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Source effects on attenuation in lined ducts. Part II: Statistical properties

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Abstract

Statistical models for acoustic attenuation in non-uniform ducts with mean flow are developed. With source descriptions based on random distributions of input modal powers and modal phase, numerical experimentation has revealed that statistical metrics of lining performance can be generated. These include probability density functions for transmitted acoustic power and attenuation, expected (mean) attenuation, and standard deviation from the mean. An important result is that for a broadband source represented by many acoustic modes, the transmitted power appears to be described by a Gaussian distribution. In the present investigation it is shown that based on arguments involving the duct transmission model and the Central Limit Theorem, the Gaussian distribution of transmitted power observed by numerical experiment is expected. For all cases, including tonal noise represented with relatively few modes, it is shown that statistical characteristics can be described by common probability density functions and conclusions about mean attenuation, deviation from the mean, and cumulative distributions are drawn. The statistical approach described has application for design of acoustic treatment in cases where knowledge of the details of the source is minimal or non-existent.

1. Introduction

In a companion paper [1], a fundamental study of the effect of the source model on realizable attenuation in an acoustically treated duct was reported. A finite element propagation model was used in a series of numerical experiments to determine transmitted acoustic power and attenuation with several statistical source models involving random descriptions of acoustic power in propagating acoustic modes and random description of the phase of these modes. It was found that with a large number of trials good estimates of the probability distribution functions could be generated yielding stable estimates of the expected (mean) attenuation and standard deviation. In the case of random acoustic power in propagating modes, with fixed phase, an apparently normal (Gaussian) distribution of transmitted power resulted, even in the case when only a modest number of acoustic modes were present, as in tonal noise. When only phase in each mode is random, with power in each mode fixed, a non-Gaussian distribution of transmitted power was found. For the case of a

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modest number of propagating modes, the case of simultaneous random power and phase yielded transmitted power that was not normally distributed. However, in all cases when a large number of modes were considered, as in the model for broadband noise, the distribution of transmitted power appeared to be normal. The study reported here is an extension of the work reported in Ref. [1]. Fundamental properties of the theoretical model for acoustic propagation in non-uniform ducts with compressible mean flow are used in conjunction with the Central Limit Theorem to argue that in the broadband model with essentially all propagating modes present, the distribution of the transmitted acoustic power should tend to be Gaussian. The analysis also explains why in the case of only a modest number of propagating modes, as in tonal noise, a random description of modal power with fixed phase yields a Gaussian distribution, while a model with equal modal power and random phase does not. In the course of the development other characteristics of the statistical distributions for transmitted power and attenuation are considered, such as relation to common non-Gaussian probability density functions and cumulative distributions.

Few studies have focused on the determination of the random noise field in ducts. In a seminal paper [2], Dyer develops expressions for the total mean square pressure and power in a frequency band as a function of the noise spectrum for circular ducts with no flow. He proved that for the purpose of calculating statistically averaged random power in hard wall sections of the duct, the energy can be assumed to be equally divided in all cut-on modes in the limit of high frequencies. His conclusion can be interpreted as an extension of the equipartition of energy principle to duct acoustics and can be applied to estimate the modal distribution. In a paper treating source distribution inside jet ejector-suppressors, Mariano [3] compares attenuation for several uniform, non-uniform and even randomly weighted radial source configurations of monopoles and also dipoles and quadrupoles with randomly selected phase angles. The equivalent of that for fan noise would be a weighted power distribution for radial modes with various phases.

In the present study, a finite element propagation code has been used to evaluate performance of acoustic treatment. The model includes the geometry of non-uniform axi-symmetric ducts with compressible, irrotational mean flow field. The duct has a hard wall source section, a lined section, and a hard wall exit section with a reflection free termination [1]. The source is modeled as a superposition of acoustic modes and the reflection free termination is obtained with a similar superposition of outward directed acoustic modes. If the incident acoustic modes are known, the complex modal pressure coefficients can be entered as input. In case there is no dependable information about the source characteristics, the modal power and/or the phase are chosen randomly. Additional assumptions which skew the power distribution toward high or low cut-off ratio modes also can be introduced. Reflected and transmitted mode amplitudes are calculated for specified input modal amplitudes. Incident, reflected and transmitted acoustic power, determined according to Morfey [4], from potential mode amplitudes at the source and termination planes, are used to calculate attenuation. Of fundamental importance to the present development is the introduction in the theoretical and numerical model of the transmission matrix, which is a linear transformation between incident modal amplitudes and transmitted modal amplitudes. This model has been used in numerical experiments to generate probability density functions and other statistical properties for transmitted power and attenuation [1].

In this paper, numerically generated statistical properties are explained and verified with an analysis based on the transmission matrix and properties of sums of randomly distributed variables.

2. Random magnitude and phase effect on attenuation

In this investigation the duct is non-uniform with compressible irrotational mean flow. A representative duct contour is shown in Fig. 1 with the source plane at $x = 0$. A uniform extension ends in the termination plane that is non-reflecting. Achievable attenuation depends strongly on the magnitude and phase of the propagating modes at the source plane, as previously shown in Refs. [1,5]. This section examines the dependence of attenuation on incident modal power and phase establishing the basis for the subsequent statistical analysis.

Consider the source at $x = 0$ in Fig. 1 modeled as a superposition of incident acoustic modes with complex modal amplitudes a_i^+. The subscript "i" enumerates the modal number (index). The complex amplitude can also be represented in terms of modal acoustic power (or modal amplitude magnitude) and phase. At the termination plane the acoustic field is a superposition of acoustic modes with complex modal amplitudes b_i^+.

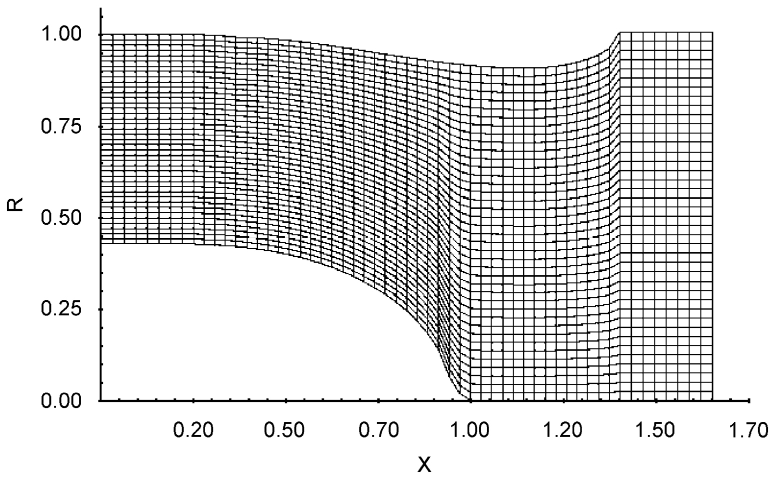

Fig. 1. Inlet geometry.

