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An artificial parameter-decomposition method for nonlinear oscillators: Applications to oscillators with odd nonlinearities

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Abstract

A method for obtaining series solutions of nonlinear second-order ordinary differential equations based on the introduction of an artificial parameter is presented and shown to be identical to the well-known Adomian's decomposition technique. The method is formulated in both integral and differential forms. For the determination of the limit cycle of oscillators with odd nonlinearities, two differential forms and one integral form of the artificial parameter method are presented. These versions are based on introducing a linear stiffness term with an unknown frequency, and the use of either the original independent variable or a new independent variable that depends linearly on the unknown frequency of the oscillator. The three formulations provide identical results, and their application to eight oscillators with odd nonlinearities shows that the artificial parameter technique presented in this paper predicts the same frequency of oscillation as the harmonic balance and iterative techniques as well as modified Linstedt–Poincaré methods. However, the method presented here is based on the introduction of an artificial parameter and does not require the presence of small perturbation parameters in the ordinary differential equation. It is also shown that two- and three-level iterative methods yield the same frequency of oscillation as the artificial parameter technique presented in this paper provided that the initial iterate of the former coincides with the leading-order solution of the latter and only one iteration of iterative techniques and only the second approximation of the artificial parameter method are determined.

1. Introduction

The determination of the frequency or limit cycles of oscillators with odd nonlinearities is a subject of great interest which has been addressed by means of perturbation methods [1,2], the harmonic balance technique [1], standard and modified Linstedt–Poincaré methods [3–5], artificial parameter techniques [5,6], two- [7–9] and three-level [10–14] iterative techniques, etc. Some of these methods, e.g., perturbation and Linstedt–Poincaré methods, do require the presence of a small parameter in the nonlinear ordinary differential equation which governs the dynamics of these oscillators and look for solutions which are series expansions of this small parameter; this series solutions are usually asymptotic. The harmonic balance method and two- and three-level iterative techniques do not require the presence of a small parameter in the governing ordinary differential

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equation. In particular, the harmonic balance method provides the frequency by assuming that the solution is a trigonometric function of this variable, and is frequently limited to only a single harmonic.

Modified Linstedt–Poincaré methods [3–5] are based on the expansion of some constants that appear in the differential equation and the solution in series of either a small perturbation parameter or an artificial one. On the other hand, two- [7–9] and three-level [10–14] iterative techniques are based on the introduction of a linear stiffness term with an unknown frequency in both the left- and right-hand sides of the governing nonlinear ordinary differential equation and provide the solution as a sequence of iterates. However, these techniques have been limited to obtain only one or two iterates because of the algebra involved, but they do provide frequencies of oscillations in accord with those of the harmonic balance method provided that the first iterate is harmonic and one limits oneself to the second iterate upon which the non-secularity condition is imposed.

Other authors [15–18] have introduced a change of independent variable so that the period of the transformed equation is 2π and expanded the linear and nonlinear terms in terms of Fourier series. By applying a first-order harmonic balance approximation, they were able to obtain an approximate frequency whose accuracy may be improved by perturbing both the solution and the frequency. When the perturbed solution and frequency are substituted into the nonlinear ordinary differential equation and the resulting equation is linearized and solved by Fourier series expansions, these authors obtained a more accurate frequency.

In a series of papers dating back to the 1970s and 1980s, Adomian proposed a decomposition method for the series solution of algebraic and ordinary differential equations which has been found to be applicable to a variety of partial differential equations, integral equations, integro-differential equations, delay-differential equations, etc. [19–21]. The decomposition method requires that the highest-order derivative in the differential equation appears in a linear fashion and provides a series solution in terms of Adomian's polynomials. It also requires that the nonlinear terms be differentiable with respect to the dependent variables.

In this paper, we present an artificial parameter method that provides the solution of nonlinear ordinary differential equations in series and that is applied to determine the frequency of oscillators with odd nonlinearities. The method is based on the introduction of an artificial parameter in the Volterra integral equation that results upon formal integration of the differential equation and a series expansion of the solution in terms of this artificial parameter. It is shown that this artificial parameter technique coincides with Adomian's decomposition method upon setting the artificial parameter to unity and, for this reason, the method is referred to as an artificial parameter-decomposition technique. Furthermore, we show that the Volterra integral equation can be written in differential form.

The artificial parameter method is then generalized for the determination of the limit cycles of oscillators with odd nonlinearities by first introducing a linear stiffness term that is proportional to the unknown frequency of oscillation in the governing equation and then introducing an artificial parameter in the right-hand side of the equation. By assuming that the solution can be expanded in terms of this artificial parameter and imposing the nonsecularity condition for each term of the series solution, we obtain the oscillation frequency. We also show that this artificial parameter method provides the same frequency as two- and three-level iterative techniques under certain conditions.

It must be pointed out that the use of artificial or book-keeping parameters for determining approximate solutions of nonlinear ordinary differential equations is not new, e.g., Refs. [6,22,23,5]. For example, Senator and Bapat [22] introduced a linear stiffness term and an artificial parameter, assumed that both the solution and the unknown frequency of oscillation can be expanded in terms of the artificial parameter, i.e., they used a modified Linstedt–Poincaré method [1,2,24], imposed a non-secularity condition at each order in the artificial parameter and determined the frequency of oscillation by minimizing the absolute value of the difference between the frequency of the linear oscillator and that of the real one, whereas Wu et al. [23] used the same procedure as Senator and Bapat [22] but minimized the square of the difference between the frequency of the linear oscillator and that of the real one.

Other authors, e.g., Amore and co-workers [25–29] and Pelster et al. [30] have proposed an alternative or improved Linstedt–Poincaré method based on the introduction of a new independent variable so that, in this new variable, the period is 2π, a linear delta expansion in terms of an artificial parameter (δ), and the introduction of a linear stiffness term. By expanding the solution in terms of the artificial parameter, these authors obtained the solution as a series which depends on the unknown frequency of oscillation and a parameter that multiplies the linear stiffness term. By eliminating secular terms at each order in the delta expansion and requiring that the sum of a finite number of terms in the series expansion for the frequency have a relative minimum with respect to the parameter that multiplies the inertia term, Amore and co-workers [25–29] and Pelster et al. [30] obtained approximations to the frequency of oscillation. By way of contrast, the artificial parameter method presented in this paper only introduces a new independent variable which depends on the unknown frequency of oscillation and a linear stiffness term and employs series expansions for the solution, and does not require any minimization procedure to determine the unknown frequency of oscillation.

