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Bifurcations of a generalized van der Pol oscillator with strong nonlinearity

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Abstract

A generalized van der Pol oscillator with parametric excitation is studied for its bifurcation. On the basis of the MLP method, we enable a strongly nonlinear system to be transformed into a small parameter system. The bifurcation response equation of a 1/2 subharmonic resonance system is determined by the multiple scales method. According to the singularity theory, the bifurcation of equilibrium points is analyzed. The stability of the zero solution is researched by the eigenvalues of the variational matrix and the bifurcation sets are constructed in various regions of the parameter plane.

1. Introduction

Some problems of nonlinear oscillator require further study, and attention has mainly focused on bifurcation and chaos, etc. Parametrically excited nonlinear oscillators are widely studied for their qualitative changes using the bifurcation theory. Bajaj [1] studied a parametrically excited oscillator which includes van der Pol as well as Duffing type nonlinearity on the basis of the bifurcation theory. Zhang and Huo [2] considered the bifurcation in a nonlinear oscillator under combined parametric and forcing excitation and obtained the transition sets and bifurcation diagrams. These analyses are limited to weakly nonlinear systems [3]. To extend the range of application of the bifurcation theories to strongly nonlinear systems is the desire of researchers.

In this paper, we consider the bifurcations of a strongly nonlinear oscillator which is a more general Mathieu–van der Pol system. On the basis of the MLP method presented by Cheung et al. [4], we enable a strongly nonlinear system to be transformed into a small parameter system [8,9]. To study the 1/2 subharmonic resonance case, we define a new transformation parameter $\alpha = \alpha(\varepsilon, 2\omega_0, \omega_1)$. Then, the bifurcation response equation is determined by the multiple scales method [10,11]. From the singularity theory, the constant solutions are obtained and the bifurcation of equilibrium points is understood. Stability analysis of the singular points gives the bifurcation sets in various regions of the parameter plane.

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2. Parameter transformation

We consider the strongly nonlinear oscillator

$$\ddot{x} + x + (\bar{\mu} + \bar{\gamma}x^2 + \bar{\beta}\dot{x}^2)\dot{x} + \bar{v}\cos 2tx = 0. \tag{1}$$

It can be written as

$$\ddot{x} + x + \varepsilon(\mu + \gamma x^2 + \beta\dot{x}^2)\dot{x} + 2\varepsilon\cos 2tx = 0, \tag{2}$$

where ε is not a small parameter. A new variable

$$\tau = \omega t \tag{3}$$

is introduced. The parametric excitation is near 1/2 subhamonic resonance, $\omega = 2$. Substituting Eq. (3) into Eq. (2) yields

$$\omega^2 x'' + x + \varepsilon(\mu + \gamma x^2 + \beta\omega^2 x'^2)\omega x' + 2\varepsilon\cos\tau x = 0, \tag{4}$$

where $x' = \mathrm{d}x/\mathrm{d}\tau$, $x'' = \mathrm{d}^2x/\mathrm{d}\tau^2$. A new parameter is defined

$$\alpha = \frac{\varepsilon\omega_1}{(2\omega_0)^2 + \varepsilon\omega_1} = \frac{\varepsilon\omega_1}{4 + \varepsilon\omega_1} \tag{5}$$

such that

$$\varepsilon = \frac{4\alpha}{\omega_1(1-\alpha)}$$

and

$$\omega^2 = 2^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots = \frac{4}{1-\alpha}(1 + \eta_2\alpha^2 + \eta_3\alpha^3 + \cdots),$$
$$\omega = 2\left[1 + \frac{1}{2}\alpha + \left(\frac{3}{8} + \frac{\eta_2}{2}\right)\alpha^2 + \cdots\right]. \tag{6}$$

Applying the mathematical operation [9], we can obtain

$$\omega_1 = \frac{4(a_0^2 - b_0^2)}{a_0^2 + b_0^2}, \tag{7}$$

where a_0 is a initial condition of Eq. (1), $x(\mathrm{o}) = a_0$, and b_0 can be determined as

$$8a_0b_0 + 4\mu(a_0^2 + b_0^2) + (\gamma + 3\beta)(a_0^2 + b_0^2)^2 = 0. \tag{8}$$

Thus, ω_1 is known which is determined in Eqs. (7) and (8). This idea comes from the result of the MLP method [4] and is suitable for other problems [5–7].