For an axi-symmetric duct, modes are classified according to their circumferential (angular) order and radial order. The source (and termination) is considered to be in a locally uniform section with hard walls where modal acoustic power depends only on modal amplitude magnitude, and not on phase. Assume that the acoustic power in each incident radial mode varies randomly and independently of other modes and only cut-on modes exist at $x = 0$. For a specified circumferential mode order, the non-dimensional acoustic power in one of its radial modes, having at the source plane section the incident complex amplitude a_i^+ is

$$\Pi_{i_0}^{\text{incident}} = a_i^{+^*} P_{i_0}^{++} a_i^+ = \left| a_i^+ \right|^2 P_{i_0}^{++} = C_i, \tag{1}$$

where the coefficient $P_{i_0}^{++}$ is extracted from the incident power matrix [6]. Complex conjugate is denoted by superscript *. The random variation in modal magnitude in each propagating incident mode is achieved by varying its modal power statistically, thus C_i is a random number. Moreover, C_i is a uniformly distributed number between 0 and 1, no value having a larger chance of occurrence than others in this interval. Assuming that only cut-on modes exist at $x = 0$, which represents the source location, the power in all N incident radial modes would be

$$\Pi_0^{\text{incident}} = \sum_{i=1}^{N} C_i. \tag{2}$$

The total power at $x = 0$ is the sum of the power in the incident modes and the power in the reflected modes: $\Pi_0 = \Pi_0^{\text{incident}} + \Pi_0^{\text{reflected}}$. Except in cases where isolated acoustic modes are reflected due to encountering local cut-off conditions, the power in the reflected modes is generally small. To demonstrate this, the following case is presented: for the inlet geometry as seen in Fig. 1, the flow Mach number at the source plane $x = 0$ is $M = -0.416$, the non-dimensional frequency $\eta = 50.75$, the circumferential mode incident is $m = 0$.

The non-dimensional frequency is determined based on the definition $\eta = 2\pi Rf/c$, where R is the duct radius at the source plane, f the frequency in Hz and c the speed of sound of the mean flow. A portion of the outer wall of the inlet, between $x = 0.51$ and 1.17, is acoustically treated, the liner impedance being $Z = 3.0\text{–}3.0i$.

One thousand cases of randomly selected magnitudes for the radial modes were tested and the maximum ratio between reflected and the total power was calculated to be 4.67×10^{-3}. That makes the reflected power an insignificant contribution to the total power at the source, so the approximation is made that

$$\Pi_0 \approx \Pi_0^{\text{incident}} = \sum_{i=1}^{N} C_i. \tag{3}$$

The modal amplitude a_i^+ can be written using the magnitude $|a_i^+|$ from Eq. (1) and the phase ϕ_i which in turn, can have a random uniform distribution as well, the range being between 0 and 2π,

$$a_i^+ = |a_i^+|\mathrm{e}^{\mathrm{j}\phi_i} = \sqrt{\frac{C_i}{P_{i_0}^{++}}}\mathrm{e}^{\mathrm{j}\phi_i}, \tag{4}$$

where $\mathrm{j} = \sqrt{-1}$.

The transmitted modal amplitude at the exit of the duct, denoted here with b_i^+, is obtained from the incident modal amplitudes

$$b_i^+ = [T]_i\{a^+\}, \tag{5}$$

where $[T]_i$; represents the row i in the transmission matrix $[T]$ [6]. By using Eq. (4), the transmitted modal amplitudes can be written as a function of the incident modal powers and phases

$$b_i^+ = \sum_{k=1}^{N} T_{ik}a_k^+ = \sum_{k=1}^{N} T_{ik}\sqrt{\frac{C_k}{P_{k_0}^{++}}}\mathrm{e}^{\mathrm{j}\phi_k}. \tag{6}$$

The elements of the complex transmission matrix T_{ik} can be expressed in polar form

$$T_{ik} = |T_{ik}|\mathrm{e}^{\mathrm{j}\psi_{ik}}. \tag{7}$$

Hence, the exit transmitted modal amplitudes are

$$b_i^+ = \sum_{k=1}^{N} \sqrt{\frac{C_k}{P_{k_0}^{++}}}|T_{ik}|\mathrm{e}^{\mathrm{j}(\phi_k+\psi_{ik})}. \tag{8}$$

The acoustic power in the transmitted radial mode b_i^+ at $x = L$ is

$$\Pi_{i_L}^{\text{transmitted}} = b_i^{+^*}P_{i_L}^{++}b_i^+ = P_{i_L}^{++}\sum_{m=1}^{N}\sum_{n=1}^{N}\sqrt{\frac{C_mC_n}{P_{m_0}^{++}P_{n_0}^{++}}}|T_{im}||T_{in}|\cos(\phi_m - \phi_n + \psi_{im} - \psi_{in}), \tag{9}$$

where (*) represents the complex conjugate operation and the power coefficient $P_{i_L}^{++}$ is extracted from the transmitted power matrix [6]. The total power at the exit of the duct is given by the sum of the transmitted modal powers since the exit of the duct is assumed reflection free

$$\Pi_L = \sum_{i=1}^{N}\Pi_{i_L}^{\text{transmitted}} = \sum_{i=1}^{N}P_{i_L}^{++}\sum_{m=1}^{N}\sum_{n=1}^{N}\sqrt{\frac{C_mC_n}{P_{m_0}^{++}P_{n_0}^{++}}}|T_{im}||T_{in}|\cos(\phi_m - \phi_n + \psi_{im} - \psi_{in}). \tag{10}$$

An equivalent formulation to Eq. (10) is

$$\Pi_L = \sum_{m=1}^{N}\sum_{n=1}^{N}\sqrt{C_mC_n}[\alpha_{mn}\cos(\phi_m - \phi_n) - \beta_{mn}\sin(\phi_m - \phi_n)], \tag{11}$$

with the coefficients α_{mn} and β_{mn} independent of the modal input, but depending on the flow conditions, geometry, frequency, etc.

