexible components [71,72], and control in active magnetic bearing systems [73]. Sliding-free beam boundary conditions have been considered for vibration isolation of structural systems from damaging earthquake ground motion [74,75], and for beams under compressive loads [76]. An analysis of four models, Euler–Bernoulli, Rayleigh, shear, and Timoshenko, for the transversely vibrating uniform beam with several boundary conditions is presented in Ref. [77]. Free edge circular plates have been considered for space structures [78] and hydroelastic analysis of very large floating platforms [79].

This paper falls in the category of analytical methods and modeling for linear vibration, benchmark solutions. The novelty of this work consists of presenting (1) a method in which the problem of transverse vibration of nonuniform elements is reduced to a singular fourth-order Sturm–Liouville eigenvalue problem, (2) the entire class of nonuniform elements, given by geometry and boundary, for which this method can be used, and (3) the exact natural frequencies and mode shapes in all these cases. Besides contributing to the continuous effort of seeking exact solutions for transverse vibrations of nonuniform one- and two-dimensional continuous systems, this paper can be very useful as reference to researchers interested in developing numerical techniques for solving boundary value singular problems. It also can be included in the existing testing packages. A shorter version of this work was presented in Ref. [80].

2. Differential equations of orthogonal polynomials and eigenvalue singular problems

Orthogonal systems play an important role in analysis, mainly because functions belonging to very general classes can be expanded in series of orthogonal functions. Classical orthogonal polynomials (Jacobi, Legendre, Hermite, Laguerre, Chebyshev) are important classes of orthogonal systems. They are commonly encountered in many applications. In addition to the orthogonal property, orthogonal polynomials are the integrals of differential equations of a simple form, and can be defined as the coefficients in expansions of powers of t of suitable chosen functions $w(x,t)$ called generating functions. Recent developments in orthogonal polynomials theory have been reported. Caruntu [1] presented a fourth-order differential equation of classical orthogonal polynomials and its associated eigenvalue singular problem. Kwon et al. [3] showed that a classical orthogonal system satisfying a second-order differential equation also satisfies a differential equation of order N, where N is an even number and the Nth-order differential operator is a linear combination of iterations of the second-order operator. Moreover, they showed that orthogonal polynomials satisfying a spectral type differential equation of order N, where N is greater than 2, must be Hermite polynomials if and only if the leading coefficient is a nonzero constant. Caruntu [2] reported self-adjoint differential equations, $2r$ order, for classical orthogonal polynomials and their associated eigenvalue singular problems, where r is any natural number.

Classical theory of orthogonal polynomials [81] shows that if $\alpha(x)$ and $\beta(x)$ are two polynomial functions given by

$$\alpha(x) = \alpha_1 x + \alpha_0, \quad \beta(x) = \beta_2 x^2 + \beta_1 x + \beta_0, \quad \alpha_1^2 + \beta_2^2 > 0 \tag{1}$$

and the following requirements are met:

$$\frac{1}{\rho(x)} \frac{\mathrm{d}\rho(x)}{\mathrm{d}x} = \frac{\alpha(x)}{\beta(x)}, \tag{2}$$

$$\lim_{x \to a} \rho(x)\beta(x) = \lim_{x \to b} \rho(x)\beta(x) = 0, \tag{3}$$

where $\rho(x)$ is the weight function of the inner product of classical orthogonal polynomials, and $[a,b]$ is the interval of orthogonality, then an eigenvalue singular problem associated with a second-order differential equation is verified by the classical orthogonal polynomials. This is known as Sturm–Liouville problem.

2.1. Second-order differential equation

The Sturm–Liouville problem consists of a second-order linear differential equation with $x = a$ and $x = b$ singular points and end conditions as follows:

$$\frac{1}{\rho(x)} \frac{\mathrm{d}}{\mathrm{d}x} \left(\rho(x)\beta(x) \frac{\mathrm{d}y(x)}{\mathrm{d}x} \right) - \lambda_1 y(x) = 0, \tag{4}$$

$$y(a), \;\; y(b) \;\; \text{finite}. \tag{5}$$

This is an eigenvalue singular problem. Its eigenvalues $\lambda_{1,n}$ and eigenfunctions $y_n(x)$ are given by Szegö [81]

$$\lambda_{1,n} = n[\alpha_1 + (n+1)\beta_2], \tag{6}$$

$$y_n(x) = Q_n(x), \tag{7}$$

where $Q_n(x)$ are classical orthogonal polynomials whose weight function is $\rho(x)$, subscript 1 of λ indicates that Eq. (4) is the first even order differential equation, and the second subscript n of λ indicates the eigenvalue order.

2.2. Fourth-order differential equation

Caruntu [1] presented a fourth-order differential equation of orthogonal polynomials and the eigenvalue singular problem associated with this equation. The eigenvalue singular problem reported by Caruntu [1] can be summarized as follows. Consider the fourth-order differential equation with $x = a$ and $x = b$ singular points

$$\frac{1}{\rho(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\rho(x)\beta^2(x)\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2}\right) - \lambda_2 y(x) = 0, \tag{8}$$

where λ_2 is a real constant and $[a,b]$ is the interval of orthogonality. If relations (1)–(3) are satisfied and end conditions (5) are met, then the eigenvalues $\lambda_{2,n}$, [1] and [2], are given by

$$\lambda_{2,n} = n(n-1)[\alpha_1 + (n+1)\beta_2][\alpha_1 + (n+2)\beta_2] \tag{9}$$

and the eigenfunctions $y_n(x) = Q_n(x)$ are classical orthogonal polynomials whose weight function is $\rho(x)$, and $n \geqslant 2$ for $\lambda_{2,n} \neq 0$.

2.3. Even-order differential equation

Caruntu [2] reported self-adjoint differential equations, $2r$ order, for classical orthogonal polynomials, where r is any natural number. He reported the eigenvalue singular problems associated with these equations as well. The eigenvalue singular problem reported by Caruntu [2] is as follows. Consider the $2r$ order differential equation with $x = a$ and $x = b$ singular points

$$\frac{1}{\rho(x)}\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(\rho(x)\beta^r(x)\frac{\mathrm{d}^r y(x)}{\mathrm{d}x^r}\right) - \lambda_r y(x) = 0, \tag{10}$$

where λ_r is a real constant, and $[a,b]$ is the interval of orthogonality. If relations (1)–(3) are satisfied and the end conditions (5) are met, then the eigenvalues $\lambda_{r,n}$ of the eigenvalue singular problem given by Eqs. (10) and (5) are as follows:

$$\lambda_{r,n} = \prod_{k=0}^{r-1}(n-k)[\alpha_1 + (n+k+1)\beta_2] \tag{11}$$

and the eigenfunctions $y_n(x) = Q_n(x)$ are classical orthogonal polynomials whose weight function is $\rho(x)$, and $n \geqslant r$ for $\lambda_{r,n} \neq 0$.

2.4. General even-order differential equation

The eigenvalue singular problem associated with a general $2r$ differential equation has been also reported by Caruntu [2]. This problem consists of the following $2r$ order differential equation:

$$\sum_{i=1}^{r} c_i \frac{\mathrm{d}^i}{\mathrm{d}x^i}\left(\rho(x)\beta^i(x)\frac{\mathrm{d}^i y(x)}{\mathrm{d}x^i}\right) - \mu_r \rho(x) y(x) = 0 \tag{12}$$

along with the end conditions (5), where c_i are real constants. The eigenvalues $\mu_{r,n}$ are given by

$$\mu_{r,n} = \sum_{i=1}^{r} c_i \lambda_{i,n}, \tag{13}$$

where $\lambda_{i,n}$ are given by Eq. (11), and the eigenfunctions $y_n(x) = Q_n(x)$ are the classical orthogonal polynomials of the considered system.

3. Beam transverse vibration. Equation and boundary

The above results and concepts can be used to study transverse vibrations of nonuniform beams. If certain requirements are met, then the differential equation of transverse vibrations of nonuniform Euler–Bernoulli beams can be reduced to a fourth-order differential equation of orthogonal polynomials. Also, certain boundary value problems of transverse vibrations of beams reduce to eigenvalue singular problems associated with fourth-order differential equation of orthogonal polynomials.

3.1. Equation

The Euler–Bernoulli differential equation of transverse vibrations of nonuniform beams of length L is as follows:

$$\frac{d^2}{dx^2}\left[EI_1(x)\frac{d^2y(x)}{dx^2}\right] - \rho_0\omega^2 A_1(x)y(x) = 0, \quad -\frac{L}{2} < x < \frac{L}{2}, \tag{14}$$

where $y(x)$ is the transverse displacement, $A_1(x)$ and $I_1(x)$ are the area and the moment of inertia of the current cross-section, respectively; E, ρ_0 and ω are Young modulus, mass density and natural frequency, respectively, and x is the current longitudinal coordinate of the beam. Using the following variable changing:

$$x = L\xi/2, \tag{15}$$

where ξ is the current dimensionless longitudinal coordinate of the beam, the dimensionless equation of transverse vibrations of nonuniform Euler–Bernoulli beam is obtained as follows:

$$\frac{1}{A(\xi)}\frac{d^2}{d\xi^2}\left[I(\xi)\frac{d^2y(\xi)}{d\xi^2}\right] - \frac{\rho_0\omega^2 A_0 L^4}{16EI_0}y(\xi) = 0, \quad -1 < \xi < 1, \tag{16}$$

where A_0 and I_0 are the cross-sectional area and moment of inertia at the reference longitudinal coordinate $\xi = 0$, Fig. 1. Eq. (16) is a fourth-order differential equation of orthogonal polynomials, given by Eq. (8), if the following conditions in terms of cross-sectional dimensionless moment of inertia and area, $I(\xi)$ and $A(\xi)$, respectively, and functions $\rho(\xi)$ and $\beta(\xi)$ of orthogonal polynomials, are met:

$$I(\xi) = \rho(\xi)\beta^2(\xi), \quad A(\xi) = \rho(\xi). \tag{17}$$

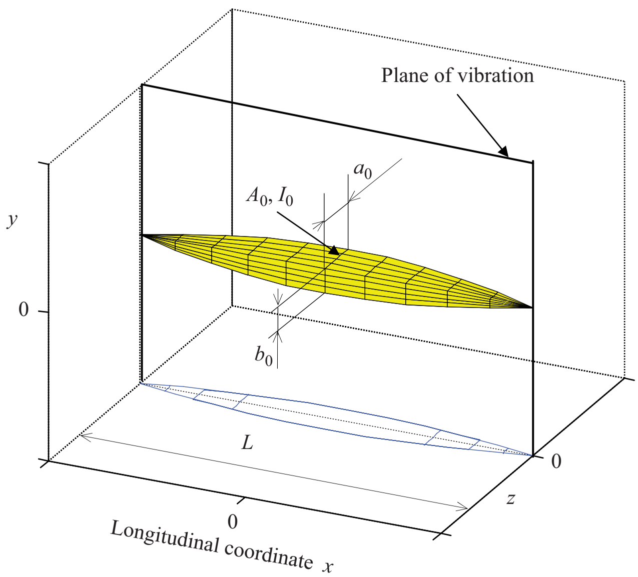

Fig. 1. Jacobi beam of parabolic width $a = a_0(1 - \xi^2)$.