3. Bifurcation response equation and analysis

The new parameter α will enable a strongly nonlinear system corresponding to ε be transformed into a small parameter system with respect to α. We have

$$(1 + \eta_2\alpha^2 + \cdots)x'' + \frac{1-\alpha}{4}x + \frac{2\alpha}{\omega_1}[\mu + \gamma x^2 + 4\beta(1 + \alpha + \alpha^2 + \cdots)$$
$$\times(1 + \eta_2\alpha^2 + \cdots)x'^2]\left[1 + \frac{1}{2}\alpha + \left(\frac{3}{8} + \frac{\eta_2}{2}\right)\alpha^2 + \cdots\right]x' + \frac{2\alpha}{\omega_1}\cos\tau x = 0. \tag{9}$$

To study Eq. (9) for small α, we use the multiple scales method. Let x be expanded in powers of α, i.e.,

$$x = x_0(T_0, T_1) + \alpha x_1(T_0, T_1) + \cdots, \tag{10}$$

where $T_0 = \tau$ and $T_1 = \alpha\tau$. The differential operators are given by

$$\frac{d}{d\tau} = D_0 + \alpha D_1 + \cdots \frac{d^2}{d\tau^2} = D_0^2 + 2\alpha D_0 D_1 + \alpha^2(D_1^2 + 2D_0 D_2) + \cdots. \tag{11}$$

Perturbation equations in this case are

$$D_0^2 x_0 + \tfrac{1}{4} x_0 = 0, \tag{12}$$

$$D_0^2 x_1 + \frac{1}{4} x_1 = \frac{1}{4} x_0 - 2D_0 D_1 x_0 - \frac{2}{\omega_1}[\mu + \gamma x_0^2 + 4\beta(D_0 x_0)^2] D_0 x_0 - \frac{2}{\omega_1} \cos T_0 x_0. \tag{13}$$

The solution of Eq. (12) is

$$x_0 = A\, e^{iT_0/2} + \bar{A}\, e^{-iT_0/2}. \tag{14}$$

Substituting Eq. (14) into Eq. (13) and eliminating the secular terms, we get

$$D_1 A = -\frac{1}{4} iA - \frac{\mu}{\omega_1} A - \frac{1}{\omega_1}(\gamma + 3\beta) A^2 \bar{A} + i\frac{\bar{A}}{\omega_1}. \tag{15}$$

Let

$$A = a\, e^{i\phi}. \tag{16}$$

Eq. (15) transforms to

$$a' = -\frac{\mu}{\omega_1} a - \frac{1}{\omega_1} \delta a^3 + \frac{1}{\omega_1} a \sin 2\phi,$$

$$a\phi' = -\frac{1}{4} a + \frac{1}{\omega_1} a \cos 2\phi, \tag{17}$$

where

$$\delta = \gamma + 3\beta. \tag{18}$$

At the singular points $a' = \phi' = 0$. Eliminating ϕ from Eq. (17), we get the bifurcation response equation

$$\delta^2 a^4 + 2\mu\delta a^2 + \mu^2 + \frac{\omega_1^2}{16} - 1 = 0, \quad a = 0. \tag{19}$$

The possible solutions of Eq. (19) are

(a) $a_1 = 0$, and

$$a_2 = \left[-\frac{\mu}{\delta} + \frac{1}{\delta} \left(1 - \frac{\omega_1^2}{16} \right)^{1/2} \right]^{1/2} \text{ for } |\omega_1| < 4, \quad \mu^2 + \frac{1}{16}\omega_1^2 < 1 \text{ and } \delta > 0. \tag{20}$$

(b) $a_1 = 0$, and

$$a_2 = \left[-\frac{\mu}{\delta} - \frac{1}{\delta} \left(1 - \frac{\omega_1^2}{16} \right)^{1/2} \right]^{1/2} \text{ for} |\omega_1| < 4, \quad \mu^2 + \frac{1}{16}\omega_1^2 < 1 \text{ and } \delta < 0. \tag{21}$$

(c) $a_1 = 0$, and

$$a_2 = \left[-\frac{\mu}{\delta} + \frac{1}{\delta} \left(1 - \frac{\omega_1^2}{16} \right)^{1/2} \right]^{1/2}, \tag{22}$$

$$a_3 = \left[-\frac{\mu}{\delta} - \frac{1}{\delta}\left(1 - \frac{\omega_1^2}{16}\right)^{1/2}\right]^{1/2} \quad \text{for} |\omega_1| < 4, \quad \mu^2 + \frac{1}{16}\omega_1^2 < 1 \text{ and } \mu\delta < 0. \tag{23}$$