The paper has been organized as follows. In Section 2, we first review Adomian's decomposition method as applied to nonlinear ordinary differential equations, and then present an artificial parameter technique in both integral and differential forms and show that this technique is identical to Adomian's decomposition method upon setting the artificial parameter to unity. In the third section, the artificial parameter method is formulated for determining the frequency of oscillators with odd nonlinearities. In Section 3, three different formulations of the artificial parameter method based on integral and differential forms and/or the introduction of a new independent variable are presented. Section 3 also contains a comparison between the method presented here and two- [7–9] and three-level [10–14] iterative techniques, whereas, in Section 4, the artificial parameter method is used to determine the frequency of eight (smooth and non-smooth) oscillators with odd nonlinearities and the results are compared with those obtained by means of other techniques. Finally, the last section summarizes the most important findings of the paper.

2. General formulation

Consider the following nonlinear ordinary differential equation:

$$u'' = f(t, u, u'), \quad u(0) = A, \quad u'(0) = B, \tag{1}$$

where the prime denotes differentiation with respect to t, and A and B are constants, and assume that f is infinitely differentiable with respect to u and u' and continuous with respect to t. Under these conditions, the Picard–Lindelof theorem [31–33] ensures a unique solution of Eq. (1). Actually, this theorem ensures a unique solution under the milder conditions that f is a continuous function of t and a Lipschitz-continuous function of u and u', and this solution may be obtained iteratively as

$$u_{k+1}(t) = A + Bt + L^{-1}(f(t, u_k, u_k')) \equiv A + Bt + \int_0^t \mathrm{d}z \int_0^z f(s, u_k(s), u_k'(s))\, \mathrm{d}s, \tag{2}$$

where $k \geqslant 0$ denotes the kth iteration and $L^{-1}(\phi) = \int_0^t \mathrm{d}z \int_0^z \phi\, \mathrm{d}s$.

The double integral in Eq. (2) may be reduced to a single one by integration by parts and the result may be written as

$$u_{k+1}(t) = A + Bt + \int_0^t (t-s) f(s, u_k(s), u_k'(s))\, \mathrm{d}s. \tag{3}$$

Eq. (2) can be written in differential form as

$$u_{k+1}'' = f(t, u_k, u_k'), \quad k \geqslant 0, \quad u_{k+1}(0) = A, \quad u_{k+1}'(0) = B, \tag{4}$$

which corresponds to a two-level fixed-point iterative technique.

Remark 1. Eqs. (3) and (4) provide a sequence, $u_k(t)$, $k \geqslant 0$, which, according to Picard–Lindelof's existence and uniqueness theorem [31–33] converges on $t \in [t_0, t_0 + \beta]$ provided that $f(t, \mathbf{u})$ is continuous with respect to t and uniform Lipschitz continuous with respect to $\mathbf{u}$ on Q, where $t_0 = 0$, $\beta = min(T, b/M)$, $M \equiv max[(\|f(t, \mathbf{u})\| : (t, \mathbf{u}) \in Q]$, $Q \equiv ((t, \mathbf{u}) : t_0 \leqslant t \leqslant t_0 + T, \|\mathbf{u} - \mathbf{u}(0)\| \leqslant b)$, $\mathbf{u} \equiv (u, u')^{\mathrm{T}}$ and the superscript T denotes transpose. This theorem also ensures that $\|\mathbf{u}_{k+1}(t) - \mathbf{u}(0)\| \leqslant b$ for $t \in [t_0, t_0 + \beta]$ if $\|\mathbf{u}_k(t) - \mathbf{u}(0)\| \leqslant b$ for $t \in [t_0, t_0 + \beta]$, and $\|\mathbf{u}_{k+1}(t) - \mathbf{u}_k(t)\| \leqslant ML^k (t - t_0)^{k+1}/(k+1)!$ for $t \in [t_0, t_0 + \beta]$ where L denotes the Lipschitz constant of $f(t, \mathbf{u})$ with respect to $\mathbf{u}$ on Q, i.e., $\|f(t, \mathbf{u}) - f(t, \mathbf{v})\| \leqslant L\|\mathbf{u}(t) - \mathbf{v}(t)\|$ on Q, and, therefore, $\|\mathbf{u}_{k+1}(t) - \mathbf{u}(0)\|$ decreases as k is increased and/or $L(t - t_0)$ is decreased for $t \in [0, \beta]$. Furthermore, for autonomous ordinary differential equations, i.e., $f(t, u, u') = f(u, u')$, using $u_0(t) = u(0) = A$ and $u_0'(t) = u'(0) = B$ (cf. Eq. (1)), Eq. (3) or Eq. (4) provide iterates which may not be power series of t.

2.1. Decomposition method

In order to solve Eq. (1), Adomian [19–21] first integrated this equation to obtain the following Volterra integral equation:

$$u(t) = A + Bt + L^{-1}(f(t,u,u')) \equiv A + Bt + \int_0^t \mathrm{d}z \int_0^z f(s,u(s),u'(s))\,\mathrm{d}s \tag{5}$$

and then proposed a decomposition method which consists in expanding the solution, u, and the nonlinear term f as

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \quad f(t,u,u') = \sum_{n=0}^{\infty} A_n, \tag{6}$$

where A_n are the so-called Adomian's polynomials which can be readily calculated [19–21].