In this case, the dimensionless natural frequencies $\bar{\omega}$ of transverse vibrations are given by

$$\bar{\omega} = \omega L^2 \sqrt{\frac{\rho_0 A_0}{EI_0}} = 4\sqrt{\lambda_2}. \tag{18}$$

3.2. Boundary conditions

Without reducing the generality, consider a beam of finite length over the interval (−1,1) of ξ. If Eqs. (17) are satisfied over this interval, then this beam has both ends sharp, Fig. 1, so it cannot sustain any end moment or shear force. Only free–free boundary conditions are allowed. As shown afterward, free boundaries, i.e. zero moment and shear force at the ends, are equivalent to finite end displacement in this case. Therefore, the necessary boundary conditions for reducing the boundary value problem of transverse vibrations to an eigenvalue singular problem of the fourth-order differential equation of orthogonal polynomials, Eqs. (8) and (5), are those of both ends free. This can be stated as follows.

If Eqs. (2) and (3), where $a = -1$ and $b = 1$, and Eqs. (17) are satisfied then the singular points $\xi = -1$ and 1 are evidently sharp ends for the beam and the vibration problem is one of free–free vibrations, which means zero boundary conditions for bending moment and shear force

$$M(\xi) = 0, \quad T(\xi) = 0 \text{ at } \quad \xi = -1 \text{ and } \xi = 1. \tag{19}$$

This is proved as follows. Since the dimensionless bending moment $M(\xi)$ and the shear force $T(\xi)$ are given by

$$M(\xi) = I(\xi)\frac{\mathrm{d}^2 y(\xi)}{\mathrm{d}\xi^2}, \quad T(\xi) = \frac{\mathrm{d}M(\xi)}{\mathrm{d}\xi} \tag{20}$$

using Eqs. (17) and (2), they become

$$M(\xi) = \rho(\xi)\beta^2(\xi)\frac{\mathrm{d}^2 y(\xi)}{\mathrm{d}\xi^2}, \tag{21}$$

$$T(\xi) = \rho(\xi)\beta^2(\xi)\frac{\mathrm{d}^3 y(\xi)}{\mathrm{d}\xi^3} + \left[\rho(\xi)\beta(\xi)\alpha(\xi) + 2\rho(\xi)\beta(\xi)\frac{\mathrm{d}\beta(\xi)}{\mathrm{d}\xi}\right]\frac{\mathrm{d}^2 y(\xi)}{\mathrm{d}\xi^2}. \tag{22}$$

Provided the displacement y is finite at both ends $\xi = -1$ and 1, the derivatives $\mathrm{d}^3y/\mathrm{d}\xi^3$, $\mathrm{d}^2y/\mathrm{d}\xi^2$ are finite. Additionally, α and $\mathrm{d}\beta/\mathrm{d}\xi$ are finite because they are polynomials. Therefore, the boundary conditions given by Eq. (19) are satisfied since $\rho\beta$ vanishes at both ends, Eqs. (3).

Consequently, in the case of finite beams, the eigenvalue singular problem of the fourth-order differential equation of orthogonal polynomials describes only free transverse vibrations of free–free nonuniform beams sharp at either end. The free–free boundary conditions are consistent with the fact that the beam cannot sustain any moment or shear force at either sharp end.

4. Transverse vibrations of Jacobi beams

Let us find the class of finite nonuniform beams of rectangular and/or elliptical cross-section whose exact transverse vibration mode shapes are classical Jacobi orthogonal polynomials. Further, the natural frequencies of this class are reported. This class hereafter called Jacobi beam class consists of nonuniform beams of parabolic thickness variation, sharp ends and free–free boundary, as shown afterward. They are beams of polynomial width variation. Since beams of finite length are studied, finite orthogonality interval of orthogonal polynomials, corresponding to Jacobi polynomials, is considered.

4.1. Jacobi beams

Weight $\rho(\xi)$ and polynomial functions $\beta(\xi)$ and $\alpha(\xi)$ of Jacobi orthogonal polynomials $J_n^{p,q}(\xi)$ [81,82] are

$$\rho(\xi) = (1-\xi)^p(1+\xi)^q, \quad \beta(z) = (1-\xi^2), \quad \alpha(\xi) = -(p+q)\xi + q - p, \tag{23}$$

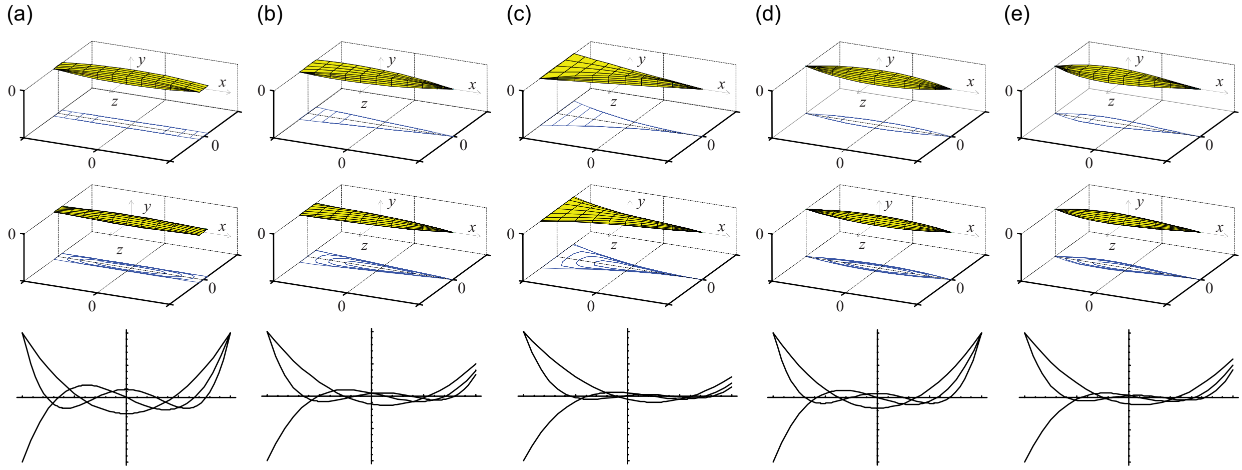

Fig. 2. Beams of rectangular and/or elliptical cross-section belonging to the class of Jacobi beams along with their first three bending mode shapes, and dimensionless width $a(\xi)$: (a) constant $a(\xi)=1$, (b) linear $a(\xi)=1+\xi$, (c) quadratic $a(\xi)=(1+\xi)^2$, (d) convex parabolic $a(\xi)=1-\xi^2$, (e) cubic $a(\xi)=(1-\xi)(1+\xi)^2$.

where p and q are real and greater than -1, and the interval of orthogonality is $(-1,1)$. Jacobi polynomials are given by Abramovitz and Stegun [82] as follows:

$$J_n^{p,q}(\xi)=\sum_{k=0}^{n}\binom{n+p}{k}\binom{n+q}{n-k}(x-1)^{n-k}(x+1)^k. \tag{24}$$

Using Eqs. (17) and (23), the dimensionless area and the dimensionless moment of inertia of the cross-section of such a beam are given by

$$I(\xi)=(1-\xi)^{p+2}(1+\xi)^{q+2},\quad A(\xi)=(1-\xi)^p(1+\xi)^q. \tag{25}$$

Consequently, the Jacobi beam class, given by its dimensionless width $a(\xi)$ and dimensionless thickness $b(\xi)$, is as follows:

$$a(\xi)=(1-\xi)^{p-1}(1+\xi)^{q-1},\quad b(\xi)=1-\xi^2,\quad -1<\xi<1, \tag{26}$$

where both rectangular and elliptical cross-sections are considered. So, Jacobi beam class consists of beams of convex parabolic thickness $b=b_0(1-\xi^2)$ and polynomial width, sharp at either end, and free–free boundary. Fig. 1 shows such a beam. Fig. 2 shows few other examples of nonuniform beams belonging to the class of Jacobi beams given by Eq. (26).

4.2. Geometric interpretation and considerations

Considerations regarding the values that the parameters p and q can get, and a geometric interpretation of the weight function $\rho(\xi)$ and the polynomial functions $\beta(\xi)$ and $\alpha(\xi)$ of orthogonal polynomials are presented next. According to the theory of classical Jacobi orthogonal polynomials, the real parameters p and q must be greater than -1. If

$$p\geqslant 1 \text{ and } q\geqslant 1, \tag{27}$$

then the width $a(\xi)$ and the cross-sectional area $A(\xi)$ vanish at both ends, see Eqs. (25) and (26). In this case, if the length of the beam is much greater than its width and thickness, the Euler–Bernoulli beam hypothesis is satisfied and consequently present transverse vibration results are reliable. If at least one of the parameters p and q is less than 1, then the width $a(\xi)$ of the beam approaches infinity at least at one of the ends. In this situation, the Euler–Bernoulli assumption is not verified and consequently the present results of transverse vibrations are not applicable. According to Eq. (17), the weight $\rho(\xi)$ of orthogonal polynomials represents the cross-section area of the beam in the dimensionless formulation of transverse vibration. The polynomial function $\beta(\xi)$, found from Eq. (17), represents the dimensionless thickness (the transversal dimension in the plane of vibration) of the beam of rectangular and/or elliptical cross-section. As resulting from Eqs. (2) and (17), the function $\alpha(\xi)$ is given by

$$\alpha(\xi) = \frac{1}{a(\xi)}\frac{\mathrm{d}A(\xi)}{\mathrm{d}\xi} = \frac{\mathrm{d}A(\xi)/\mathrm{d}\xi}{\mathrm{d}A_{xy}(\xi)/\mathrm{d}\xi}, \tag{28}$$

where $A(\xi)$ and $A_{xy}(\xi)$ are the cross-section area and the longitudinal area (xy plane). The function $\alpha(\xi)$ represents the ratio of the rate of change of cross-section area to the rate of change of longitudinal area. The sum $p+q$ is the absolute value of the rate of change of $\alpha(\xi)$ with respect to the longitudinal coordinate, see Eq. (23). Furthermore, $(q-p)$ is the rate of change of cross-sectional area $\mathrm{d}A/\mathrm{d}\xi$ at zero dimensionless longitudinal coordinate $\xi = 0$ as obtained using Eq. (28) along with Eqs. (23) and (26). Also, $(q-p)/(q+p)$ is the longitudinal coordinate where cross-sectional area has a maximum as resulting from Eq. (28). Meanwhile, $(q-p)/(q+p)$ is the longitudinal location of the centroid of the longitudinal sectional area perpendicular to the plane of vibration as can be found by integrating Eq. (28) between $\xi = -1$ and 1. In the particular case $q = p$, the maximum of the cross-sectional area and the centroid of the longitudinal sectional area perpendicular to the plane of vibration, both occur at the zero longitudinal coordinate $\xi = 0$, which is consistent in this case with the symmetry of the beam with respect to the plane $\xi = 0$.