(d) $a_1 = 0$, and

$$a_2 = \sqrt{\frac{-\mu}{\delta}} \quad \text{for} |\omega_1| = 4 \text{ and } \mu\delta < 0. \tag{24}$$

(e) $a_1 = 0$, and

$$a_2 = \left[\delta^2\left(1 - \frac{\omega_1^2}{16}\right)\right]^{1/4} \quad \text{for} |\omega_1| < 4, \quad \mu = 0 \text{ and } \delta \neq 0. \tag{25}$$

(f) $a_1 = 0$ everywhere else in (ω_1, μ, δ) space.

One or two non-trivial solutions of Eq. (19) depend on the parameters ω_1, μ and δ. The origin $a = 0$ is always a singular point. We divide (ω_1, μ) plane into 12 regions. The relation of a–μ, a–δ, a–ω_1 is shown in Fig. 1. The curve a–δ in the fixed parameters μ and ω_1 are obtained numerically. Two curve shapes are not same, but to turn inside out is same. The curve a–μ in the fixed parameters δ and ω_1 are similar to the curve a–δ. The curve a–ω_1 in the fixed parameters δ and μ is four and more complex.

4. Stability of the zero solution

The stability of the constant solutions is determined by the eigenvalues. Let

$$A = u + iv, \tag{26}$$

where $u = a\cos\phi$ and $v = a\sin\phi$. Eq. (15) transform to

$$\begin{aligned} u' &= -\frac{1}{\omega_1}\mu u + \left(\frac{1}{4} + \frac{1}{\omega_1}\right)v - \frac{1}{\omega_1}\delta u(u^2 + v^2), \\ v' &= \left(-\frac{1}{4} + \frac{1}{\omega_1}\right)u - \frac{1}{\omega_1}\mu v - \frac{1}{\omega_1}\delta v(u^2 + v^2). \end{aligned} \tag{27}$$

The Jacobi matrix of Eq. (27) is

$$D_w f = \begin{bmatrix} -\frac{\mu}{\omega_1} - \frac{1}{\omega_1}\delta(3u^2 + v^2) & \frac{1}{4} + \frac{1}{\omega_1} - \frac{2}{\omega_1}\delta uv \\ -\frac{1}{4} + \frac{1}{\omega_1} - \frac{2}{\omega_1}\delta uv & -\frac{\mu}{\omega_1} - \frac{1}{\omega_1}\delta(u^2 + 3v^2) \end{bmatrix}, \tag{28}$$

where

$$w = (u, v)^{\mathrm{T}}, \quad f = \begin{bmatrix} -\frac{1}{\omega_1}\mu u + (\frac{1}{4} + \frac{1}{\omega_1})v - \frac{1}{\omega_1}\delta u(u^2 + v^2) \\ (-\frac{1}{4} + \frac{1}{\omega_1})u - \frac{1}{\omega_1}\mu v - \frac{1}{\omega_1}\delta v(u^2 + v^2) \end{bmatrix}.$$

For the zero solution, the Jacobi matrix is

$$D_w f\big|_{a=0} = \begin{bmatrix} -\frac{\mu}{\omega_1} & \frac{1}{4} + \frac{1}{\omega_1} \\ -\frac{1}{4} + \frac{1}{\omega_1} & -\frac{\mu}{\omega_1} \end{bmatrix}. \tag{29}$$