$$\alpha_{mn} = \frac{1}{\sqrt{P_{m_0}^{++}P_{n_0}^{++}}}\sum_{i=1}^{N}P_{i_L}^{++}|T_{im}||T_{in}|\cos(\psi_{im} - \psi_{in}),$$

$$\beta_{mn} = \frac{1}{\sqrt{P_{m_0}^{++}P_{n_0}^{++}}}\sum_{i=1}^{N}P_{i_L}^{++}|T_{im}||T_{in}|\sin(\psi_{im} - \psi_{in}). \tag{12}$$

If the reflected modal power at $x = 0$ is neglected, the acoustic power attenuation (in dB) can be written as

$$\text{Atten} \approx -10\log_{10}\frac{\Pi_L}{\Pi_0} = -10\log_{10}\left(\frac{\sum_{m=1}^{N}\sum_{n=1}^{N}\sqrt{C_m C_n}[\alpha_{mn}\cos(\phi_m - \phi_n) - \beta_{mn}\sin(\phi_m - \phi_n)]}{\sum_{i=1}^{N} C_i}\right). \tag{13}$$

When incident modal power is allowed to change randomly, only the parameters C_m, C_n vary randomly, and, similarly, when the incident phase is random, only ϕ_m, ϕ_n vary randomly while the rest of the parameters in Eq. (13) remain constant.

By using the array of attenuations that results from Eq. (13), one may establish a conservative lower bound on attenuation, or determine an average attenuation obtained over many trials. This approach can thus provide impedance that responds best to the encountered least favorable modal power and phase combination, or it can supply optimum impedance that will respond well to a wide range of power and phase distributions by optimizing the lining for the median or mean attenuation case.

2.1. Analysis of probability distribution for acoustic power and attenuation, random magnitude case

In this section the noise source is characterized by random incident modal magnitudes, while the modal phases are kept fixed, all equal for instance. Under these circumstances, in the companion paper [1] it was observed that the total power at the exit of the duct, as given by Eq. (11), approaches a normal distribution as the number of incident modes increases. A similar case is presented subsequently.

For the inlet of Fig. 1, the Mach number is taken as $M = -0.288$ at the source plane. The broadband noise frequency considered is 1250 Hz, which in non-dimensional form is $\eta = 37.814$. This generates 8 propagating radial modes at the source plane in spinning mode $m = 0$, 5 cut-on modes in spinning mode $m = 18$ and only 2 cut-on modes in spinning mode $m = 30$. 100,000 sets of incident radial modes with randomly distributed modal power and equal phases are tested. The liner non-dimensional impedance $Z = 1.72 + 1.2i$ is investigated.

The randomly powered sets of incident modes determine the acoustic power at the exit of the duct. The resulting range of normalized exit power (fraction of incident power) is partitioned in bins of equal size (0.3 times the standard deviation). The fractional number of occurrences in each bin is plotted versus the exit power range in Fig. 2. This type of graph is a scaled version of the probability density function. This is because of the bin size. In a true probability density function the area under the curve is unity and the ordinate is the probability of occurrence within a unit slice of the abscissa. In the version here the interpretation is the probability of occurrence within the defined bin width. In this form the total number of occurrences in all bins must equal the number of trials, or in its normalized form it should be unity. The result is that the ordinate is

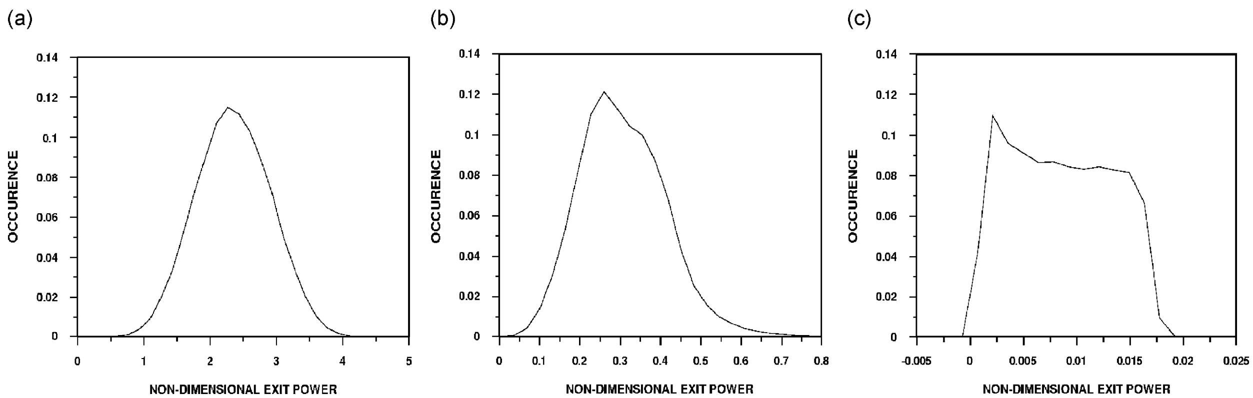

Fig. 2. Exit power distribution for three different angular modes. Non-uniform duct $\eta_r = 37.914$, $M = -0.288$, $Z = 1.72 + 1.2i$, equal incident modal phase, random power. (a) $m = 0$, (b) $m = 18$, and (c) $m = 30$.

scaled by the inverse of the bin width as compared to a traditional density function. This is an important consideration when comparing density functions with different standard deviations, and hence, different bin widths.

A Gaussian distribution tendency of the total power at the exit of the duct with the increase of the number of incident propagating modes can be observed in Fig. 2. For $m = 0$, there are eight propagating modes, a relatively modest number, yet in Fig. 2 it is seen that the probability density function appears to be normal. For $m = 18$ and 30 five and two modes propagate, respectively and the density function tends to deviate from a normal distribution, drastically in the $m = 30$ case. This trend toward Gaussian statistics with increasing number of propagating modes is addressed subsequently.