Substitution of Eq. (6) into Eq. (5) and identifying $u_0(t)$ with all the terms that arise from the initial conditions, one can easily obtain

$$u_0(t) = A + Bt \tag{7}$$

and

$$u_{k+1}(t) = L^{-1}(A_k), \quad k \geqslant 0 \tag{8}$$

and one can, in principle, determine the terms of the series for $u(t)$ although the integrals may be cumbersome if f is a highly nonlinear function of either u and u' or t. When this occurs, one may resort to symbolic algebraic package manipulators such as Maple and Mathematica.

Remark 2. A comparison between the iterative Picard–Lindelof method, i.e., Eq. (3) or Eq. (4), and Adomian's decomposition technique, i.e., Eqs. (5)–(8), clearly indicates that the former provides a sequence of iterates for the solution, whereas the latter provides the solution as a series, cf. Eq. (6). Moreover, as stated above, the Picard–Lindelof theorem ensures the local convergence of the iterative Eqs. (3) and (4) provided that $f(t,u,u')$ is continuous with respect to t and Lipschitz continuous with respect to u and u', whereas Adomian's decomposition method demands that $f(t,u,u')$ have an infinite number of partial derivatives with respect to u and u' because such derivatives are required for the determination of Adomian's polynomials, i.e., Eq. (6). Furthermore, if $f(t,u,u')$ is continuous with respect to t and first-order differentiable with respect to u and u' as required by Adomian's decomposition method, then $f(t,u,u')$ is Lipschitz continuous with respect to u and u', and the Picard–Lindelof theorem ensures the local existence and uniqueness of Eq. (1), as well as the convergence of the iterative equations (3) and (4). This means that the decomposition method puts greater demands on $f(t,u,u')$ than those required for the local existence and uniqueness of the solution to Eq. (1) and the convergence of Eq. (3) or Eq. (4).

Remark 3. For autonomous ordinary differential equations, one can easily show that the decomposition method provides a convergent power series solution which coincides with the Taylor series expansion of the solution of Eq. (1) about $t = 0$ [19–21,35]. Such a power series is not very useful, especially for large t. For large t, one can use an analytical continuation procedure whereby the interval of integration $[t_0 = 0, T]$ is divided into a series of non-overlapping intervals, $I_i \equiv [t_i, t_{i+1}]$, $i = 1, 2, \cdots N$, such that $t_1 = t_0 = 0$, $t_{N+1} = T$ and $[0, T] = \bigcup_{i=1}^{N} I_i$, and Eqs. (3) and (4) and Eq. (5) are solved in each interval I_i so that u and u' are continuous at the end points of each interval. The reason for this analytical continuation is that the Picard–Lindelof theorem [31–33] shows that $\|\mathbf{u}_{k+1}(t) - \mathbf{u}_k(t)\| \leqslant ML^k(t - t_i)^{k+1}/(k+1)!$ for $t \in [t_i, \beta_i]$ (cf. Remark 1), provided that $f(t, \mathbf{u})$ is continuous with respect to t and uniform Lipschitz continuous with respect to $\mathbf{u}$ on Q_i, where $\beta_i = min(t_{i+1} - t_i, b_i/M_i)$ and $Q_i \equiv ((t, \mathbf{u}) : t_i \leqslant t \leqslant t_{i+1}, \|\mathbf{u}(t) - \mathbf{u}(0)\| \leqslant b_i)$ and M_i and L_i denote the supremum and the Lipschitz constant of f on Q_i, and, therefore, the convergence is accelerated by decreasing the size of the intervals I_i.

For both autonomous and non-autonomous ordinary differential equations, Adomian's decomposition method may introduce what are referred to as "noise" terms [34], i.e., terms that appear in successive terms of the series solution obtained by the decomposition method and they cancel each other when determining the final series solution for $u(t)$. This phenomenon may cause problems in determining the convergence of the decomposition method, if it does converge at all. Although, some authors have claimed to have derived the necessary conditions for the appearance of noise terms in decomposition series solutions [36], it turns out that such an appearance is related to both the differential equation and the manner in which the decomposition method is applied [35], e.g., it depends on whether Eq. (1) is treated as such or is written as two first-order differential equations, and these terms may be absent if a judicious implementation of the decomposition method is employed as illustrated by the author for the Lane–Emden equation [35]. In addition, since the decomposition method provides a series solution, it is, in general, very difficult to establish its convergence or the number of terms required to achieve a specified accuracy. On the other hand, Picard's iterative technique clearly indicates that, provided that $f(t, u, u')$ is Lipschitz continuous, there is a unique solution to Eq. (1) which may be obtained iteratively (cf. Eq. (3) or Eq. (4)), and the number of iterations required by Picard's iterative procedure to achieve a user's specified convergence tolerance is related to the product $L(t_{i+1} - t_i)$ and, therefore, may be controlled by employing sufficiently small intervals I_i. It must also be noted that the decomposition method may also exhibit very slow convergence in some cases [37] and that is very difficult to prove its convergence for general equations despite some claims to the contrary based on the use of Banach's fixed-point theory and contractive mappings arguments, e.g., Refs. [38–43]. Such claims have been based on rather strong assumptions such as, for example, that the nonlinear terms can be expanded as entire series with an infinite radius of convergence. Such as an assumption is the same as that of the Cauchy–Kovaleskaya theorem for the solution of initial-value (or Cauchy) problems which states that if f in Eq. (1) is an analytic function of its arguments in a neighborhood of $(t_0 = 0, u(0) = A, u'(0) = B)$, the solution of Eq. (1) can be expanded in a series of non-negative integer powers of $(t - t_0)$, and such a solution exists and is unique. As stated previously in Remark 1, the Picard–Lindelof theorem ensures the existence of a local unique solution to Eq. (1) under much milder conditions on f than that of analyticity. Moreover, for functions, f, in Eq. (1), which are not analytic, it may stated that the decomposition method cannot be applied.