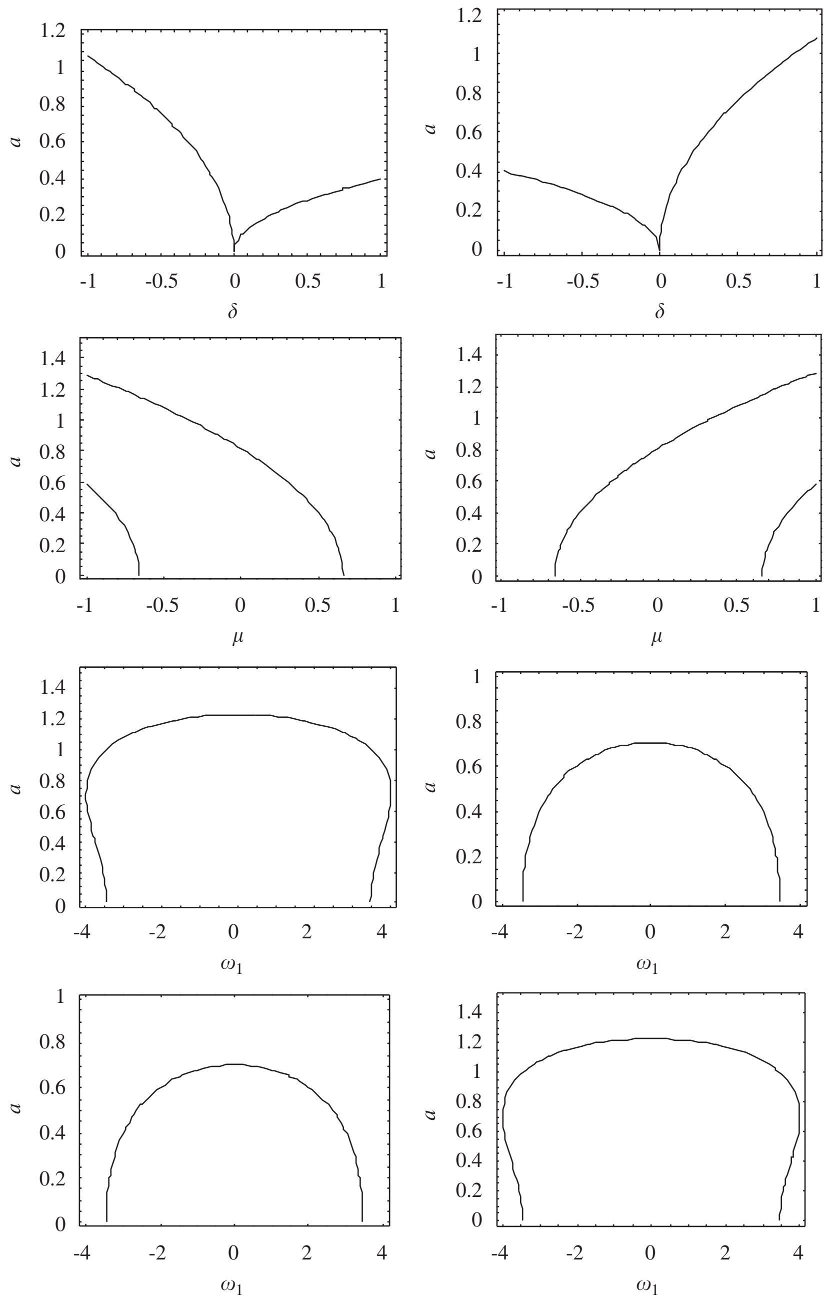

Fig. 1. The relation of a and parameters μ, δ, ω_1.

The characteristic equation for the zero solution can be shown to be

$$\lambda^2 + \frac{2\mu}{\omega_1}\lambda + \frac{1}{16} + \frac{\mu^2}{\omega_1^2} - \frac{1}{\omega_1^2} = 0. \tag{30}$$

The characteristic equation for a non-zero solution is

$$\lambda^2 + \frac{2}{\omega_1}(\mu + 2\delta a^2)\lambda - \frac{4}{\omega_1^2}\mu\delta a^2 - 4\left(\frac{1}{16} + \frac{\mu^2}{\omega_1^2} - \frac{1}{\omega_1^2}\right) = 0. \tag{31}$$

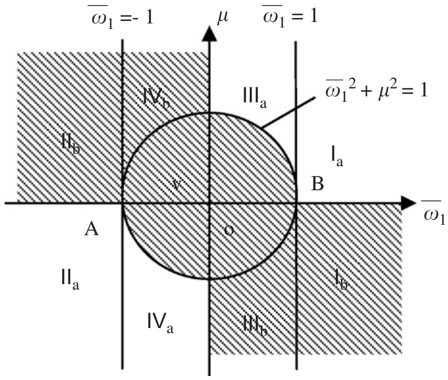

Fig. 2. The bifurcation set in $(\overline{\omega_1}, \mu)$ plane.