The sum $p+q$, up to a constant coefficient, gives the average rate of relative decrease of the cross-section area from mid-beam toward ends. It also gives, up to a constant coefficient and an additional term, the average rate of the relative decrease of beam rigidity from mid-beam toward ends. The rate of relative decrease of area A with respect to the longitudinal coordinate ξ is $\mathrm{d}A/A\,\mathrm{d}\xi$. The average rate of the relative decrease of the cross-section area A toward the ends is as follows:

$$\frac{[\int_0^1 b(\xi)(\mathrm{d}A(\xi)/A(\xi)) + \int_0^{-1} b(\xi)(\mathrm{d}A(\xi)/A(\xi))]}{\int_{-1}^1 b(\xi)\,\mathrm{d}\xi} = -\frac{3}{4}(p+q), \tag{29}$$

where the thickness $b(\xi)$, given by Eq. (26), has been used as weight for finding the average rate. Analogously, the average rate of relative decrease of the rigidity EI from mid-beam toward ends results as follows:

$$\frac{[\int_0^1 b(\xi)(\mathrm{d}EI(\xi)/EI(\xi)) + \int_0^{-1} b(\xi)(\mathrm{d}EI(\xi)/EI(\xi))]}{\int_{-1}^1 b(\xi)\,\mathrm{d}\xi} = -\frac{3}{4}(p+q) - 3. \tag{30}$$

4.3. Differential equation and boundary conditions

The fourth-order differential equation of transverse vibrations of a beam belonging to the class given by Eqs. (26), results from Eq. (16) as follows:

$$\frac{1}{(1-\xi)^p(1+\xi)^q}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[(1-\xi)^{p+2}(1+\xi)^{q+2}\frac{\mathrm{d}^2 y(\xi)}{\mathrm{d}\xi^2}\right] - \lambda_2 y(\xi) = 0, \quad -1<\xi<1, \tag{31}$$

where λ_2 is given by Eq. (18), and parameters p and q are real and greater than or equal to 1. Since Jacobi beams are sharp at either end, they cannot sustain boundary moments and shear forces. So, only free–free boundary conditions are allowed. Consequently, the boundary conditions given by Eqs. (19) reduce to the following:

$$y(-1), \; y(1) \text{ finite.} \tag{32}$$

The reference cross-sectional area and moment of inertia, A_0 and I_0, respectively, are considered at the longitudinal coordinate $\xi = 0$, and are given by

$$A_0 = a_0 b_0, \quad I_0 = \frac{a_0 b_0^3}{12} \quad \text{(rectangular cross-section)}, \tag{33}$$

$$A_0 = \pi a_0 b_0, \quad I_0 = \frac{\pi a_0 b_0^3}{4} \quad \text{(elliptical cross-section)}, \tag{34}$$

where a_0 and b_0 are the reference width and thickness, also considered at $\xi = 0$.

4.4. Natural frequencies and mode shapes

Boundary value problem given by Eqs. (31) and (32) is the eigenvalue singular problem of orthogonal polynomials given by Eqs. (8) and (5). The eigenvalues $\lambda_{2,n}$ and the eigenfunctions $y_n(\xi)$ of this eigenvalue singular problem are given by Eqs. (9) and (7), respectively, where $\alpha_1 = -(p+q)$ and $\beta_2 = -1$. Therefore, the dimensionless natural frequencies $\bar{\omega}_n$, Eq. (18), and the mode shapes $y_n(\xi)$ of the transverse vibration boundary value problem given by Eqs. (16) and (32), are as follows:

$$\bar{\omega}_n = 4\sqrt{\lambda_{2,n+1}} = 4\sqrt{n(n+1)(n+2+p+q)(n+3+p+q)}, \tag{35}$$

$$y_n(\xi) = J_{n+1}^{p,q}(\xi), \tag{36}$$

where n is any natural number. The Jacobi orthogonal polynomials $J_n^{p,q}(\xi)$, given by Eq. (24), can be also found using the Rodriguez formula [81,82]. Fig. 2 shows a few types of beams belonging to this class. The parameters p and q and the corresponding dimensionless natural frequencies $\bar{\omega}_n = \omega_n L^2\sqrt{(\rho_0 A_0)/(EI_0)}$ and mode shapes $y_n(\xi)$ of transverse vibrations of the free–free beams plotted in Fig. 2 are: (a) $p = q = 1$, $\bar{\omega}_n = 4\sqrt{n(n+1)(n+3)(n+4)}$, $y_n(\xi) = J_{n+1}^{1,1}(\xi)$, (b) $p = 1$, $q = 2$, $\bar{\omega}_n = 4\sqrt{n(n+1)(n+4)(n+5)}$, $y_n(\xi) = J_{n+1}^{1,2}(\xi)$, (c) $p = 1$, $q = 3$, $\bar{\omega}_n = 4\sqrt{n(n+1)(n+5)(n+6)}$, $y_n(\xi) = J_{n+1}^{1,3}(\xi)$, (d) $p = q = 2$, $\bar{\omega}_n = 4\sqrt{n(n+1)(n+5)(n+6)}$, $y_n(\xi) = J_{n+1}^{2,2}(\xi)$, (e) $p = 2$, $q = 3$, $\bar{\omega}_n = 4\sqrt{n(n+1)(n+6)(n+7)}$, $y_n(\xi) = J_{n+1}^{2,3}(\xi)$. One can note from Eq. (36) that the mode shapes depend on the parameters p and q while the natural frequencies $\bar{\omega}_n$ given by Eq. (35) depend only on the sum $p+q$. Since the parameters p and q give the geometry of the beam, see Eqs. (26), different beams have different mode shapes but the same natural frequencies as long as the sum $p+q$ is the same.

4.5. Discussion

Table 1 shows numerical values of the first ten natural frequencies of Jacobi beams for values of the parameter $p+q$ between 2 and 5. Fig. 3 shows the dependence of the first six natural frequencies on the parameter $p+q$. It also shows for a comparison the first six dimensionless natural frequencies of a uniform

Table 1
First ten dimensionless natural frequencies $\omega L^2\sqrt{(\rho_0 A_0)/(EI_0)}$ of transverse vibrations of Jacobi beams versus parameter $p+q$

$p+q$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$	$\bar{\omega}_7$	$\bar{\omega}_8$	$\bar{\omega}_9$	$\bar{\omega}_{10}$
2	30.98	63.50	103.7	151.8	207.8	271.9	343.9	423.9	511.9	607.9
2.25	32.40	65.95	107.2	156.3	213.3	278.4	351.4	432.4	521.4	618.4
2.5	33.82	68.41	110.6	160.7	218.8	284.9	358.9	440.9	530.9	628.9
2.75	35.24	70.87	114.1	165.2	224.3	291.3	366.4	449.4	540.4	639.4
3	36.66	73.32	117.6	169.7	229.8	297.8	373.9	457.9	549.9	649.9
3.25	38.08	75.78	121.0	174.2	235.3	304.3	381.4	466.4	559.4	660.4
3.5	39.50	78.23	124.5	178.7	240.7	310.8	388.8	474.9	568.9	670.9
3.75	40.91	80.68	128.0	183.1	246.2	317.3	396.3	483.4	578.4	681.4
4	42.33	83.14	131.5	187.6	251.7	323.8	403.8	491.9	587.9	691.9
4.25	43.75	85.59	134.9	192.1	257.2	330.3	411.3	500.3	597.4	702.4
4.5	45.17	88.05	138.4	196.6	262.7	336.7	418.8	508.8	606.9	712.9
4.75	46.58	90.50	141.9	201.0	268.2	343.2	426.3	517.3	616.4	723.4
5	48.00	92.95	145.3	205.5	273.6	349.7	433.8	525.8	625.8	733.9

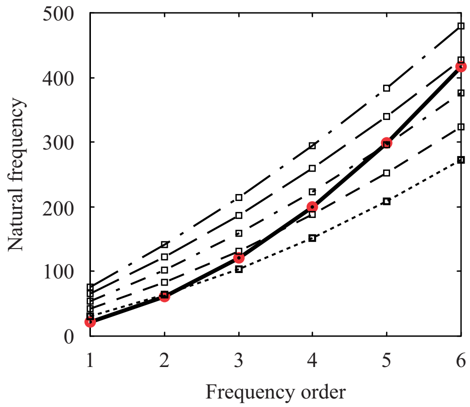

Fig. 3. First six dimensionless natural frequencies $\bar{\omega} = \omega L^2\sqrt{(\rho_0 A_0)/(EI_0)}$ of free–free vibrations of five sets of Jacobi beams given by ·········, $p+q=2$; - - - -, $p+q=4$; - · - · -, $p+q=6$; — — ·, $p+q=8$; — · —, $p+q=10$; and ▬▬, a uniform beam.

Table 2
First ten dimensionless natural frequencies $\bar{\omega} = \omega L^2\sqrt{(\rho_0 A_0)/(EI_0)}$ of free–free vibrations of three Jacobi beams $b(\xi)=1-\xi^2$ three compound beams [11] of dimensionless thickness varying linearly $b_C(\xi) = \begin{cases} 1+\xi, & \text{if } -1<\xi\leqslant 0 \\ 1-\xi, & \text{if } 0<\xi<1 \end{cases}$, and a uniform beam

Ref.	Thickness	Width	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$	$\bar{\omega}_7$	$\bar{\omega}_8$	$\bar{\omega}_9$	$\bar{\omega}_{10}$
Present	$b(\xi)$	1	30.98	63.50	103.7	151.8	207.8	271.9	343.9	423.9	511.9	607.9
		$[b(\xi)]^{1/2}$	36.66	73.32	117.6	169.7	229.8	297.8	373.9	457.9	549.9	649.9
		$b(\xi)$	42.33	83.14	131.5	187.6	251.7	323.8	403.8	491.9	587.9	691.9
[11]	$b_C(\xi)$	1	26.37	43.60	70.85	98.5	135.0	172.8	218.9	266.7	322.6	380.3
		$[b_C(\xi)]^{1/2}$	33.21	52.32	82.72	112.4	151.9	191.8	240.7	290.7	348.1	409.3
		$b_C(\xi)$	40.70	61.62	95.28	127.0	169.4	211.4	263.2	315.4	376.7	439.0
Uniform	1	1	22.38	61.60	120.9	200.0	298.6	416.0	555.0	712.0	890.6	1088

beam. Jacobi beam class, Eqs. (26), is characterized by an average rate of relative variation of rigidity EI given by Eq. (30). As one can see, Table 1 and Fig. 3, the rigidity decrease from the mid-beam toward the ends leads to greater values of the lower-order natural frequencies and lesser values of the higher-order natural frequencies of the nonuniform beams when compared with uniform beams. Moreover, an increase of $p+q$, which results in an increase of the average rate of relative rigidity reduction toward beam ends, see Eq. (30), leads within the class of Jacobi beams to an increase of all dimensionless natural frequencies. Table 2 and Fig. 4 allow for a comparison with data available in the literature. Table 2 presents the first ten dimensionless natural frequencies of three Jacobi beams, three compound beams of linear thickness variation [11], and a uniform beam for comparison. The three cases of the nonuniform beams are: constant width, width proportional to the thickness square root, and width proportional to the thickness. All beams of Table 2 have the same length L and maximum cross-section (A_0, I_0). Fig. 4 shows a graphical representation of the first four natural frequencies of the beams presented in Table 2. As one can see, for free–free boundary conditions, beams of parabolic thickness variation (Jacobi beams) in all three-width cases have greater values for first-order natural frequency and lesser values for higher-order natural frequencies when compared with uniform beams. This is in agreement with the results for compound beams of linear thickness variation reported in Ref. [11] and graphically represented in Fig. 4. The increase of relative rigidity reduction toward ends leads within each family of beams, parabolic thickness, and linear thickness [11], to an increase of all dimensionless natural frequencies as well.