In the previously developed Eq. (11), if only the incident modal powers $C_{m,n}$ vary randomly, the total power at the exit of the duct would be the sum of independent distributions

$$\Pi_L = \sum_{m=1}^{N} \sum_{n=1}^{N} X_{mn}, \tag{14}$$

where

$$X_{mn} = \sqrt{C_m C_n}(\alpha_{mn} \cos(\phi_m - \phi_n) - \beta_{mn} \sin(\phi_m - \phi_n)), \tag{15}$$

with α_{mn} and β_{mn} given by Eqs. (12). X_{mn} in Eq. (15) are not identical, each depending on the distributions of power C_m, C_n in modes m and n. The other parameters in Eq. (15) are independent of the modal input. α_{mn},β_{mn} depend on the duct and the input modal phases ϕ_m, ϕ_n are assigned and held fixed.

A fundamental result in probability theory, the Central Limit Theorem [7] in its simplest form demonstrates that sums of a sufficiently large number of identically distributed independent random variables will tend to produce a normal distribution, no matter the shape of the original distribution. Several generalizations (Lindeberg Condition) to the Central Limit Theorem [8] do not require identical distributions, but include the condition that none of the variables X_{mn} exert a much larger influence than the others, in the sense that the individual variances be small compared to their sum. Additionally, a normal distribution is approached more quickly as the number of variates that go into the computation of the sum increases.

A general rule of thumb for the Central Limit Theorem is that as long as the distribution X_{mn} is not too skewed, a normal approximation to the sum is achieved for a number of 30 terms. In the case in Fig. 2, for $m = 0$ with eight incident modes, there would be 64 terms in the total exit power in Eq. (14). As a consequence, the power at the exit plane will tend to be normally distributed provided that relatively many modes contribute to it and none of the radial modes exercises much more power than the others. This could be accomplished by a low angular mode and/or a high frequency. For similar reasons as the ones stated above, the total power at the source plane Π_0 will also approach a normal distribution. Eq. (3) shows that Π_0 is the sum of identically distributed random values C_i.

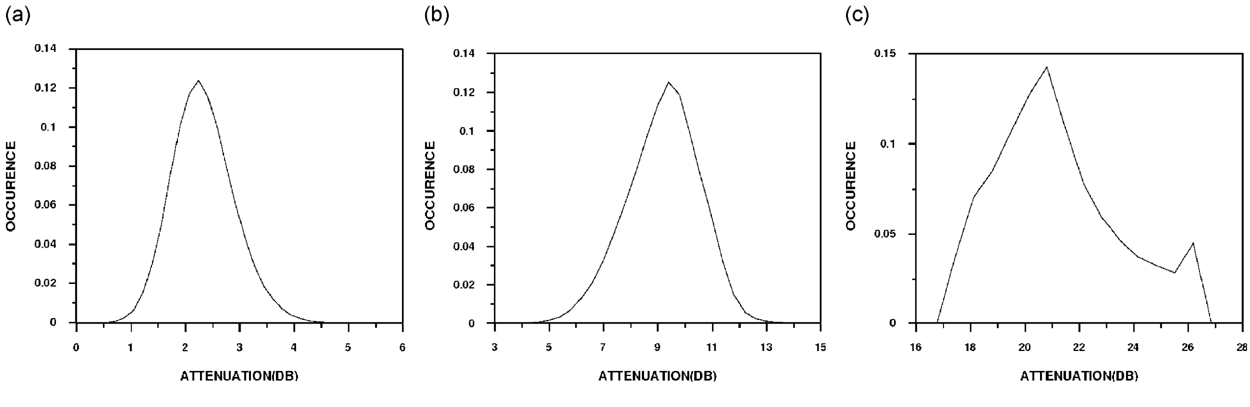

Fig. 3. Attenuation distribution for three different angular modes. Non-uniform duct $\eta_r = 37.914$, $M = -0.288$, $Z = 1.72 + 1.2i$, equal incident modal phase, random power. (a) $m = 0$, (b) $m = 18$, and (c) $m = 30$.

The power transmission coefficient is the quotient of the acoustic power at the exit of the duct and the power at the source TC $= \Pi_L/\Pi_0$. It is a ratio of two normal distributions for sufficiently numerous incident modes. There is no proof available that TC is normally distributed. Attenuation is logarithmically related to the power transmission coefficient in Eq. (13). In Fig. 3, it is noted that attenuation is not Gaussian, and is skewed, although not strongly in the case $m = 0$ with eight propagating modes. Possible non-Gaussian distribution of power transmission coefficient and the logarithmic mapping from transmission coefficient to attenuation introduce the non-Gaussian behavior.

2.2. Analysis of probability distribution for acoustic power and attenuation, random phase case

In this section modal powers are assumed equal, while modal phases are allowed to vary randomly. Each modal phase can assume any value in the interval between 0 and 2π with equal chance of occurrence such that the phases correspond to a continuous uniform random distribution. Non-uniform distributions can also be used in case pre-existent information favors certain phase values. The following results do not depend on the type of random phase distributions (uniform or non-uniform).

In the companion paper [1] it was suggested that having the incident modal phases picked randomly has more affect on the range of achievable attenuation than having the magnitudes picked randomly. It was also found that the transmitted power at the exit of the duct is not Gaussian when the phase varies randomly. A supporting example is shown in Fig. 4.

Noise propagation is analyzed for the inlet in Fig. 1, a portion of which is lined with impedance $Z = 1.72 + 1.2i$, with the same flow conditions as the case illustrated in Fig. 2, $M = -0.288$ at the source plane, at the same broadband noise frequency 1250 Hz, or $\eta = 37.814$ non-dimensionally. Spinning modes $m = 0$, 18 and 30 are considered. This time 100,000 sets of incident radial modes with randomly distributed phases and equal modal powers are considered.

In Fig. 4 are shown the power distributions at the exit of the duct for the three angular mode cases. The presence of modes with high cut-off ratios, which are harder to attenuate, skews the distribution of exit power in favor of high values for the lower order spinning mode $m = 0$.