It must also be pointed out that there are no rigorous and formal proofs on the convergence of modified Linstedt–Poincaré methods based on the introduction of a linear stiffness term and artificial parameters, the elimination of secular terms, and the use of variational principles that minimize either the absolute value [22] or the square [23] of the difference between the frequency of the linear oscillator and that of the real one, or require that the sum of a finite number of terms in the series expansion for the frequency have a relative minimum with respect to the parameter that multiplies the inertia term [25–29]. For example, in modified Linstedt–Poincaré methods based on the minimization of the frequency with respect to the artificial parameter that multiplies the inertia terms, one cannot even show rigorously that the resulting series for the frequency converges, neither can one state à priori the number of terms of the series required to achieve a user's specified accuracy.

2.2. Artificial parameter method: integral formulation

In this section, we present an artificial parameter method for the solution of Eq. (1) and show that this technique is identical to Adomian's decomposition method upon setting the artificial parameter to unity.

Eq. (1) can be written as the following nonlinear Volterra integral equation:

$$u(t) = A + Bt + \int_0^t (t - s) f(s, u(s), u'(s))\,\mathrm{d}s. \tag{9}$$

We now introduce an artificial parameter p in Eq. (9) as

$$u(t) = A + Bt + p\int_0^t (t - s) f(s, u(s), u'(s))\,\mathrm{d}s, \tag{10}$$

so that Eq. (10) is identically equal to Eq. (9) when $p = 1$ and assume that the solution to Eq. (10) can be obtained as a series expansion of p as

$$u(t) = \sum_{n=0}^{\infty} p^n u_n(t). \tag{11}$$

Substitution of Eq. (11) into Eq. (10), expansion of $f(t,u,u')=f(t,\sum_{n=0}^{\infty}p^n u_n(t),\sum_{n=0}^{\infty}p^n u_n'(t))$ about (t,u_0,u_0') and setting the coefficients of the monomials p^n, $n\geqslant 0$, in the resulting series to zero, one obtains a sequence of equations that may be written as

$$u_0(t)=A+Bt,\quad u_1(t)=\int_0^t (t-s)f(s,u_0(s),u_0'(s))\,\mathrm{d}s \tag{12}$$

and

$$u_{k+1}(t)=\int_0^t (t-s)A_k\,\mathrm{d}s=\int_0^t \mathrm{d}z\int_0^z A_k\,\mathrm{d}s=L^{-1}(A_k),\quad k\geqslant 1, \tag{13}$$

which coincide with Eqs. (7) and (8) of the Adomian decomposition method and the solution, i.e., Eq. (11), coincides with that of the decomposition method, i.e., Eq. (6), upon setting $p=1$ in Eq. (11). We, therefore, have proved the following theorem.

Theorem 1. *The artificial parameter method presented above is identical to Adomian's decomposition method.*

In addition, we have

Theorem 2. *The Adomian's polynomials, A_k, $k=0,1,2,\ldots$, can be obtained from the expansion of $f(t,u,u')=f(t,u_0(t)+pu_1(t)+p^2u_2(t)+O(p^3),u_0'(t)+pu_1'(t)+p^2u_2'(t)+O(p^3))$ about $(t,u_0(t),u_0'(t))$ and correspond to the coefficients of the monomials p^k, $k=0,1,2,\ldots$, in the resulting expansion.*

Proof. Using Eq. (11) and expanding $f(t,u,u')$ about $(t,u_0(t),u_0'(t))$, one can obtain

$$f(t,u,u')=A_0(t)+pA_1(t)+p^2A_2(t)+p^3A_3(t)+O(p^4), \tag{14}$$

where

$$\begin{aligned} A_0(t)&=f(t,u_0,u_0'),\quad A_1=\frac{\partial f}{\partial u}(t,u_0,u_0')u_1(t)+\frac{\partial f}{\partial u'}(t,u_0,u_0')u_1'(t),\\ A_2&=\frac{\partial f}{\partial u}(t,u_0,u_0')u_2(t)+\frac{1}{2!}\frac{\partial^2 f}{\partial u^2}(t,u_0,u_0')u_1^2(t)+\frac{\partial f}{\partial u'}(t,u_0,u_0')u_2(t)\\ &\quad+\frac{1}{2!}\frac{\partial^2 f}{\partial u'^2}(t,u_0,u_0')u_1'^2(t)+\frac{\partial^2 f}{\partial u\partial u'}(t,u_0,u_0')u_1u_1',\ldots, \end{aligned} \tag{15}$$

which are the Adomian's polynomials [19–21] and coincide with the coefficients of the monomials p^n, $n\geqslant 0$, in the series expansion of $f(t,u,u')$ about $(t,u_0(t),u_0'(t))$. □

Remark 4. Since, in Theorem 1, it has been shown that the artificial parameter method presented in this section is identical to Adomian's decomposition method and that the convergence of the latter can only be ensured rigorously for the case that f in Eq. (1) is an analytic function of its arguments, the convergence of the former cannot be proved in a formal and rigorous manner and has to be established in a case-by-case manner.

2.3. Artificial parameter method: differential formulation

Instead of working with the nonlinear Volterra integral Eq. (10), one can differentiate this equation twice with respect to t, make use of Leibnitz's rule and eliminate the integral that appears in Eq. (10) by using that equation and the corresponding one for u'' to obtain

$$u''=pf(t,u,u'),\quad u(0)=A,\quad u'(0)=B, \tag{16}$$

which is exactly the same equation as the one that one would obtain by introducing the parameter p in the right-hand side of Eq. (1).

By expanding $u(t)$ as in Eq. (11) and the nonlinear function f about (t,u_0,u_0'), substituting these expansions into Eq. (16) and setting the coefficients of the monomials p^n, $n\geqslant 0$, in the resulting series to zero, one can obtain a hierarchy of equations such as

$$u_0''(t)=0,\quad u(0)=A,\quad u'(0)=B, \tag{17}$$

$$u_1'' = f(t, u_0, u_0'), \quad u_1(0) = 0, \quad u_1'(0) = 0, \ldots, \tag{18}$$

whose solutions are Eqs. (12) and (13), and, therefore, the solution of Eq. (1) is obtained by setting $p = 1$ in Eq. (11).