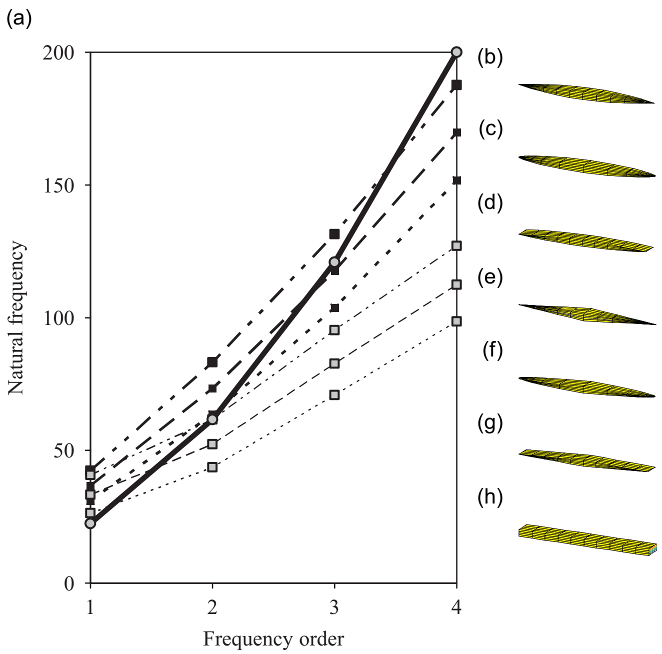

Fig. 4. (a) First four dimensionless natural frequencies from Table 3, $\bar{\omega} = \omega L^2\sqrt{(\rho_0 A_0)/(EI_0)}$, of three Jacobi beams of width $a(\xi)$ given by —··— $a(\xi) = b(\xi)$, shown in (b), – – – $a(\xi) = [b(\xi)]^{1/2}$, shown in (c), - - - - - $a(\xi) = 1$, shown in (d); three compound beams [11] of linear thickness $b_C(\xi) = \begin{cases} 1+\xi, & \text{if} \quad -1<\xi\leqslant 0 \\ 1-\xi, & \text{if} \quad 0<\xi<1 \end{cases}$ and width $a(\xi)$ given by ·· ·–· ···· ·· $a(\xi) = b_C(\xi)$, shown in (e), –––––– $a(\xi) = [b_C(\xi)]^{1/2}$, shown in (f), -------- $a(\xi) = 1$, shown in (g); and —— uniform beam, shown in (h).

4.6. Jacobi beam of parabolic width variation

This is a beam belonging to the class of Jacobi beams. First natural frequency and mode shape are found using Eqs. (35), (36), and (24). They are also found using an approximate method, namely Galerkin, for comparison. Jacobi beams are characterized by parabolic thickness variation $b(\xi) = 1 - \xi^2$. The parabolic width variation $a(\xi) = 1 - \xi^2$ is obtained from Eq. (26) taking $p = 2$ and $q = 2$. This type of beam is shown in Figs. 1 and 2(d). Its natural frequencies $\bar{\omega}_n$ and mode shapes $y_n(\xi)$ for transverse vibrations with free–free boundary are given by $\bar{\omega}_n = 4\sqrt{n(n+1)(n+5)(n+6)}$ and $y_n(\xi) = J_{n+1}^{2,2}(\xi)$. The first dimensionless natural frequency $\bar{\omega}_1$ and mode shape $y_1(\xi)$ in this case are found as

$$\bar{\omega}_1 = 4\sqrt{\lambda_{2,2}} = 16\sqrt{7}, \quad y_1(\xi) = J_2^{2,2}(\xi) = b_1(1 - 7\xi^2). \tag{37}$$

Next, the Galerkin method is used to find the first dimensionless natural frequency and mode shape of this beam. It is shown that the same values as those given by Eq. (37) are found. We look for a trial solution for the Eq. (31) as follows:

$$\bar{y}(\xi) = \sum_{i=1}^{3} b_i \varphi_i(\xi), \tag{38}$$

where $\varphi_i(\xi) = \xi^{i-1}$, and b_i are adjustable constants, $i = 1,2,3$. We determine these constants by the requirement:

$$\int_{-1}^{1} \varphi_i(\xi)\left\{\frac{d^2}{d\xi^2}\left[(1-\xi^2)^4\frac{d^2\bar{y}}{d\xi^2}\right] - \lambda_2(1-\xi^2)^2\bar{y}\right\} d\xi = 0, \quad i = 1,2,3, \tag{39}$$

where λ_2 is given by Eq. (18). Indeed, if $\bar{y}(\xi)$ is the true solution, it satisfies Eq. (31) identically; the Galerkin method requires $L_2[\bar{y}]$ to be orthogonal to the linear span of $\{\varphi_i\}$, where L_2 is the differential operator of Eq. (31). Eq. (39) lead to the following system of three equations in three unknowns b_i, $i = 1,2,3$:

$$b_1\lambda_2 P_{k1} + b_2\lambda_2 P_{k2} + b_3(\lambda_2 P_{k3} - 16M_k) = 0, \quad k = 1,2,3, \tag{40}$$

where $M_2 = 0$, $P_{ki} = 0$ if $k+i = 3$ and/or $k+i = 5$, otherwise

$$M_k = -B\left(\frac{k}{2},4\right) + 6B\left(\frac{k+2}{2},3\right), \quad P_{ki} = B\left(\frac{k+i}{2},3\right). \tag{41}$$

Here, $B(p,q)$ is Beta Function. This system of equations has a nontrivial solution if and only if the determinant of the coefficients of system (40) is zero, which represents the eigenvalue equation. Solving this equation (MATLAB has been used), a value of 112.00 of the eigenvalue λ_2 results. Therefore, the value of the first dimensionless natural frequency $\bar{\omega}_1$, according to Eq. (18), can be found as 42.332, which is exactly the value given by Eq. (37). Substituting the value of λ_2 into Eq. (40), the trial solution given by Eq. (38) results as $\bar{y}(\xi) = b_1(1 - 7\xi^2)$, which is exactly the first mode shape given by Eq. (37).

5. Transverse vibrations of Jacobi half-beams

The subset of symmetric Jacobi beams of rectangular and/or elliptical cross-section whose bending vibration mode shapes are Jacobi polynomials is given by Eqs. (26) where $p = q$ and $p \geqslant 1$.

5.1. Jacobi half-beam class

Let us now consider half of such a symmetric beam, Fig. 5, hereafter called Jacobi half-beam. It is sharp at one end only. The differential equation of transverse vibrations of the half-beam is given by Eq. (14), where $0 < x < l$ and $l = L/2$. Consequently, its dimensionless differential equation is Eq. (31), where $0 < \xi < 1$ and λ_2 is given by Eq. (18). The Jacobi half-beam class is given by the width $a(\xi)$ and thickness $b(\xi)$ as follows:

$$a(\xi) = (1 - \xi^2)^{p-1}, \quad b(\xi) = 1 - \xi^2, \quad \text{where} \quad 0 < \xi < 1 \tag{42}$$

and sliding- and hinged-free boundary conditions as shown afterward. It can be noticed that the dimensionless longitudinal coordinate is $-1 < \xi < 1$ for beams, Eqs. (26), and $0 < \xi < 1$ for half-beams as given by Eq. (42). One can find that the average rate of the relative decrease of the cross-section area A toward the sharp end

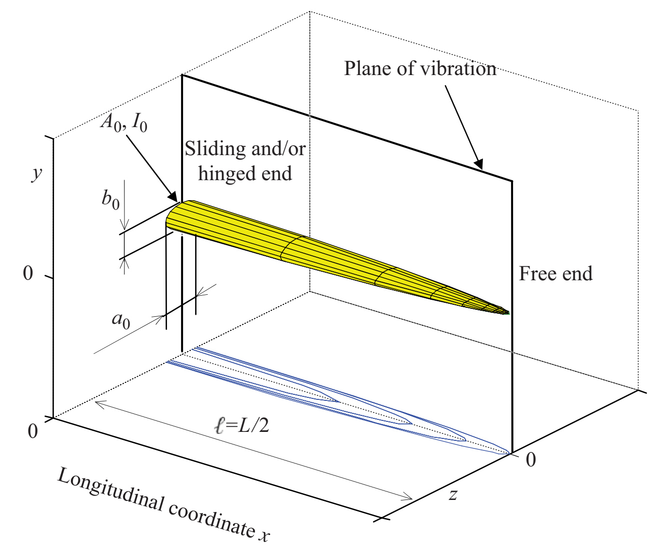

Fig. 5. Jacobi half-beam of elliptic cross-section and width $a = a_0\sqrt{1 - \xi^2}$.

is as follows:

$$\frac{\int_0^1 b(\xi)(\mathrm{d}A(\xi)/A(\xi))}{\int_0^1 b(\xi)\,\mathrm{d}\xi} = -\frac{3}{2}p, \tag{43}$$

where the thickness $b(\xi)$ given by Eq. (42) has been used as weight for finding the average rate. Analogously, the average rate of relative decrease of the rigidity EI from mid-beam toward ends results as follows:

$$\frac{\int_0^1 b(\xi)(\mathrm{d}EI(\xi)/EI(\xi))}{\int_0^1 b(\xi)\,\mathrm{d}\xi} = -\frac{3}{2}p - 3. \tag{44}$$

The parameter p, up to a constant coefficient and an additional term, gives the average rate of the relative decrease of beam rigidity toward the sharp end.

5.2. Boundary conditions

Jacobi half-beams have Jacobi orthogonal polynomials as exact mode shapes in two cases of boundary conditions. These two cases are (1) large end sliding and sharp end free (SF) given by

$$\frac{\mathrm{d}y}{\mathrm{d}\xi}(0) = \frac{\mathrm{d}^3y}{\mathrm{d}\xi^3}(0) = 0, \quad y(1) \quad \text{finite} \tag{45}$$

and (2) large end hinged and sharp end free (HF) given by

$$y(0) = \frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}(0) = 0, \quad y(1) \quad \text{finite.} \tag{46}$$

Under these conditions, the dimensionless natural frequencies of the Jacobi half-beams can be found from those of the Jacobi beams (both halves).

5.2.1. SF boundary conditions

The mode shapes of transverse vibrations of SF Jacobi half-beams ($0 \leqslant \xi \leqslant 1$) are the same with the symmetric modes of free–free (FF) vibrations of the corresponding Jacobi beam ($-1 \leqslant \xi \leqslant 1$), except the domain (as one can notice the domain of ξ is different). In the mentioned conditions, dimensionless natural frequencies of Jacobi half-beams are one fourth of those of the corresponding Jacobi beams; this is due to the reduction in half of the beam length. Let us verify that this is true. The symmetric modes of FF transverse vibrations of a symmetric Jacobi beam are even degree Jacobi polynomials $y_{2n-1}(\xi) = J_{2n}^{p,p}(\xi)$, see Eq. (36), where n is any natural number and $-1 < \xi < 1$. These Jacobi polynomials are even functions, $J_{2n}^{p,p}(-\xi) = (-1)^{2n} J_{2n}^{p,p}(\xi)$, [82], i.e. symmetric functions with respect to the longitudinal coordinate ξ. So they can be written as polynomials in ξ^2, $J_{2n}^{p,p}(\xi) = P_n(\xi^2)$. Using this last form, it can be seen that the first derivative and the third derivative of $J_{2n}^{p,p}(\xi)$ vanish at $\xi = 0$. So, these Jacobi polynomials of even degree satisfy both boundary conditions (45) and differential Eq. (31). The length of the Jacobi the half-beam is the length of the symmetric Jacobi beam reduced by half. The reference cross-sectional area A_0 and moment of inertia I_0 are considered at $\xi = 0$ for both beams. So, the mode shapes $\breve{y}_n(\xi)$ and dimensionless natural frequencies $\breve{\omega}_n$ of half-beam SF boundary value problem of transverse vibrations are expressed in terms of those of the corresponding symmetric Jacobi beam as follows:

$$\breve{y}_n(\xi) = J_{2n}^{p,p}(\xi), \quad \breve{\omega}_n = \omega_{2n-1}\ell^2\sqrt{\frac{\rho_0 A_0}{EI_0}} = \frac{\bar{\omega}_{2n-1}}{4} = \sqrt{\lambda_{2,2n}} = 2\sqrt{n(2n-1)(2n+2p+1)(n+p+1)}, \tag{47}$$

where n is any natural number and $0 < \xi < 1$.