Eq. (27) is a normal form of codimension one, therefore, the bifurcation of codimension one may be yielded in the system (1). We discuss the stability of the zero solution. Let $\overline{\omega_1} = \omega_1/4$, the equations

$$\frac{1}{\omega_1^2} - \frac{1}{16} = 0, \quad \frac{1}{16} + \frac{\mu^2}{\omega_1^2} - \frac{1}{\omega_1^2} = 0$$

become, respectively,

$$\overline{\omega_1} = 1, \ \overline{\omega_1} = -1 \text{ and } \overline{\omega_1}^2 + \mu^2 = 1.$$

The lines $\overline{\omega_1} = 1$ and $= -1$ the axes $\overline{\omega_1}$ and μ and the circle $\overline{\omega_1}^2 + \mu^2 = 1$ divide $(\overline{\omega_1}, \mu)$ plane into 12 regions. The bifurcation set of the singular point is shown in Fig. 2. The zero solution is unstable in the shaded regions.

(a) In region I_a, I_b, II_a, and II_b the singular point is the focus point. Moving from I_a (II_a) across the axis $\overline{\omega_1}$ into region II_b (II_b) the zero solution loses the stability.
(b) In region III_a, III_b, IV_a, and IV_b the origin is the node point. It is a sink type and stable in region III_a and IV_a. Moving from III_a (IV_a) across the axis μ into region III_b (IV_b) the zero solution loses the stability.
(c) Across the circle $\overline{\omega_1}^2 + \mu^2 = 1$ the singular point becomes a saddle type. It is source type and unstable in region V. The oscillator exhibits a pitchfork bifurcation.
(d) Along the circle the singular point is a degeneracy case, the oscillator is structurally unstable.
(e) Along the $\overline{\omega_1}$ $(-\infty < \overline{\omega_1} < -1)$ or $(1 < \overline{\omega_1} < \infty)$ both eigenvalues are a pair of pure imaginary numbers, therefore, we expect Hopf bifurcations of the origin into a limit cycle.
(f) At points A and B two eigenvalues are all equal to zero, the system exhibits the bifurcation of codimension two. The degenerate bifurcation of codimension two will be studied in detail in the next paper.

On the basis of Eq. (31) the stability of the non-zero solutions is similarly analyzed.

5. Conclusions

A strongly nonlinear with parameter excitation has been transformed into a small parameter system on the basis of the MLP method presented by Chueng et al. and its bifurcation response equation were determined by using the multiple scale method. The possible solution of this oscillator has been discussed and the number of fixed points has been analyzed. Stability analysis of the singular points has given the bifurcation set using normal form theory. These techniques are applicable to other oscillators with strong nonlinearity.

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References

[1] A.K. Bajaj, Bifurcations in a parametrically excited nonlinear oscillator, *International Journal of Non-Linear Mechanics* 22 (1987) 47–59.
[2] W. Zhang, Q.Z. Huo, Bifurcations of nonlinear oscillation system under combined parametric and forcing excitation, *Acta Mechanica Sinica* 23 (1991) 464–474.
[3] P. Yu, K. Huseyin, Parametrically excited nonlinear systems: a comparison of certain methods, *International Journal of Non-Linear Mechanics* 33 (1998) 967–978.
[4] Y.K. Cheung, S.H. Chen, S.L. Lau, A modified Lindstedt–Poincaré method for certain strongly nonlinear oscillators, *International Journal of Non-Linear Mechanics* 26 (1991) 367–378.
[5] J.S. Tang, W.B. Fu, K.A. Li, Bifurcations of a parametrically excited oscillator with strong nonlinearity, *Chinese Physics* 11 (2002) 1004–1007.
[6] Z.H. Wang, H.Y. Hu, Robust Hurwitz stability test for linear systems with uncertain commensurate time delays, *IEEE Transactions on Automatic Control* 49 (2004) 1389–1393.
[7] Q. Zheng, P. Yue, L.X. Gong, New exact solutions of nonlinear Klein–Gordon equation, *Chinese Physics* 15 (2006) 35–38.
[8] J.S. Tang, A method for parameter identification of strongly nonlinear systems, *Journal of Sound and Vibration* 232 (2000) 993–996.
[9] J.S. Tang, C.Z. Qian, The asymptotic solution of the strongly nonlinear Klein–Gordon equation, *Journal of Sound and Vibration* 268 (2003) 1036–1040.
[10] C.Z. Qian, J.S. Tang, Asymptotic solution for a kind of boundary layer problem, *Nonlinear Dynamics* 45 (2006) 15–24.
[11] J.S. Tang, Z.L. Chen, Amplitude control of limit cycle in van der Pol system, *International Journal of Bifurcation and Chaos* 16 (2006) 487–495.