Remark 5. The Adomian's decomposition method reviewed above can also be written in differential form by simply differentiating Eqs. (12) and (13) twice with respect to t, to obtain

$$u_0'' = 0, \quad u_0(0) = A, \quad u_0'(0) = B \tag{19}$$

and

$$u_k'' = A_k, \quad u_k(0) = 0, \quad u_k'(0) = 0, \quad k \geqslant 1, \tag{20}$$

which coincide with Eqs. (17) and (18).

Remark 6. As indicated above, the artificial parameter and decomposition methods described above provide series solutions of $u(t)$ in terms of t, cf. Eqs. (11) and (6), respectively, which are not power series of t. But even if they were power series of t, it may be difficult to deduce that, when Eq. (1) has, for example, trigonometric solutions, the resulting series correspond to these trigonometric solutions.

3. Formulation for oscillators with odd nonlinearities

In this section, the artificial parameter method presented in the previous section is formulated for obtaining the limit cycles of oscillators with odd nonlinearities governed by

$$u'' = f(u, u'), \quad u(0) = A, \quad u'(0) = 0, \tag{21}$$

where $f(-u, -u') = -f(u, u')$. To this end we first introduce a linear stiffness term with an unknown (constant) frequency ω in both sides of Eq. (21) as

$$u'' + \omega^2 u = f(u, u') + \omega^2 u \equiv g(u, u'; \omega), \quad u(0) = A, \quad u'(0) = 0 \tag{22}$$

and apply the method of variation of parameters to Eq. (22) to obtain

$$u(t) = A\cos(\omega t) + \frac{1}{\omega}\int_0^t \sin(\omega(t-s))g(u(s).u'(s), \omega)\,\mathrm{d}s. \tag{23}$$

We now introduce an artificial parameter p in Eq. (23) so that this equation can be written as

$$u(t) = A\cos(\omega t) + p\frac{1}{\omega}\int_0^t \sin(\omega(t-s))g(u(s), u'(s), \omega)\,\mathrm{d}s, \tag{24}$$

which coincides with Eq. (23) upon setting $p = 1$, and assume that the solution can be expressed as in Eq. (11). By substituting Eq. (11) into Eq. (24), expanding $g(u, u'; \omega)$ about (u_0, u_0') and setting the coefficients of the monomials p^n, $n \geqslant 0$, in the resulting series to zero, it is an easy exercise to show that

$$u_0(t) = A\cos(\omega t), \quad u_1(t) = \frac{1}{\omega}\int_0^t \sin(\omega(t-s))g(u_0(s), u_0'(s), \omega)\,\mathrm{d}s, \ldots. \tag{25}$$

In order to determine the frequency of the limit cycle, we require that $u_k(t)$, $k \geqslant 1$, do not contain secular terms. Note that the solution of Eq. (21) is obtained from Eq. (11) upon setting $p = 1$ in that equation.

Remark 7. Instead of working with integral expressions as in Eq. (24), one can eliminate the integral term that appears in Eq. (24) by using this equation and its second-order derivative with respect to time to obtain

$$u'' + \omega^2 u = pg(u, u'; \omega), \quad u(0) = A, \quad u'(0) = 0, \tag{26}$$

which is identical to Eq. (22) when p is introduced in the right-hand side of that equation.

Eq. (26) can be solved by employing Eq. (11), expanding $g(u, u'; \omega)$ about (u_0, u_0') and setting the coefficients of the monomials p^n, $n \geqslant 0$, in the resulting series to zero to obtain

$$u_0'' + \omega^2 u_0 = 0, \quad u_0(0) = A, \quad u_0'(0) = 0, \tag{27}$$

$$u_1'' + \omega^2 u_1 = g(u_0, u_0'; \omega), \quad u_1(0) = 0, \quad u_1'(0) = 0, \tag{28}$$

whose solutions are Eq. (25) and, for the determination of the frequency of the limit cycle, one demands that $u_k(t)$, $k \geqslant 1$, be free from secular terms. The solution to Eq. (1) is obtained by setting $p = 1$ in that equation.

Remark 8. Instead of working with the independent variable t, one may work with $\theta = \omega t$, so that Eq. (26) may be written as

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{\omega^2} f\left(u, \frac{du}{d\theta}\right) + u \equiv G\left(u, \frac{du}{d\theta}; \omega\right), \quad u(0) = A, \quad \frac{du}{d\theta}(0) = 0, \tag{29}$$

which, upon using

$$u(\theta) = \sum_{n=0}^{\infty} p^n u_n(\theta), \tag{30}$$

expansion of $G(u, du/d\theta; \omega)$ about $(u_0, du_0/d\theta)$ and setting the coefficients of the monomials p^n, $n \geqslant 0$, in the resulting series to zero yields

$$\frac{d^2u_0}{d\theta^2} + u_0 = 0, \quad u_0(0) = A, \quad \frac{du_0}{d\theta}(0) = 0. \tag{31}$$

$$\frac{d^2u_1}{d\theta^2} + u_1 = G\left(u_0, \frac{du_0}{d\theta}; \omega\right), \quad u_1(0) = 0, \quad \frac{du_1}{d\theta}(0) = 0, \ldots, \tag{32}$$

whose solutions are

$$u_0(t) = A\cos(\theta), \quad u_1(t) = \frac{1}{\omega^2} \int_0^{\theta} \sin(\theta - \mu) G(u_0(\mu), u_0'(\mu); \omega)\, d\mu, \ldots, \tag{33}$$

which are identical to Eq. (25), and, for the determination of the frequency of the limit cycle, one demands that $u_k(\theta)$, $k \geqslant 1$, be free from secular terms. The solution to Eq. (21) is obtained by setting $p = 1$ in that equation.

Remark 9. Eq. (23) may be solved iteratively by writing it as

$$u_{k+1}(t) = A\cos(\omega t) + \frac{1}{\omega} \int_0^t \sin(\omega(t - s)) g(u_k(s), u_k'(s); \omega)\, ds, \quad k \geqslant 0 \tag{34}$$

and this iterative technique converges if $g(u, u'; \omega)$ is a Lipschitz continuous function of u and u' as a consequence of the Picard–Lindelof's theorem [31–33], as indicated previously in Section 2.