5.2.2. HF boundary conditions

The mode shapes of transverse vibrations of HF Jacobi half-beams ($0 \leqslant \xi \leqslant 1$) are the same with the antisymmetric modes of FF vibrations of the corresponding symmetric Jacobi beam ($-1 \leqslant \xi \leqslant 1$), except the

Table 3
First ten dimensional natural frequencies $\omega\ell^2\sqrt{(\rho_0 A_0)/(EI_0)}$ of transverse vibrations of Jacobi half-beams for sliding-free (SF) boundary conditions versus parameter p

p	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$	$\bar{\omega}_7$	$\bar{\omega}_8$	$\bar{\omega}_9$	$\bar{\omega}_{10}$
1	7.75	25.92	51.96	85.98	128.0	178.0	236.0	302.0	376.0	458.0
1.125	8.10	26.79	53.33	87.85	130.4	180.9	239.4	305.9	380.4	462.9
1.25	8.46	27.66	54.70	89.72	132.7	183.7	242.7	309.7	384.7	467.7
1.375	8.81	28.53	56.07	91.59	135.1	186.6	246.1	313.6	389.1	472.6
1.5	9.17	29.39	57.45	93.47	137.5	189.5	249.5	317.5	393.5	477.5
1.625	9.52	30.26	58.82	95.34	139.9	192.4	252.9	321.4	397.9	482.4
1.75	9.87	31.13	60.19	97.21	142.2	195.2	256.2	325.2	402.2	487.2
1.875	10.23	32.00	61.56	99.08	144.6	198.1	259.6	329.1	406.6	492.1
2	10.58	32.86	62.93	101.0	147.0	201.0	263.0	333.0	411.0	497.0
2.125	10.94	33.73	64.30	102.8	149.3	203.9	266.4	336.9	415.4	501.9
2.25	11.29	34.60	65.67	104.7	151.7	206.7	269.7	340.7	419.7	506.7
2.375	11.65	35.46	67.04	106.6	154.1	209.6	273.1	344.6	424.1	511.6
2.5	12.00	36.33	68.41	108.4	156.5	212.5	276.5	348.5	428.5	516.5

Table 4
First ten dimensional natural frequencies $\omega\ell^2\sqrt{(\rho_0 A_0)/(EI_0)}$ of transverse vibrations of Jacobi half-beams for hinged-free (HF) boundary conditions versus parameter p

p	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$	$\bar{\omega}_7$	$\bar{\omega}_8$	$\bar{\omega}_9$	$\bar{\omega}_{10}$
1	15.87	37.95	67.97	106.0	152.0	206.0	268.0	338.0	416.0	502.0
1.125	16.49	39.07	69.59	108.1	154.6	209.1	271.6	342.1	420.6	507.1
1.25	17.10	40.19	71.21	110.2	157.2	212.2	275.2	346.2	425.2	512.2
1.375	17.72	41.31	72.84	112.3	159.9	215.4	278.9	350.4	429.9	517.4
1.5	18.33	42.43	74.46	114.5	162.5	218.5	282.5	354.5	434.5	522.5
1.625	18.94	43.55	76.08	116.6	165.1	221.6	286.1	358.6	439.1	527.6
1.75	19.56	44.67	77.70	118.7	167.7	224.7	289.7	362.7	443.7	532.7
1.875	20.17	45.78	79.32	120.8	170.4	227.9	293.4	366.9	448.4	537.9
2	20.78	46.90	80.94	123.0	173.0	231.0	297.0	371.0	453.0	543.0
2.125	21.40	48.02	82.57	125.1	175.6	234.1	300.6	375.1	457.6	548.1
2.25	22.01	49.14	84.19	127.2	178.2	237.2	304.2	379.2	462.2	553.2
2.375	22.62	50.26	85.81	129.3	180.8	240.4	307.9	383.4	466.9	558.4
2.5	23.24	51.38	87.43	131.5	183.5	243.5	311.5	387.5	471.5	563.5

domain. In the mentioned conditions, dimensionless natural frequencies for half-beams are one fourth of those of the corresponding symmetric Jacobi beams. Let us prove this. The antisymmetric modes of symmetric Jacobi beams $y_{2n}(\xi) = J^{p,p}_{2n+1}(\xi)$, Eq. (36), are odd functions, $J^{p,p}_{2n+1}(-\xi) = (-1)^{2n+1} J^{p,p}_{2n+1}(\xi)$, [82], and therefore satisfy the boundary conditions given by Eqs. (46). Consequently, the mode shapes $\widehat{y}_n(\xi)$ and natural frequencies $\widehat{\omega}_n$ of the half-beam vibrations in the PF case are as follows:

$$\widehat{y}_n(\xi) = J^{p,p}_{2n+1}(\xi), \quad \widehat{\omega}_n = \omega_{2n}\ell^2\sqrt{\frac{\rho_0 A_0}{EI_0}} = \bar{\omega}_{2n}/4 = \sqrt{\lambda_{2,2n+1}} = 2\sqrt{n(2n+1)(n+p+1)(2n+2p+3)}. \qquad (48)$$

Tables 3 and 4 show the first ten dimensionless natural frequencies of Jacobi half-beams for SF and HF boundary conditions, respectively, versus the parameter p. The parameter p, which gives the width variation, Eq. (42), also gives the average rate of the relative decrease of beam rigidity toward the sharp end. The larger the parameter p, the larger the average rate of relative reduction. Table 5 shows for comparison the first ten dimensionless natural frequencies of three Jacobi half-beams (parabolic thickness variation), three beams of linear thickness variation [11], and a uniform beam [83]. The three cases for each thickness variation

Table 5
First ten dimensionless natural frequencies $\omega\ell^2\sqrt{(\rho_0 A_0)/(EI_0)}$ of hinged-free (HF) and/or sliding-free (SF) boundary conditions (BC) for Jacobi half-beams $b(\xi)=1-\xi^2$, beams of linear thickness variation $b_C(\xi)=1-\xi$, and uniform beams

BC	Ref.	Thickness	Width	$\bar\omega_1$	$\bar\omega_2$	$\bar\omega_3$	$\bar\omega_4$	$\bar\omega_5$	$\bar\omega_6$	$\bar\omega_7$	$\bar\omega_8$	$\bar\omega_9$	$\bar\omega_{10}$
HF	Present	$b(\xi)$	1	15.87	37.95	67.97	106.0	152.0	206.0	268.0	338.0	416.0	502.0
			$[b(\xi)]^{1/2}$	18.33	42.43	74.46	114.5	162.5	218.5	282.5	354.5	434.5	522.5
			$b(\xi)$	20.78	46.90	80.94	123.0	173.0	231.0	297.0	371.0	453.0	543.0
	[11]	$b_C(\xi)$	1	10.90	24.63	43.20	66.68	95.08	—	—	—	—	—
			$[b_C(\xi)]^{1/2}$	13.08	28.11	47.95	72.68	102.3	—	—	—	—	—
			$b_C(\xi)$	15.41	31.75	52.86	78.85	109.7	—	—	—	—	—
	[83]	1	1	15.40	50.00	104.0	178.0	272.0	—	—	—	—	—
SF	Present	$b(\xi)$	1	7.75	25.92	51.96	85.98	128.0	178.0	236.0	302.0	376.0	458.0
			$[b(\xi)]^{1/2}$	9.17	29.39	57.45	93.47	137.5	189.5	249.5	317.5	393.5	477.5
			$b(\xi)$	10.58	32.86	62.93	101.0	147.0	201.0	263.0	333.0	411.0	497.0
	[11]	$b_C(\xi)$	1	6.59	17.71	33.76	54.72	80.64	—	—	—	—	—
			$[b_C(\xi)]^{1/2}$	8.30	20.68	37.96	60.18	87.03	—	—	—	—	—
			$b_C(\xi)$	10.18	23.82	42.35	65.80	94.18	—	—	—	—	—
	[83]	1	1	5.59	30.23	74.64	138.8	222.7	—	—	—	—	—

are: (1) constant width, (2) width proportional to the thickness square root, and (3) width proportional to the thickness. All beams have the same length ℓ and maximum cross-section (A_0, I_0). Figs. 6 and 7 show graphically the first three dimensionless natural frequencies from Table 5 for HF and SF boundary conditions, respectively. Fig. 6 shows that for SF boundary the first frequency of Jacobi half-beams (parabolic thickness variation) is greater, and higher order frequencies lesser, than those of uniform beams. Within the Jacobi half-beam class, the larger the parameter p, the larger the values of all frequencies, i.e. the greater the average rate of rigidity relative reduction toward the beam ends, the higher values of all frequencies. This is in agreement with data published in the literature. The same pattern can be seen for beams of linear thickness variation [11] in Fig. 6. Also, the frequencies of beams of parabolic thickness variation have been obtained to be greater than those of corresponding beams of linear thickness variation. Fig. 7 shows for HF boundary of Jacobi half-beams the same properties as for SF boundary. The only difference is that while the first frequency for parabolic thickness variation is greater than that of the uniform beam, the first frequency for linear thickness variation [11] is lesser.

6. Transverse vibrations of uniformly rotating Jacobi half-beams

Jacobi polynomials are also mode shapes of transverse vibrations of uniformly rotating half-beams, Fig. 8, belonging to the class given by Eqs. (42). This occurs when (1) the rotation axis passes through the reference section $\xi=0$ and is perpendicular to the beam and in the plane of vibration and (2) the boundary conditions are HF. The differential equation of uniformly rotating beams is given by

$$\frac{d^2}{dx^2}\left(EI_1(x)\frac{d^2Y(x)}{dx^2}\right)-\Omega^2\rho_0\frac{d}{dx}\left(\int_x^\ell \zeta A_1(\zeta)\,d\zeta\,\frac{dY(x)}{dx}\right)-\rho_0\omega^2A_1(x)Y(x)=0,\quad 0<x<\ell=\frac{L}{2},\tag{49}$$

where the nomenclature is the same as in Section 3. In addition, Ω is the angular velocity. The axial distributed force is $N(x)=\Omega^2\rho_0\int_x^\ell \zeta A_1(\zeta)\,d\zeta$. The boundary conditions [84] for beams subject to axial force are

$$\left\{\frac{d}{dx}\left[EI_1(x)\frac{d^2y}{dx^2}\right]-N(x)\frac{dy}{dx}\right\}\delta y\Bigg|_0^\ell-EI_1(x)\frac{d^2y}{dx^2}\delta\left(\frac{dy}{dx}\right)\Bigg|_0^\ell=0.\tag{50}$$

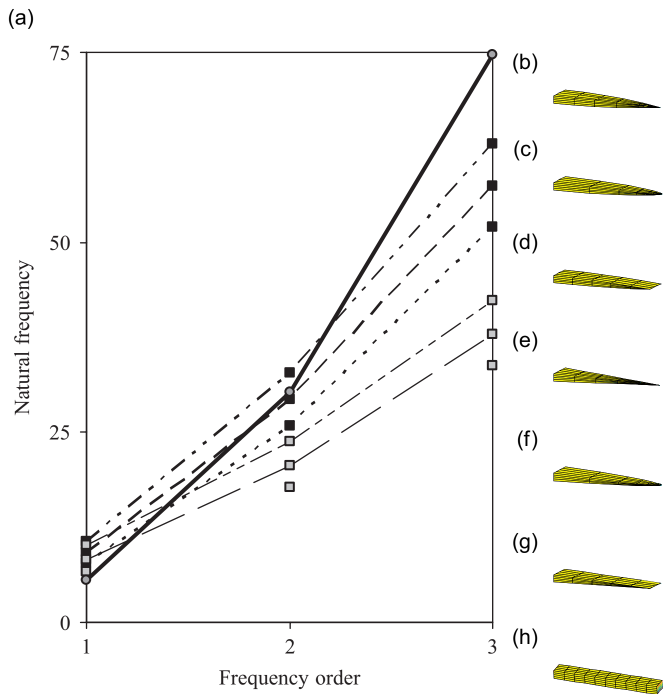

Fig. 6. (a) First three dimensionless natural frequencies $\omega\ell^2\sqrt{(\rho_0 A_0)/(EI_0)}$ from Table 6 for sliding-free (SF) boundary conditions of three Jacobi half-beams of width $a(\xi)$ given by —··— $a(\xi)=b(\xi)$, shown in (b), – – – $a(\xi)=[b(\xi)]^{1/2}$, shown in (c), - - - - - $a(\xi)=1$, shown in (d); three half-beams of linear thickness $b_C(\xi)=1-\xi$ [11] and width $a(\xi)$ given by –· ·–· ·–· ·– $a(\xi)=b_C(\xi)$, shown in (e), – – – – – – $a(\xi)=[b_C(\xi)]^{1/2}$, shown in (f), - - - - - - - $a(\xi)=1$, shown in (g); and —— uniform beam, shown in (h).