The demonstration that the exit power distribution in a single circumferential mode (even with multiple propagating radial modes) is non-Gaussian when phase is allowed to vary randomly involves a similar statistical approach as in the previous section. It was shown using Eq. (11) that the power at the exit of the duct depends on the incident modal powers, as well as the phases. The total exit acoustic power $\prod_L$ can be written as the difference

$$\Pi_L = \sum_{m=1}^{N}\sum_{n=1}^{N} Y_{mn} - \sum_{m=1}^{N}\sum_{n=1}^{N} Z_{mn}, \tag{16}$$

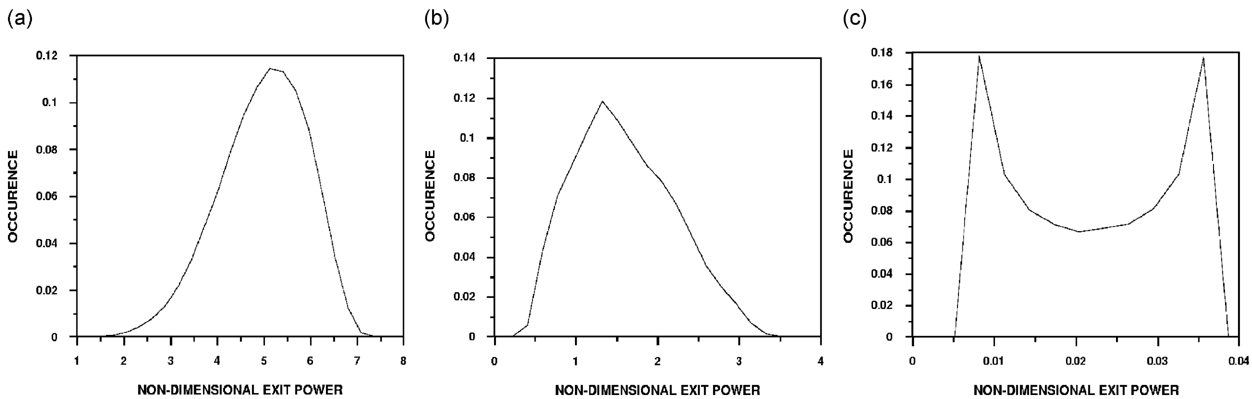

Fig. 4. Exit power distribution for three different angular modes. Non-uniform duct $\eta_r = 37.814$, $M = -0.288$, $Z = 1.72 + 1.2i$, equal incident modal power, random phase. (a) $m = 0$, (b) $m = 18$, and (c) $m = 30$.

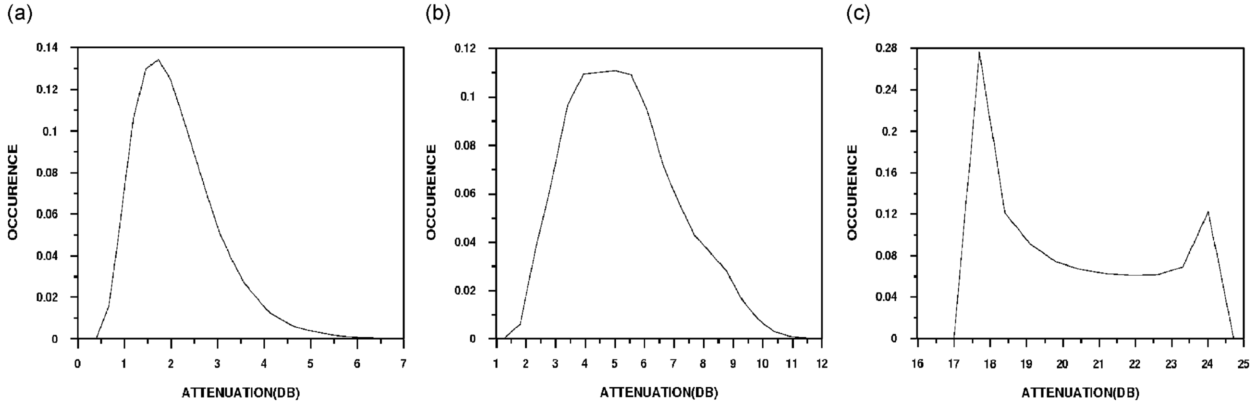

Fig. 5. Attenuation distribution for three different angular modes. Non-uniform duct $\eta_r = 37.814$, $M = -0.288$, $Z = 1.72 + 1.2i$, equal incident modal power, random phase. (a) $m = 0$, (b) $m = 18$, (c) $m = 30$.

with Y_{mn} and Z_{mn} defined by

$$Y_{mn} = \sqrt{C_m C_n}\alpha_{mn}\cos(\phi_m - \phi_n),$$
$$Z_{mn} = \sqrt{C_m C_n}\beta_{mn}\sin(\phi_m - \phi_n). \tag{17}$$

From Eqs. (17), it can be observed that Y_{mn} and Z_{mn} are functions only of the incident random phases ϕ_m and ϕ_n. The incident modal power coefficients C_m, C_n are assumed known, all equal for example, and the coefficients α_{mn}, β_{mn} do not depend on the modal input (see Eqs. (12)).

Eqs. (17) reveal the interdependence of Y_{mn} and Z_{mn} due to the presence of the trigonometric functions of the relative modal phase $\phi_m - \phi_n$. By using the Lindeberg generalization of the Central Limit Theorem argument, each of the sums $\sum_{m=1}^{N}\sum_{n=1}^{N} Y_{mn}$, $\sum_{m=1}^{N}\sum_{n=1}^{N} Z_{mn}$ should tend to normal distributions. However, the distributions of the two sums are not independent, because they both depend on the same relative phases. The difference of two normal distributions can only be normal with the condition that the distributions are independent. It is therefore not expected that the exit power should be normally distributed.

The factor that produces the skewed character of the probability distribution for the exit power when phase is modified randomly can be found by analyzing Eq. (10). If $\psi_{im} - \psi_{in} = 0$, then $Z_{mn} = 0$ and $\Pi_L = \sum_{m=1}^{N}\sum_{n=1}^{N} Y_{mn}$. This sum has been argued to be normally distributed, but the combination of the two terms in Eq. (17) is not. The modal phase shift produced by transmission through the lined section of the duct destroys the normal distribution for the total exit power. From Eq. (3), the modal acoustic power at the source plane does not depend on modal phase, so when input incident modal phase is allowed to fluctuate randomly, the total power at $x = 0$ does not vary. Thus the power transmission coefficient, $\mathrm{TC} = \Pi_\mathrm{L}/\Pi_0$, follows the distribution of the exit power. In Fig. 5 are shown the corresponding attenuation distributions for the same cases as presented in Fig. 4. Again, the logarithm mapping influences the distribution of attenuation.