Eq. (34) can also be written as

$$u_{k+1}'' + \omega^2 u_{k+1} = g(u_k, u_k''; \omega), \quad k \geqslant 0, \quad u(0) = A, \quad u''(0) = 0, \tag{35}$$

which has been previously used to determine the limit cycles of oscillators with odd nonlinearities [7–9] upon imposing the condition that u_k be free from secular terms for $k \geqslant 1$.

If $f(u, u')$ is differentiable with respect to u and u', one may approximate $g(u_k, u_k'; \omega)$ in Eq. (35) by its linearization about u_{k-1} and u_{k-1}' to obtain

$$\begin{aligned} u_{k+1}'' + \omega^2 u_{k+1} = {} & g(u_{k-1}, u_{k-1}') + \frac{\partial g}{\partial u}(u_{k-1}, u_{k-1}')(u_k - u_{k-1}) \\ & + \frac{\partial g}{\partial u'}(u_{k-1}, u_{k-1}')(u_k' - u_{k-1}'), \quad k \geqslant 0, \quad u_{k+1}(0) = A, \quad u_{k+1}'(0) = 0, \end{aligned} \tag{36}$$

which corresponds to a three-level iterative technique and has been previously applied for the determination of the limit cycle of oscillators with odd nonlinearities upon imposing the condition that u_k for $k \geqslant 1$ does not contain secular terms [10–14]. This three-level iterative technique demands that $\partial g/\partial u$ and $\partial g/\partial u'$ exist and the Picard–Lindelof's theorem [31–33] ensures the convergence of the iterative Eq. (36) provided that g, $\partial g/\partial u$ and $\partial g/\partial u'$ be Lipschitz continuous with respect to u and u'.

Eqs. (35) and (36) can also be written in integral form

$$u_{k+1}(t) = A\cos(\omega t) + \int_0^t \sin(\omega(t-s))g(u_k, u'_{k-1})\,\mathrm{d}s, \quad k \geqslant 0 \tag{37}$$

and

$$u_{k+1}(t) = A\cos(\omega t) + \int_0^t \mathrm{d}z \sin(\omega(t-s))$$

$$\left(g(u_{k-1}, u'_{k-1}) + \frac{\partial g}{\partial u}(u_{k-1}, u'_{k-1})(u_k - u_{k-1}) + \frac{\partial g}{\partial u'}(u_{k-1}, u'_{k-1})(u'_k - u'_{k-1})\right)\mathrm{d}s, \quad k \geqslant 0, \tag{38}$$

respectively. Therefore, we have shown that

Theorem 3. *Two- and three-level iterative techniques for the solution of Eq.* (1) *correspond to the Picard–Lindelof method applied to the original differential equation and the linearization of the nonlinear terms with respect to the previous and next to the previous iteration, respectively*.

Remark 10. If, in the two- and three-level iterative methods, one chooses $u_0(t) = A\cos(\omega t)$ and $u_{-1}(t) = u_0(t) = A\cos(\omega t)$, respectively, then Eqs. (35) and (36) for $k = 0$ are identical to Eq. (28) and, therefore, these iterative methods predict the same $u_1(t)$ as the artificial parameter-decomposition method presented in this paper. Moreover, since these three techniques require that u_1 does not contain secular terms, they predict the same frequency of the limit cycle for $k = 1$. However, as indicated above, iterative techniques provide a sequence of iterates to the solution, whereas the artificial parameter method presented in this paper is based on a series expansion, i.e., Eq. (11) or Eq. (30), and setting $p = 1$ into that expansion.

Remark 11. The two-level iterative technique is a special case of the quasilinearization method developed by Bellman and Kalaba [44] and Lakshmikantham [45] which linearizes Eq. (22) as

$$u''_{k+1} + \omega^2 u_{k+1} = g(u_k, u'_k; \omega) + \frac{\partial g}{\partial u}(u_k, u'_k; \omega)(u_{k+1} - u_k) + \frac{\partial g}{\partial u'}(u_k, u'_k; \omega)(u'_{k+1} - u'_k), \quad k \geqslant 0 \tag{39}$$

and, therefore, requires that $g(u, u'; \omega)$ be differentiable with respect to u and u', and results in linear ordinary differential equations with, in general, time-dependent coefficients, provided that one imposes the condition of non-secular terms in $u_k(t)$, $k \geqslant 1$. However, the quasilinearization method is more general than the two- and three-level iterative procedures mentioned above and may be applied to obtain the solution of nonlinear ordinary differential equations, although it may have to be implemented numerically in either an iterative [46,47] or a time-linearized [48–51] fashion. It must be noted that if $g(u, u')$ is differentiable with respect to u and u', then the Picard–Lindelof theorem ensures the local existence and uniqueness of the solution of Eqs. (21), (26) and (34), as indicated previously in Section 2.

Remark 12. In nonlinear dynamical systems, non-dimensionalization of the governing equations usually results in the appearance of small parameters, the presence of which can be used to obtain series solutions in terms of small parameters, e.g., the method of multiple scales, the Linstedt–Poincaré technique, etc. [1,2]; these series solutions are usually asymptotic. When there are no small parameters in the governing equations, one may resort to numerical techniques, iterative procedures or methods such as the one presented in this paper. Numerical methods including quasilinearization [44–47] and piecewise time linearization [48–51] are very useful but they do require a lot of thought and work for determining, for example, the dependence of the frequency on the amplitude for nonlinear oscillators. Two-[7–9] and three-level [10–14] iterative techniques have the advantage that, provided that the conditions of the Picard–Lindelof theorem or Banach's fixed-point theory for contractive mappings are satisfied, they ensure local convergence to the solution and that analytical continuation may be used to determine the solution as indicated previously. Their main disadvantage, however, is that it is difficult to determine analytically iterates greater than the second one even when one employs symbolic algebraic package manipulators such as Maple and Mathematica, unless the differential equation to be solved is simple enough. On the other hand, the artificial parameter method presented in this paper and Adomian's decomposition technique provide power series approximations when used with the formulation presented in Section 2 and the convergence of the resulting series has to be dealt with in a case-by-case basis because there are no formal and rigorous proofs on the convergence of these methods, except for autonomous ordinary differential equations for which it can be easily shown that these techniques provide a series solutions identical to the Taylor series expansion of the solution. However, as indicated above, such a series is not very useful for long times even when analytical continuation is employed and does not allow, for example, to identify easily a limit cycle when it exists.