Using the variable changing given by Eq. (15) where $\ell = L/2$, Eq. (49) becomes

$$\frac{d^2}{d\xi^2}\left((1-\xi^2)^{p+2}\frac{d^2y(\xi)}{d\xi^2}\right)-\eta^2\frac{1}{2(p+1)}\frac{d}{d\xi}\left((1-\xi^2)^{p+1}\frac{dy(\xi)}{d\xi}\right)-\hat{\omega}^2(1-\xi^2)^p y(\xi)=0,\quad 0<\xi<1, \tag{51}$$

where $\hat{\omega}$ and η are dimensionless natural frequency and rotational speed parameter, respectively, given by

$$\hat{\omega}=\omega\ell^2\sqrt{\frac{\rho_0 A_0}{EI_0}}\quad\text{and}\quad \eta=\Omega\ell^2\sqrt{\frac{\rho_0 A_0}{EI_0}}. \tag{52}$$

Also, the boundary conditions given by Eq. (50) become

$$\left\{\frac{d}{d\xi}\left[(1-\xi^2)^{p+2}\frac{d^2y}{d\xi^2}\right]-\eta^2\frac{1}{2(p+1)}(1-\xi^2)^{p+1}\frac{dy}{d\xi}\right\}\delta y\Bigg|_0^1-(1-\xi^2)^{p+2}\frac{d^2y}{d\xi^2}\delta\left(\frac{dy}{d\xi}\right)\Bigg|_0^1=0. \tag{53}$$

As one can see, Eq. (51) is Eq. (12) where $r=2$, $\mu_2=\hat{\omega}^2$, and $\rho(\xi)$, $\beta(\xi)$, $\alpha(\xi)$ and the constants c_1 and c_2 are given by

$$\rho(\xi)=(1-\xi^2)^p,\ \ \beta(\xi)=(1-\xi^2),\ \ \alpha(\xi)=-2p\xi,\ \ c_1=1\ \text{and}\ c_2=-\eta^2\frac{1}{2(p+1)}. \tag{54}$$

According to Eq. (13), the eigenvalue $\hat{\omega}^2$ of the eigenvalue problem associated with Eq. (51) can be found as

$$\hat{\omega}^2=\lambda_2-\eta^2\frac{1}{2(p+1)}\lambda_1, \tag{55}$$

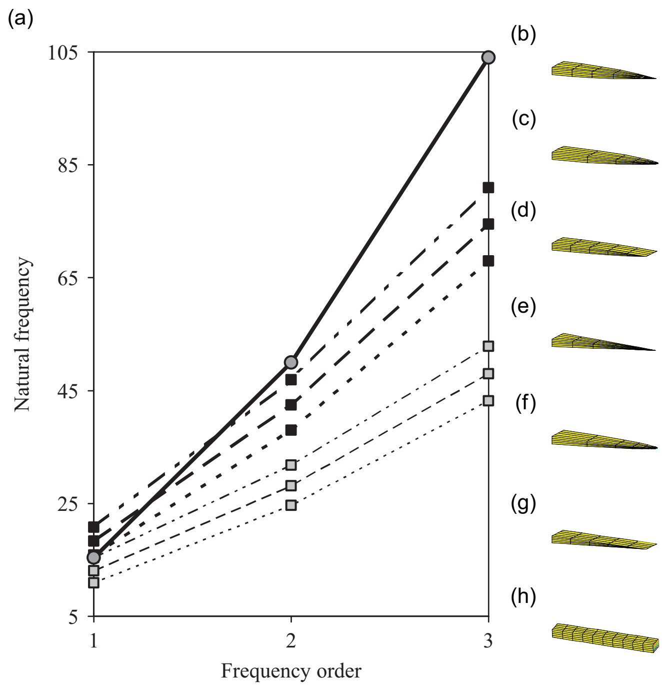

Fig. 7. (a) First three dimensionless natural frequencies $\omega\ell^2\sqrt{(\rho_0 A_0)/(EI_0)}$ from Table 6 for hinged-free (HF) boundary conditions of three Jacobi half-beams $b(\xi) = 1-\xi^2$ of width $a(\xi)$ given by —··— $a(\xi) = b(\xi)$, shown in (b), – – – $a(\xi) = [b(\xi)]^{1/2}$, shown in (c), - - - - - $a(\xi) = 1$, shown in (d); three half-beams of linear thickness $b_C(\xi) = 1-\xi$ [11] and width $a(\xi)$ given by –· ·–· ·–· ·– $a(\xi) = b_C(\xi)$, shown in (e), – – – – – $a(\xi) = [b_C(\xi)]^{1/2}$, shown in (f), - - - - - - - $a(\xi) = 1$, shown in (g); and —— uniform beam, shown in (h).

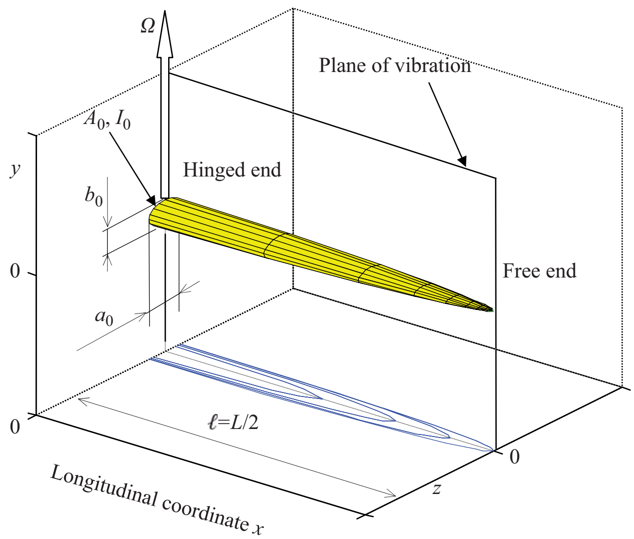

Fig. 8. Uniformly rotating Jacobi half-beam, one sharp end and thickness $b = b_0(1-\xi^2)$, of elliptic cross-section and width $a = a_0\sqrt{1-\xi^2}$.

where λ_2 and λ_1 are eigenvalues of similar eigenvalue problems associated with Eqs. (4) and (8), respectively. Boundary value problem associated with Eq. (51) gives Jacobi polynomials as exact mode shapes if the boundary conditions are HF.

Table 6
First three dimensionless natural frequencies $\hat{\omega} = \omega \ell^2 \sqrt{\rho_0 A_0 / EI_0}$ versus the rotational speed parameter $\eta = \Omega \ell^2 \sqrt{\rho_0 A_0 / EI_0}$ for hinged-free (HF) bounday conditions of Jacobi half-beams, $b(\xi) = 1 - \xi^2$, in three width $a(\xi)$ cases, and uniform beams [85]

Ref.	Width		η										
			0	1	2	3	4	5	6	7	8	9	10
Present	1	$\bar{\omega}_1$	15.88	16.02	16.43	17.10	18.00	19.09	20.35	21.74	23.24	24.83	26.50
		$\bar{\omega}_2$	37.95	38.08	38.47	39.12	40.00	41.11	42.43	43.93	45.61	47.43	49.40
		$\bar{\omega}_3$	67.97	68.10	68.48	69.12	70.00	71.12	72.46	74.01	75.76	77.70	79.81
Present	$[b(\xi)]^{1/2}$	$\bar{\omega}_1$	18.33	18.45	18.78	19.33	20.08	21.00	22.07	23.28	24.59	26.00	27.50
		$\bar{\omega}_2$	42.43	42.53	42.85	43.37	44.09	45.00	46.09	47.34	48.74	50.29	51.96
		$\bar{\omega}_3$	74.46	74.56	74.87	75.38	76.10	77.00	78.09	79.36	80.81	82.41	84.17
Present	$b(\xi)$	$\bar{\omega}_1$	20.79	20.88	21.17	21.63	22.27	23.07	24.00	25.06	26.23	27.50	28.84
		$\bar{\omega}_2$	46.90	46.99	47.26	47.70	48.31	49.08	50.00	51.07	52.28	53.62	55.08
		$\bar{\omega}_3$	80.94	81.03	81.29	81.72	82.32	83.08	84.00	85.08	86.30	87.67	89.17
[85]		$\bar{\omega}_1$	15.42	15.62	16.23	17.18	18.43	19.92	21.59	23.41	25.34	27.36	29.44
		$\bar{\omega}_2$	49.97	50.14	50.68	51.55	52.75	54.24	56.01	58.02	60.25	62.67	65.26
		$\bar{\omega}_3$	104.2	104.4	104.9	105.8	107.0	108.5	110.3	112.4	114.7	117.3	120.1

6.1. HF boundary conditions

Jacobi polynomials of odd degree $J^{p,p}_{2n+1}(\xi)$ are eigenfunctions of the eigenvalue singular problems given by Eqs. (4) and (5), and Eqs. (8) and (5). Moreover, they are odd functions. Therefore they verify both Eq. (51) and HF boundary conditions (53) since they verify Eqs. (46). Consequently, the mode shapes $\tilde{y}_n(\xi)$ and dimensionless natural frequencies $\tilde{\omega}_n$ of transverse vibrations of the uniformly rotating half-beam for the boundary value problem given by Eqs. (51) and (46) are as follows:

$$\tilde{y}_n(\xi) = J^{p,p}_{2n+1}(\xi) \quad \text{and} \quad \tilde{\omega}_n = \sqrt{\lambda_{2,2n+1} - \eta^2 \frac{1}{2(p+1)} \lambda_{1,2n+1}}, \tag{56}$$

where

$$\lambda_{2,2n+1} = 2n(2n+1)(2n+2p+2)(2n+2p+3), \quad \lambda_{1,2n+1} = -(2n+1)(2n+2p+2). \tag{57}$$

Eqs. (57) resulted from Eqs. (9) and (6) where $\alpha_1 = -2p$ and $\beta_2 = -1$ due to Eqs. (1) and (54).

Table 6 shows the influence of the rotational speed parameter η on the first three dimensionless natural frequencies of three uniformly rotating Jacobi half-beams of parabolic thickness variation. These are beams of (1) constant width, (2) width proportional to the thickness square root, and (3) width proportional to the thickness. This table also includes for comparison the case of HF uniformly rotating beam [85]. Fig. 9 presents graphically the results given in Table 6. As one can see the stiffening effect is shown. The larger the rotational speed, the larger the values of natural frequencies within the class of rotating beams. When compared to a uniform beam, the stiffening effect is somewhat reduced.