2.3. *Analysis of probability distribution for acoustic power and attenuation with multiple circumferential modes*

In case many or all propagating angular modes at a certain frequency need to be taken into account when evaluating lining performance, acoustic power attenuation can be calculated in a manner similar to the attenuation for a given angular mode. Incident powers in each circumferential (angular) mode k, calculated according to Eq. (3) and transmitted powers at the exit of the duct, as in Eqs. (10) or (11) can be summed up to form the total incident and transmitted power for all spinning modes

$$P_{x=0}^{\text{incident}} = \sum_{k=0}^{N\ \text{modes}} \Pi_0^k, \tag{18}$$

$$P_{x=L}^{\text{transmitted}} = \sum_{k=0}^{N \text{ modes}} \Pi_L^k. \tag{19}$$

Up to this point in the present analysis, the acoustic power transmission of only a single circumferential mode, with associated propagating radial modes, was considered at a given frequency. In case modal amplitudes at the source plane are unknown, both magnitude and phase picked randomly can provide a statistical distribution for the attainable attenuation range. It has been proved that total acoustic power at the source in circumferential mode k, Π_0^k as well as the total power at the exit of the duct, Π_L^k, have a normal (Gaussian) distribution when the incident radial mode powers, and hence mode magnitudes, are allowed to vary randomly. It has also been shown that the total power at the source, Π_0^k, is fixed when incident modal phase varies randomly, while at the duct exit plane, Π_L^k, has a non-Gaussian distribution.

From Eqs. (18) and (19) conclusions can be drawn about distributions of acoustic power when many circumferential modes are considered. When phase is allocated randomly, the summation of constant incident powers, Π_0^k, over a large number of circumferential modes, enumerated by k, will result in a constant total power $P_{x=0}^{\text{incident}}$. The summation of normally distributed incident powers Π_0^k over a large number of circumferential modes, enumerated by k, when incident modal power is randomly chosen, will produce a normal distribution, according to the Central Limit Theorem. Similarly, the result of summing over a large number of circumferential mode powers Π_L^k to obtain $P_{x=L}^{\text{trasmitted}}$ will generate a normal distribution for any distribution of Π_L^k, Gaussian or not.

In summary, in case many or all incident propagating modes at a given frequency are taken into consideration in calculating attenuation, the total acoustic power at the source section $P_{x=0}^{\text{incident}}$ is either constant, when modal phases are selected randomly, or normally distributed if modal powers are random. At the same time, the total acoustic power at the exit of the duct $P_{x=L}^{\text{transmitted}}$ is normally distributed if either the incident modal phase or power is taken randomly. The following examples will support these results.

The inlet geometry of Fig. 1 is again considered with a section of the outer wall lined with impedance $Z = 1.72 + 1.2i$. The mean flow Mach number is $M = -0.288$ at the source plane, the frequency is 1250 Hz, or $\eta = 37.814$. 100,000 sets of incident radial modes with randomly distributed power and equal phases are considered. In Fig. 6, the plot of total power distributions at both the source plane (right-hand curve) and at the exit of the duct (left-hand curve) shows their Gaussian character. In this comparative case, both attenuation occurrence curves were based on the same bin width such that the areas under the graphs are equal.

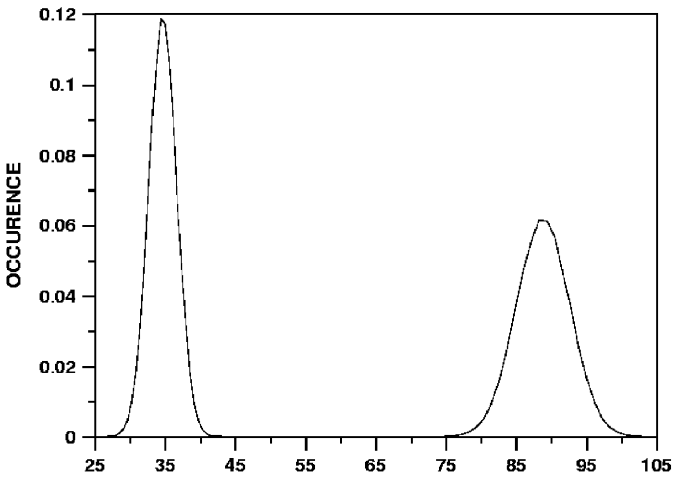

Fig. 6. Total power distributions at the exit of the duct (left) and at the source plane (right). Non-uniform duct, $\eta_r = 37.814$, $M = -0.288$, $Z = 1.72 + 1.2i$, equal incident modal phase, random power.

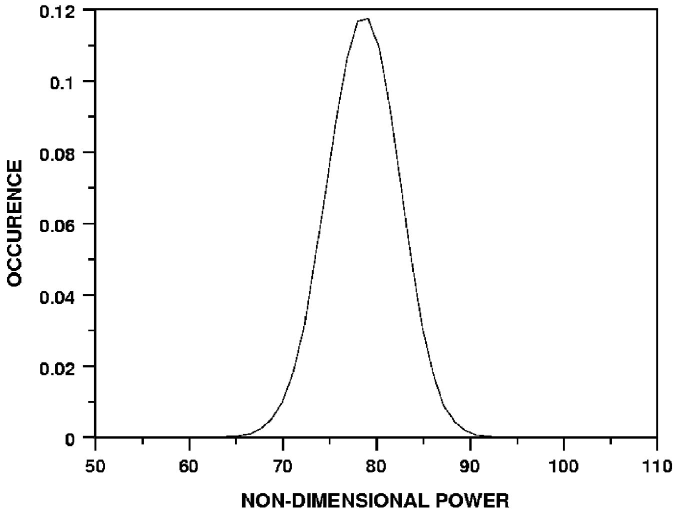

Fig. 7. Total power distribution at the exit of the duct. Non-uniform duct, $\eta_r = 37.914$, $M = -0.288$, $Z = 1.72 + 1.2i$, equal incident modal power, random phase.