Remark 13. As one of the referees of this paper pointed out, the choice of an artificial parameter for the determination of the solution is not unique. For example, one could use $p_1 = p$ as above, $p_2 = 2p/(p+1)$, $p_3 = \ln(1+p)/\ln(2)$, etc., because $p_1 = p_2 = p_3 = 1$ for $p = 1$ and the parameter is set to unity once the series solution has been determined (cf. Sections 2 and 3). Since there are no formal and rigorous proofs for the convergence of the artificial parameter technique presented in this paper (cf. Remarks 3 and 4), one cannot state what is the best parameter to determine the solution of single-degree-of-freedom problems. The use of $p_1 = p$ above and in the examples presented in the next section has been motivated by the author's experience with nonlinear dynamical systems and perturbation methods, as well as for simplicity.This remark also applies to other modified Linstedt–Poincaré methods based on the introduction of a linear stiffness term and artificial parameters, the elimination of secular terms, and the use of variational principles that minimize either the absolute value [22] or the square [23] of the difference between the frequency of the linear oscillator and that of the real one, or require that the sum of a finite number of terms in the series expansion for the frequency have a relative minimum with respect to the parameter that multiplies the inertia term [25–29].

4. Applications

In this section, the artificial parameter-decomposition method presented in Section 3 is applied to eight oscillators with odd nonlinearities in the displacement and the results are compared with those predicted by other techniques. Five of the oscillators considered in this section are smooth, whereas the other three are not.

Example 1. This example corresponds to

$$u'' = -\text{sign}(u), \quad u(0) = A, \quad u'(0) = 0, \tag{40}$$

where sign(u) is $+1$ and -1 for $u>0$ and $u<0$, respectively, which can be written as Eq. (22) with $g(u, u'; \omega) = \omega^2 u - \text{sign}(u)$.

By using either the integral or differential formulation of the artificial parameter-decomposition method presented above, it is easy to show that

$$u_0(t) = A\cos(\omega t) \tag{41}$$

and

$$u_1'' + \omega^2 u_1 = \omega^2 u_0 - \text{sign}(u_0), \quad u_1(0) = 0, \quad u_1'(0) = 0. \tag{42}$$

The absence of secular terms in u_1 requires that the right-hand side of Eq. (42) be orthogonal to the solution of the corresponding homogeneous equation in $[0, 2\pi/\omega]$ and this condition can be implemented by expanding $\text{sign}(u_0) = \text{sign}(A\cos(\omega t))$ in Fourier series as

$$\text{sign}(A\cos(\omega t)) = \frac{4}{\pi}\cos(\omega t) + \sum_{k=1}^{\infty} a_{2k+1}\cos((2k+1)\omega t) \tag{43}$$

and results in

$$\omega^2 = \frac{4}{\pi A}. \tag{44}$$

Eq. (44) is identical to the result that can be obtained from a harmonic balance method upon approximating u by $A\cos(\omega t)$. It also coincides with the result of a modified Linstedt–Poincaré method [52] based on writing Eq. (40) as $u'' + 0u = -p\,\text{sign}(u)$ and expanding the solution u and the coefficient 0 in series of the parameter p. The exact value of $\omega A^{1/2}$ is 1.110721 and that of Eq. (44) is 1.12838, whereas that obtained by means of a three-level iterative technique is 1.11003 [13] and the first, second and third approximations of a linearized harmonic balance technique are $2/\sqrt{\pi} \approx 1.12838$, $\sqrt{104/27\pi} \approx 1.10729$ and $\sqrt{13108/3375\pi} \approx 1.11188$, respectively [15].

Example 2. This example corresponds to

$$u'' = -\alpha u - \varepsilon u^3, \quad u(0) = A, \quad u'(0) = 0, \tag{45}$$

which is first written as

$$\frac{d^2u}{d\theta^2} + u = u - \frac{\alpha}{\omega^2}u - \frac{\varepsilon}{\omega^2}u^3, \quad u(0) = A, \quad \frac{du}{d\theta}(0) = 0, \tag{46}$$

where $\theta = \omega t$ and $\alpha \geqslant 0$ is a constant.

Application of the artificial parameter-decomposition method to Eq. (46) yields

$$u_0(\theta) = A\cos(\theta). \tag{47}$$

and

$$\frac{d^2u_1}{d\theta^2} + u_1 = A\cos(\theta) - \frac{\alpha A}{\omega^2}\cos(\theta) - \frac{\varepsilon^3}{4\omega^2}(\cos(3\theta)) + \cos(\theta)), \quad u_1(0) = 0, \quad \frac{du_1}{d\theta}(0) = 0. \tag{48}$$

The absence of secular terms in u_1 requires that

$$\omega^2 = \alpha + \tfrac{3}{4}\varepsilon A^2, \tag{49}$$

which is the same frequency as that resulting from the application of the harmonic balance method to Eq. (45) upon assuming that $u(t) = A\cos(\omega t)$.

For $\alpha = 0$, Eq. (49) provides the same frequency as the one obtained by a modified Linstedt–Poincaré method [53] based on writing Eq. (45) as $u'' + 0u + \varepsilon u^3 = 0$ and expanding both u and the coefficient 0 in series of ε. It is also identical to that of a three-level iterative technique [10] which, at second order, yields, for $\varepsilon = 1$, $\omega^2 = 66A^2/92$, whereas the first- and second-order approximations of a linearized harmonic balance method yield $\omega = 2\pi A/7.2552$ and $2\pi A/7.4278$, respectively [18], for $\varepsilon = 1$, and the Linstedt–Poincaré method yields $\omega = 0.84695A$ [54], for $\varepsilon = 1$. Hu [54] also obtained $\omega^2 = \frac{3}{4}\varepsilon A^2 - 3\varepsilon^2 A^4/128\omega^2$ by means of a modified Linstedt–Poincaré method based on writing Eq. (45) as $u'' + 0u + \varepsilon u^3 = 0$ and expanding the solution and 0 in series of ε. Note that the method proposed in this paper is based on the expansion of the solution in power series of the book-keeping parameter p which is not related at all to any small parameter that may appear in the ordinary differential equation under study.