7. Axisymmetrical vibrations of Jacobi circular plates

Nonuniform plates of $\upsilon = 1/3$, Poisson ratio applicable to many materials, whose exact axisymmetrical transverse vibration mode shapes are classical Jacobi orthogonal polynomials are reported. Their natural frequencies are reported as well. This class hereafter called Jacobi plate class consists of nonuniform circular plates of parabolic thickness variation, zero thickness at zero and outer radii, and free boundary as shown afterward.

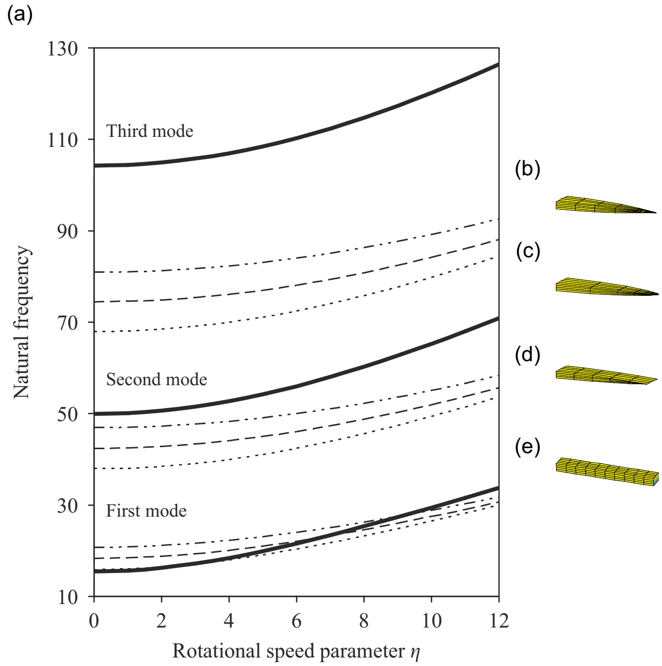

Fig. 9. (a) Rotational speed parameter $\eta = \Omega \ell^2 \sqrt{\rho_0 A_0 / EI_0}$ influence on the first three dimensionless natural frequencies $\hat{\omega} = \omega \ell^2 \sqrt{\rho_0 A_0 / EI_0}$ from Table 7, for Jacobi half-beams, $b(\xi) = 1-\xi^2$, of width $a(\xi)$ given by –· ·–· ·–· ·– $a(\xi) = b(\xi)$, shown in (b), –––––– $a(\xi) = [b(\xi)]^{1/2}$, shown in (c), ------- $a(\xi) = 1$, shown in (d); and —— uniform beam, shown in (e).

7.1. Differential equation

The class of circular plates whose differential equation of transverse vibrations can be reduced to a differential equation of orthogonal polynomials, Eq. (12), is found. Consider the free axisymmetrical vibrations of a circular nonuniform plate the flexural rigidity of which varies with the radius. Such a plate is governed by the partial differential equation [86]

$$D \frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \frac{\partial D}{\partial r}\left(\frac{\partial^2 D}{\partial r^2} + \frac{\upsilon}{r}\frac{\partial w}{\partial r}\right) = -\frac{1}{r}\int_0^r \rho_0 h \frac{\partial^2 w}{\partial t^2} r \, \mathrm{d}r, \tag{58}$$

where r, $w(r,t)$, $h(r)$ and $D(r)$ are the radius, deflection, axial thickness, and flexural rigidity, respectively. Multiplying Eq. (58) by r and then differentiating it with respect to r, the following form of this partial differential equation is found:

$$\frac{1}{hr}\frac{\partial^2}{\partial r^2}\left(rD\frac{\partial^2 w}{\partial r^2}\right) + \frac{1}{hr}\frac{\partial}{\partial r}\left[\left(-\frac{D}{r} + \upsilon\frac{\partial D}{\partial r}\right)\frac{\partial w}{\partial r}\right] = -\rho_0 \frac{\partial^2 w}{\partial t^2}, \quad 0 < r < r_1, \tag{59}$$

where r_1 is the outer radius. The flexural rigidity is given by

$$D(r) = \frac{Eh^3(r)}{12(1-\upsilon^2)}. \tag{60}$$

Then, Eq. (59) is transformed into a dimensionless differential equation in a new variable ξ (dimensionless radius) that is given by

$$\xi = \frac{r}{r_1}, \quad 0 < \xi < 1. \tag{61}$$

This variable changing leads to

$$h(r) = h_0 h(\xi), \quad D(r) = D_0 h^3(\xi), \tag{62}$$

where $D_0 = Eh_0^3/[12(1-\upsilon^2)]$, and $h(\xi)$ is the dimensionless thickness. Separating variables

$$w(t,\xi) = W(\xi)\cos(\omega t + \varphi), \tag{63}$$

where $W(\xi)$ and ω are the mode shape and natural frequency, respectively, the dimensionless form of Eq. (59) is as follows:

$$\frac{1}{\xi h(\xi)}\frac{d^2}{d\xi^2}\left[\xi h^3(\xi)\frac{d^2W}{d\xi^2}\right] + \frac{1}{\xi h(\xi)}\frac{d}{d\xi}\left\{\left[-\frac{h^3(\xi)}{\xi} + 3vh^2(\xi)\frac{dh(\xi)}{d\xi}\right]\frac{dW}{d\xi}\right\} - \bar{\omega}^2 W = 0, \tag{64}$$

where $\bar{\omega}$ is the dimensionless natural frequency given by

$$\bar{\omega} = \omega r_1^2\sqrt{\frac{\rho_0 h_0}{D_0}}. \tag{65}$$

Next, Eq. (64) is reduced to a differential equation of orthogonal polynomials. The domain of the dimensionless radius $0<\xi<1$ is transformed into the domain $(-1,1)$ of the independent variable of differential equations of Jacobi orthogonal polynomials, Eq. (12), by the following variable changing:

$$\eta = 2\xi - 1, \quad -1<\eta<1. \tag{66}$$

Therefore, Eq. (64) becomes

$$\frac{1}{(\eta+1)h(\eta)}\frac{d^2}{d\eta^2}\left[(\eta+1)h^3(\eta)\frac{d^2W}{d\eta^2}\right] + \frac{1}{(\eta+1)h(\eta)}\frac{d}{d\eta}\left\{\left[-\frac{h^3(\eta)}{(\eta+1)} + 3vh^2(\eta)\frac{dh(\eta)}{d\eta}\right]\frac{dW}{d\eta}\right\} - \frac{\bar{\omega}^2}{16}W = 0. \tag{67}$$

Differential equations (12) and (67) are identical if $r = 2$ into Eq. (12) and the following conditions are satisfied along with Eqs. (1)–(3):

$$\rho(\eta) = (\eta+1)h(\eta), \tag{68}$$

$$\rho(\eta)\beta^2(\eta) = (\eta+1)h^3(\eta), \tag{69}$$

$$\rho(\eta)\beta(\eta) = -\frac{h^3(\eta)}{(\eta+1)} + 3vh^2(\eta)\frac{dh(\eta)}{d\eta}. \tag{70}$$

Also, the dimensionless natural frequency $\bar{\omega}$ is as follows:

$$\bar{\omega} = 4\sqrt{\mu_2}, \tag{71}$$

where μ_2 is given by Eq. (13).

7.2. Boundary condition

Free boundary of circular plate satisfying Eqs. (68)–(70) and (1)–(3) leads to a boundary value problem associated with Eq. (67) that is identical with the singular value problem of orthogonal polynomials Eqs. (12) and (5). The bending moment M_r and the shear force V_r corresponding to Eq. (59) are given by Elishakoff [44] as

$$M_r = -D(r)\left[\frac{d^2w}{dr^2} + \frac{v}{r}\frac{dw}{dr}\right], \tag{72}$$

$$V_r = -D(r)\frac{d}{dr}\left[\frac{d^2w}{dr^2} + \frac{v}{r}\frac{dw}{dr}\right] - \frac{dD(r)}{dr}\left[\frac{d^2w}{dr^2} + \frac{v}{r}\frac{dw}{dr}\right]. \tag{73}$$

According to Eqs. (3) and (70), the thickness $h(\eta)$ vanishes at $\eta = -1$ and 1. So, the dimensionless thickness $h(\xi)$ vanishes at zero and outer dimensionless radii, $\xi = 0$ and 1, respectively, due to Eq. (66). Therefore, the

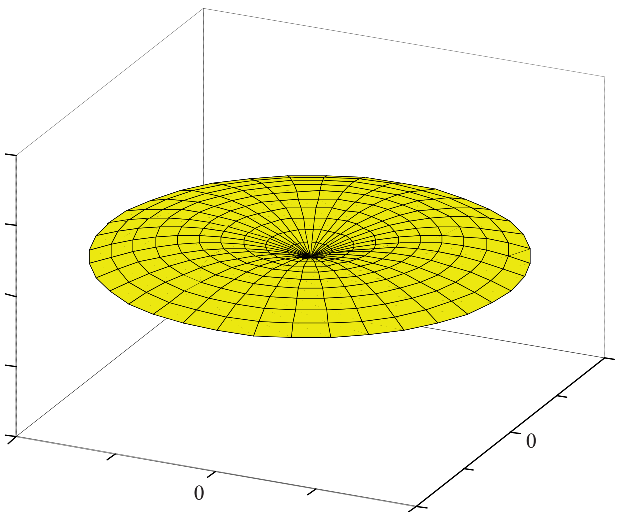

Fig. 10. Jacobi circular plate ($h = 4h_0\xi(1-\xi)$, $\upsilon = 1/3$, and free boundary).

thickness $h(r)$ vanishes at $r = 0$ and r_1, Fig. 10. This circular plate, which has zero thickness at zero and outer radii, allows only free boundary conditions for the transverse vibration boundary value problem. This is consistent with the fact that this plate cannot sustain any outer boundary moment or shear force. The finite displacement of the center of the plate, and the free outer boundary consisting of zero bending moment M_r and shear force V_r, are given by

$$w(t,0) \text{ finite and } M_r(t,r_1) = V_r(t,r_1) = 0. \tag{74}$$

Since the thickness vanishes at the outer radius $r = r_1$, and the flexural rigidity $D(r)$ is proportional to the third power of thickness, Eq. (60), both the flexural rigidity $D(r)$ and its derivative $\mathrm{d}D(r)/\mathrm{d}r$ also vanish at the outer radius

$$D(r_1) = \frac{\mathrm{d}D}{\mathrm{d}r}(r_1) = 0. \tag{75}$$

Therefore the bending moment M_r and shear force V_r, given by Eqs. (72) and (73), satisfy the free outer boundary given by Eq. (74) as long as the displacement $w(t,r_1)$ is finite. Consequently, the boundary conditions (74) reduce to

$$w(t,0) \text{ and } w(t,r_1) \text{ finite.} \tag{76}$$

7.3. Jacobi circular plates

The class of plates of Poisson ratio $\upsilon = 1/3$ whose mode shapes are Jacobi polynomials is found. As mentioned above the differential equation of transverse vibrations of circular plates Eq. (58) can be reduced to an equation, Eq. (67), identical with a differential equation of orthogonal polynomials, Eq. (12), if the conditions given by Eqs. (68)–(70) and (1)–(3) are met. These conditions along with Poisson ratio $\upsilon = 1/3$ lead to this class of plates. This class consists of plates of thickness $h(r)$ and flexural rigidity $D(r)$ given by

$$h(r) = \frac{4h_0}{r_1^2} r(r_1 - r), \quad D(r) = \frac{Eh^3(r)}{12(1-\upsilon^2)}, \quad 0 < r < r_1, \tag{77}$$