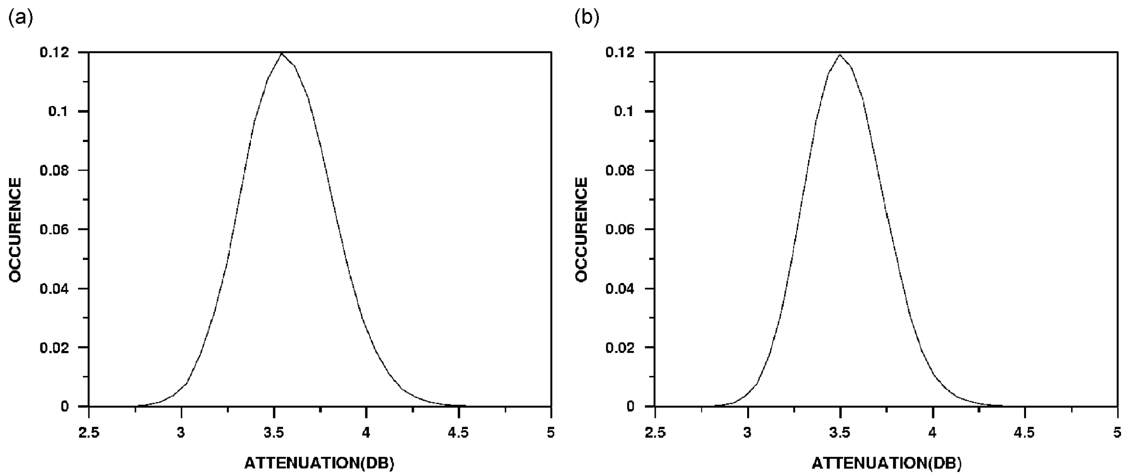

Fig. 8. Attenuation distribution. Non-uniform duct, $\eta_r = 37.914$, $M = -0.288$, $Z = 1.72 + 1.2i$, (a) equal incident modal phase, random modal power, and (b) equal incident modal power, random phase.

The same inlet with the same flow conditions and frequency as in the previous example is considered, but the modal input is characterized by random modal phase and equal power. The distribution of the total power at the exit of the duct shows the same Gaussian nature, as can be seen in Fig. 7. The total power at the source plane, not shown in the graph, is constant and equal to 177 (177 modes with unit non-dimensional power). The distribution of the power transmission coefficient will thus be either Gaussian in case modal phases are selected randomly, or the ratio of two normal distributions in case the incident modal powers are random when all incident radial modes are taken into consideration. Attenuation distributions for the two cases presented are shown in Fig. 8. It can be seen that the distributions are slightly skewed, but less so than in the cases in which just one angular mode was used to calculate attenuation (see Figs. 3 and 5 for comparison).

3. Acoustic attenuation probability distribution

If the input modal amplitudes at the source are unknown or unreliable, random sets of magnitude and phase can be considered and attenuation can be modeled statistically. Because only relatively small samples from all possible modal magnitude and phase combinations are investigated, the reported attenuation results must reflect the uncertainty through the use of probability statements and ranges of attenuation.

Model identification for the attenuation distribution, parameter estimation, model diagnostic checking, in addition to some useful statistical measures will be demonstrated for the duct presented in Fig. 1. Both assumptions of random power with equal phase and then random phase with equal modal power will be exemplified. The outer wall has non-dimensional impedance $Z = 1.72 + 1.2i$, the mean flow velocity is $M = -0.288$ at the source plane. At the 1250 Hz frequency, power attenuation was previously calculated for both angular mode $m = 0$ and for the combination of all propagating modes.

The attenuation distribution plots in Fig. 3 (random power, fixed phase, $m = 0$), Fig. 5 (fixed power, random phase, $m = 0$), and Fig. 8 (a) (random power, fixed phase, all modes) and (b) (fixed power, random phase, all modes), indicate that attenuation results follow a skewed distribution with the right tail longer (less visible in Fig. 8, but still present). A number of right skewed (also known as positively skewed) probability density functions could be suitable representations of these distributions obtained by numerical experimentation. The search was restricted to four that fit the right skewed probability distribution. They were the Weibull, Gamma, Lognormal, and Inverse Gaussian density functions. These are three parameter (scale, shape, location) distributions [9]. The criterion that was used for selecting the best distributional match to the data is the probability plot correlation coefficient method (PPCC) [10]. For each of the selected members of the three parameter distributional group a probability plot is generated by plotting the correlation coefficient of the probability plot against the shape parameter. The value of the shape parameter that maximizes the correlation coefficient, making the straightest plot on the probability graph, is the optimal shape parameter for that particular distribution. The correlation of the probability plots for the distributions of Figs. 3,5, and 8 are summarized using the probability plot correlation coefficient, denoted PPCC in Table 1. A PPCC value of 1.0 corresponds to theoretically perfect correlation.

The Gamma probability density function appears to better-fit attenuation for mode $m = 0$ when phase was randomly picked, while the Lognormal is better when incident powers were chosen randomly. When all propagating modes are considered at the specified 1250 Hz frequency, the Gamma or the Lognormal perform comparably. The differences, however, are small and it is possible to successfully fit the numerically obtained attenuation distribution with any of the four listed probability density functions. In fact for a frequency of 1000 Hz, the Weibull and the Inverse Gaussian proved to fit better the attenuation of mode $m = 0$.

The Gamma and Lognormal probability density functionss [9] can be written in terms of the location, shape and scale parameters as follows:

$$\mathrm{PDF}_G(x) = \frac{\left(\frac{x-\mu}{\alpha}\right)^{\gamma-1} \mathrm{e}^{-((x-\mu)/\alpha)}}{\alpha\Gamma(\gamma)}, \tag{20}$$

$$\mathrm{PDF}_L(x) = \frac{\mathrm{e}^{-\left(\ln((x-\mu)/\alpha)\right)^2/2\gamma^2}}{(x-\mu)\gamma\sqrt{2\pi}}, \tag{21}$$

Table 1
Probability plot correlation coefficients for four candidate probability distributions obtained from numerical trials

Distribution	PPCC value			
	$m = 0$		All modes	
	Random phase	Random power	Random phase	Random power
Weibull	0.99958	0.99881	0.99912	0.99921
Gamma	0.99983	0.99993	0.99997	0.99999
Lognormal	0.99915	0.99994	0.99997	0.99999
Inverse Gaussian	0.99954	0.99993	0.99996	0.99998