where r_1 and h_0 are the outer radius and the maximum thickness that occurs at $r = r_1/2$, respectively. So, the boundary value problem of transverse vibrations of circular plates can be reduced to an eigenvalue singular problem of Jacobi orthogonal polynomials as long as the plates are of parabolic thickness variation Eq. (77) and free boundary, Eq. (76), Fig. 10. The thickness, the flexural rigidity, and the boundary value problem written in terms of the dimensionless radius ξ, given by Eq. (61), are as follows:

$$h = 4h_0\xi(1-\xi), \quad D = 64D_0\xi^3(1-\xi)^3, \tag{78}$$

$$\frac{1}{\xi^2(1-\xi)}\frac{d^2}{d\xi^2}\left[\xi^4(1-\xi)^3\frac{d^2W}{d\xi^2}\right] - \frac{1}{\xi^2(1-\xi)}\frac{d}{d\xi}\left[\xi^3(1-\xi)^2\frac{dW}{d\xi}\right] - \frac{\bar{\omega}^2}{16}W = 0, \tag{79}$$

$$W(0) \text{ and } W(1) \text{ finite.} \tag{80}$$

Eqs. (78)–(80) become as follows when using the variable changing given by Eq. (66)

$$h = h_0(1-\eta^2), \quad D = D_0(1-\eta^2)^3, \tag{81}$$

$$\frac{1}{(1+\eta)^2(1-\eta)}\frac{d^2}{d\eta^2}\left[(1+\eta)^4(1-\eta)^3\frac{d^2W}{d\eta^2}\right] - \frac{1}{(1+\eta)^2(1-\eta)}\frac{d}{d\eta}\left[(1+\eta)^3(1-\eta)^2\frac{dW}{d\eta}\right] - \frac{\bar{\omega}^2}{16}W = 0, \tag{82}$$

$$W(-1) \text{ and } W(1) \text{ finite.} \tag{83}$$

7.4. Natural frequencies and mode shapes

The eigenvalue problem given by Eqs. (82) and (83) is identical with an eigenvalue singular problem of orthogonal polynomials given by Eqs. (12) and (5), where the last value of the summation index of Eq. (12) is $r = 2$, the coefficients c_i are $c_1 = 1$ and $c_2 = -1$, and the functions $\rho(\eta)$, $\beta(\eta)$ and $\alpha(\eta)$ as resulting from Eqs. (81), (68), (69) and (2) are as follows:

$$\rho(\eta) = (1-\eta)(1+\eta)^2, \quad \beta(\eta) = 1-\eta^2, \quad \alpha(\eta) = 1-3\eta. \tag{84}$$

The function $\rho(\eta)$ is the weight of Jacobi polynomials $J_n^{1,2}(\eta)$ [81,82]. From Eqs. (1) and (84) the leading coefficients of the polynomials $\alpha(\eta)$ and $\beta(\eta)$ are $\alpha_1 = -3$ and $\beta_2 = -1$. According to Caruntu [2], the eigenfunctions $W_n(\eta)$ of the eigenvalue singular problem given by Eqs. (82) and (83) are Jacobi orthogonal polynomials $J_{n+1}^{1,2}(\eta)$, and the eigenvalues are $\mu_{2,n+1} = \lambda_{2,n+1} - \lambda_{1,n+1}$. Therefore, the boundary value problem of the free transverse axisymmetrical vibration of the above nonuniform plate, given by Eqs. (79) and (80), has the following mode shapes $W_n(\xi)$ and dimensionless natural frequencies $\bar{\omega}_n$:

$$W_n(\xi) = J_{n+1}^{1,2}(2\xi-1), \quad \bar{\omega}_n = 4\sqrt{\lambda_{2,n+1} - \lambda_{1,n+1}}, \tag{85}$$

where

$$\lambda_{2,n+1} = n(n+1)(n+5)(n+6), \quad \lambda_{1,n+1} = -(n+1)(n+5). \tag{86}$$

Table 7 shows the first ten dimensionless natural frequencies $\bar{\omega} = \omega r_1^2\sqrt{\rho_0 h_0/D_0}$ of the Jacobi circular plate (parabolic thickness variation and free boundary) and uniform circular plates. Fig. 11 shows the first five natural frequencies presented in Table 7. Fig. 12 shows the first three mode shapes of the Jacobi circular plates.

Table 7
First ten dimensionless natural frequencies $\bar{\omega} = \omega r_1^2\sqrt{\rho_0 h_0/D_0}$ for free boundary and $\upsilon = 1/3$ of Jacobi circular plate and uniform circular plate [55]

Ref.	Thickness	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$	$\bar{\omega}_7$	$\bar{\omega}_8$	$\bar{\omega}_9$	$\bar{\omega}_{10}$
Present	Parabolic	39.19	75.58	119.7	171.8	231.9	299.9	375.9	459.9	551.9	652.0
[55]	Uniform	9.07	38.51	87.82	156.9	245.8	354.5	483.1	631.9	801.0	—

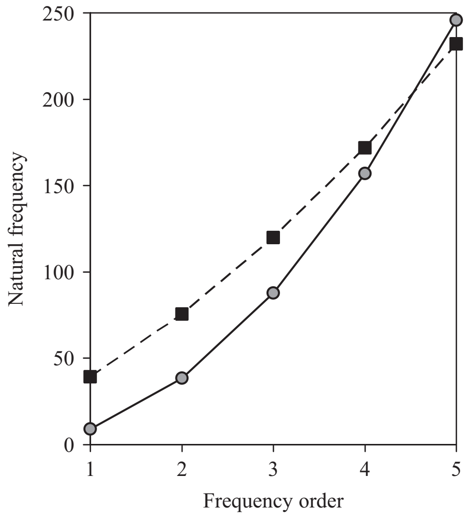

Fig. 11. First five dimensionless natural frequencies $\bar{\omega} = \omega r_1^2 \sqrt{\rho_0 h_0 / D_0}$ of ----- Jacobi circular plate, and ——— uniform plate, from Table 7.

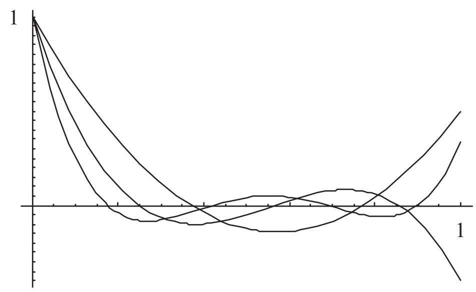

Fig. 12. First three mode shapes of transverse vibration of the Jacobi circular plate.

8. Discussion and conclusions

Studies dedicated to the mass distribution influence to the response of the structure became very important. The trend of present-day structures is to be lightweight and to avoid resonant frequencies. A review of the literature reveals that this is a difficult task, although several investigators focused on the transverse vibration analysis of nonuniform beams and plates. They used either analytical or approximate methods. Few of them reported closed-form analytical solutions either in terms of Bessel functions, hypergeometric series or power series by Frobenius method. To the best of our knowledge, classical orthogonal polynomials were not reported in the literature to be exact mode shapes of transverse vibrations of nonuniform beams and plates until now, except the paper of Caruntu [1] that presented a case of nonuniform beam of circular cross-section. Recent developments of the orthogonal polynomials' theory [1–3], allowed for this study of bending vibration of nonuniform beams and plates.

This paper studied free transverse vibrations in one principal plane of nonuniform beams and free transverse axisymmetrical vibrations of nonuniform plates, developed on the Euler–Bernoulli hypothesis and classical plate theory respectively, using an approach in which vibration boundary value problems were reduced to eigenvalue singular problems of orthogonal polynomials [1,2]. This approach led to finding (1) classes of beams and plates that have Jacobi polynomials as closed-form solutions of the mode shape equation, (2) boundary conditions that correspond to the eigenvalue singular problem of orthogonal polynomials, and (3) natural frequencies of these boundary value problems. Specific to these classes was that the equation of motion is a linear differential equation with two regular singularities, and consequently certain boundary conditions were required. Both the class of nonuniform beams and the class of circular nonuniform plates, found and reported, are characterized by geometry and boundary conditions.

The beam class geometry consisted of convex parabolic thickness variation $b(\xi) = 1 - \xi^2$ and polynomial width variation $a(\xi) = (1-\xi)^{p-1}(1+\xi)^{q-1}$ with the dimensionless longitudinal coordinate ξ, where p and q are real parameters greater than or equal to 1. Both, rectangular and elliptic cross-sections were considered. Four boundary value problems of transverse vibrations of beams belonging to this class have been reported: (1) Jacobi beams (sharp at either end) with free-free boundary conditions, (2) Jacobi half-beams, i.e. halves of symmetric Jacobi beams, with the large end sliding and sharp end free (SF), (3) Jacobi half-beams with the large end hinged and sharp end free (HF), and (4) uniformly rotating Jacobi half-beams with (HF).

The circular plate class geometry consisted of convex parabolic thickness variation $h(\xi) = \xi(1-\xi)$, where ξ is the current dimensionless radius.

One boundary value problem has been considered, namely Jacobi plate (zero thickness at zero and outer radii) with free-free boundary conditions.

The results presented in this paper are in agreement with data obtained using approximate methods and data reported in the literature. Transverse vibrations of a Jacobi beam has been studied using the Galerkin method also. An agreement between results obtained by using the method described in the paper and this approximate method has been found.

The dimensionless natural frequencies of three Jacobi beams are compared to those of the uniform beam, Fig. 4. The values of lower order natural frequencies of these Jacobi beams were found to be greater than those of the uniform beam. Also, the values of higher order natural frequencies were found to be less than those of the uniform beam. This is in agreement with data reported in the literature regarding the natural frequencies of the free–free transverse vibrations of some double tapered complete-beams [11].

The particular case of a Jacobi beam with an elliptical cross-section, with both major and minor diameters varying parabolically with the axial coordinate, reduces to a beam with a circular cross-section if its diameters are equal. The mode shapes and natural frequencies of this beam, reported in this paper, are the same as those reported by Caruntu [1]. When comparing the natural frequencies of a half-beam that is uniformly rotating with the same non-rotating beam, an increase in natural frequencies was found for the rotating beam, see Eqs. (41) and (48), and (42) and (50), for (HF) boundary conditions case. This is in agreement with Du et al. [85]. They reported an increased stiffness of the beam due to rotation.

This paper is relevant in a few aspects. First, it reports the exact mode shapes and the natural frequencies of a large family of nonuniform beams and circular nonuniform plates for certain boundary conditions. The classes of beams and circular plates, whose boundary value problems lead to Jacobi polynomials as mode shapes, are found, and moreover natural frequencies are reported. The advantages of reporting expressions for natural frequencies and mode shapes, which are Jacobi polynomials, consist of great ease of numerical calculations and parametric studies. Second, results reported in this paper could be used for studying forced and nonlinear vibrations of these classes of beams and circular plates. Third, the Jacobi polynomials reported here as exact mode shapes of the presented boundary value problems can also be used as admissible functions for approximate methods.

The results presented here allow for finding natural frequencies and mode shapes of beams and circular plates with similar geometry, and boundary conditions like those given in the paper. Fourth, if used along with Kantorovich method, the approach presented in this paper can be employed to find the mode shapes and the natural frequencies of transverse vibration of certain rectangular plates with certain boundary conditions. Fifth, the results of this paper can serve as test cases for the development of computational methods, and not just as a source of basic design for situations which happen to fall within the geometry and boundary conditions presented here.

The results of this paper are limited to Euler–Bernoulli beams and classical circular plates, i.e. they are not accurate in the case of short beams and/or thick plates.

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