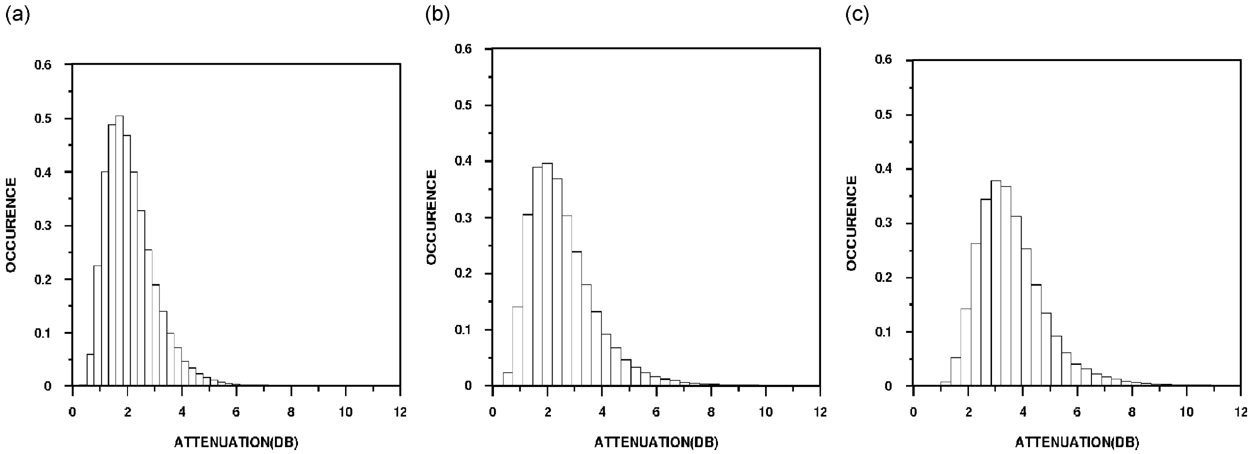

Fig. 9. Attenuation histograms for three linings. Non-uniform duct, $\eta_r = 37.914$, $M = -0.288$, $Z_1 = 1.72+1.2i$, $Z_2 = 2.34-0.18i$, $Z_3 = 2.69-0.32i$, equal incident modal power, random phase. (a) Lining 1, (b) lining 2, (c) lining 3.

Table 2
Probability of achieving an attenuation range for mode $m = 0$ for three different lining

Propagation of model equal modal power 100,000 random phase cases	Lining 1	Lining 2	Lining 3
Outer wall impendence	$1.72+1.20i$	$2.34-0.18i$	$2.69-0.32i$
Propagation of attenuation between 4 and 6 dB	3.7%	9.6%	26.0%

Non-uniform duct, $\eta_r = 37.914$, $M = -0.288$, $Z_1 = 1.72+1.2i$, $Z_2 = 2.34-0.18i$, $Z_3 = 2.69-0.32i$.

where γ, μ and α are the shape, location and the scale parameters, respectively. In Eq. (20), Γ is the standard gamma function. Changing the value of any of the three parameters will produce a change in the shape of the probability density function.

After designating the probability density function that best fits the attenuation data, in the present case either Gamma or Lognormal, the three parameters that define the probability density function can be determined. A least square fit process that tries to match the probability distribution function governed by the mentioned parameters with the distributional data can be used.

A final example shows how the acoustic performance of three different linings can be compared on a statistical basis. For geometry and flow conditions of previous cases, the attenuation probability density function for circumferential mode $m = 0$ for the case 1250 Hz ($\eta = 37.914$), is shown in the form in histograms in Fig. 9. The modal amplitudes considered are characterized by equal power and random phase. It is clear from Fig. 9 that lining 1 will produce the lowest expected (mean) attenuation and lining 3 will produce the highest expected attenuation. An additional statistical assessment of the merit of these linings is obtained by using the cumulative distribution function defining the probability that attenuation is between specified lower and upper bounds. For the three linings, the probabilities of achieving an attenuation between 4 and 6 dB are shown in Table 2.

4. Conclusions

In previous studies, the details of the source are proven critical in predicting the performance of the acoustic lining [1,5]. Unknown or unreliable information about the source can be overcome by applying a statistical model to the incident modal phases as well as acoustic power for modes representing the source [1]. The present study formalizes previous results obtained by numerical experimentation and shows that there is a firm basis for approaching the prediction of performance of acoustic treatment and optimization of performance in a statistical sense.

Some of the most important results found are:

- For a given geometry, frequency, and flow conditions, randomly chosen sets of source modal magnitude and/or phase produce a range of attenuation results with a similar statistical distribution.
- When source modal power is chosen randomly, the distribution of the overall power at the source, as well as that of the total power at the exit of the duct is shown to have a Gaussian distribution under the condition that at least a modest number of cut-on source modes are incident.
- Variation of the incident modal phase has no effect on the total power calculated at the source plane. Random modal phase at the source creates total power at the exit of the duct with a non-Gaussian character if attenuation is calculated for a single angular mode; if all incident angular modes are considered in calculating attenuation, then the distribution of the total power at the exit of the duct is Gaussian.
- With the use of a statistical source model lining performance can be predicted in terms of expected (mean) attenuation, standard deviation from the mean, probability of achieving attenuation between specified limits, or maximum and minimum achievable attenuation.

References

[1] G. Zlavog, W. Eversman, Source effects on attenuation in lined ducts. Part I: a statistically based computational approach, *Journal of Sound and Vibration*.

[2] I. Dyer, Measurement of noise sources in ducts, *Journal of Acoustical Society of America* 30 (1958) 833–841.

[3] S. Mariano, Sound source location effects on the attenuation in acoustically lined rectangular ducts, *Journal of Sound and Vibration* 41 (1975) 473–491.

[4] C.L. Morfey, Acoustic energy in non-uniform flows, *Journal of Sound and Vibration* 14 (1971) 15–170.

[5] G. Zlavog, W. Eversman, Source effects on realized attenuation in lined ducts, *Ninth AIAA/CEAS Aerobacoustics Conference*, Hutton Head, NC, 2003, pp. 2003–3247.

[6] W. Eversman, A reverse flow theorem and acoustic reciprocity in compressible potential flow in ducts, *Journal of Sound and Vibration* 246 (l) (2001) 71–95.

[7] P.G. Hoel, *Mathematical Theory of Probability and Statistics*, third ed., Wiley, New York, 1971, pp. 115–137.

[8] R. von Mises, *Mathematica Theory of Probability and Statistics*, Academic Press Inc., New York, 1964, pp. 289–305.

[9] N. Johnson, S. Kotz, N. Balakrishnan, *Continuous Univariate Distributions*, second ed., Wiley, New York, 1974, pp. 207–259, 337–415.

[10] J. Filliben, The probability plot correlation coefficient test for normality, *Technometrics* 17 (1975) 111–117.