For $\alpha = 1$, Eq. (49) yields the same value as the one that results from the first-order approximation obtained by a modified Linstedt–Poincaré method based on the expansion of both the solution and the coefficient of the linear stiffness term in Eq. (45), i.e., $\alpha = 1$, in terms of ε [3], the first iteration of a two-level iterative technique [7], the first iteration of a three-level iterative method [11], a convolution integral method [9], an equivalent linearization procedure [55] and the standard and modified Linstedt–Poincaré methods [56].

Upon imposing Eq. (49), Eq. (48) becomes

$$\frac{d^2u_1}{d\theta^2} + u_1 = -\frac{\varepsilon A^3}{\omega^2}\cos(3\theta), \quad u_1(0) = 0, \quad \frac{du_1}{d\theta}(0) = 0, \tag{50}$$

whose solution is

$$u_1(\theta) = \frac{\varepsilon A^3}{32\omega^2}(\cos(3\theta) - \cos(\theta)). \tag{51}$$

Example 3. This example corresponds to

$$u'' = -\frac{u^3}{1+u^2}, \quad u(0) = A, \quad u'(0) = 0, \tag{52}$$

which is first written as

$$\frac{d^2u}{d\theta^2} + u = u - \frac{1}{\omega^2}u^3 - u^2\frac{d^2u}{d\theta^2}, \quad u(0) = A, \quad \frac{du}{d\theta}(0) = 0, \tag{53}$$

where $\theta = \omega t$.

Application of the artificial parameter-decomposition method to Eq. (53) yields

$$u_0(\theta) = A\cos(\theta) \tag{54}$$

and

$$\frac{d^2u_1}{d\theta^2} + u_1 = A\cos(\theta) - \frac{A^3}{4\omega^2}(3\cos(\theta) + \cos(3\theta)) + \frac{A^3}{4}(3\cos(\theta) + \cos(3\theta)), \quad u_1(0) = 0, \quad \frac{du_1}{d\theta}(0) = 0 \tag{55}$$

and the absence of secular terms requires that

$$\omega^2 = \frac{3A^2}{4 + 3A^2}, \tag{56}$$

which is the same frequency as that resulting from the application of the harmonic balance method to Eq. (52) upon assuming that $u(t) = A\cos(\omega t)$ and the same result as that obtained with a Ritz procedure [57–59]. On the other hand, a three-level iterative method where the linearization is performed with respect to u_0 rather than with respect to u_k predicts $\omega^2 = a_1/A$ where a_1 can be obtained from $u_0^3/(1 + u_0^2) = \sum_{k=0}^{\infty} a_{2k+1}\cos((2k+1)\theta)$ and depends on A [60]. The same result, i.e., $\omega^2 = a_1/A$, is obtained from the application of a linearized harmonic balance method [61] and an iterative harmonic balance-Newton procedure [62].

Upon employing Eq. (56), Eq. (55) becomes

$$\frac{d^2u_1}{d\theta^2} + u_1 = \frac{A^3}{4}\left(1 - \frac{1}{\omega^2}\right)\cos(3\theta), \quad u_1(0) = 0, \quad \frac{du_1}{d\theta}(0) = 0, \tag{57}$$

whose solution is

$$u_1(\theta) = -\frac{A^3}{32}\left(1 - \frac{1}{\omega^2}\right)(\cos(3\theta) - \cos(\theta)). \tag{58}$$

Example 4. This example corresponds to

$$u'' = -|u|u, \quad u(0) = A, \quad u'(0) = 0, \tag{59}$$

which is first written as

$$\frac{d^2u}{d\theta^2} + u = u - \frac{1}{\omega^2}|u|u, \quad u(0) = A, \quad \frac{du}{d\theta}(0) = 0, \tag{60}$$

where $\theta = \omega t$.

Application of the artificial parameter-decomposition method to Eq. (60) yields

$$u_0(\theta) = A\cos(\theta) \tag{61}$$

and

$$\frac{d^2u_1}{d\theta^2} + u_1 = u_0 - \frac{1}{\omega^2}|u_0|u_0, \quad u_1(0) = 0, \quad \frac{du_1}{d\theta}(0) = 0 \tag{62}$$

and the absence of secular terms in $u_1(\theta)$ requires that the right-hand side of Eq. (62) be orthogonal in $[0, 2\pi]$ to the solutions of the corresponding homogeneous differential equation. This orthogonality condition can be applied by first expanding $|u_0|$ in Fourier series as

$$|u_0| = A\left(\frac{2}{\pi} + \frac{4}{3\pi}\cos(2\theta) + \sum_{k=2}^{\infty} a_{2k}\cos(2k\theta)\right) \tag{63}$$

and yields

$$\omega^2 = \frac{8A}{3\pi}, \tag{64}$$

which is the same frequency as that resulting from the application of the harmonic balance method to Eq. (59) upon assuming that $u(t) = A\cos(\omega t)$ and using Eq. (63), and the same value as the one that results from the application of an equivalent linearization technique to Eq. (59) [55].

Example 5. This example corresponds to

$$u'' = -\frac{1}{\varepsilon u}, \quad u(0) = A, \quad u'(0) = 0, \tag{65}$$

which is first written as

$$\frac{d^2u}{d\theta^2} + u = u - \frac{1}{\omega^2}u - \varepsilon u^2\frac{d^2u}{d\theta^2} + \frac{d^2u}{d\theta^2}, \quad u(0) = A, \quad \frac{du}{d\theta}(0) = 0, \tag{66}$$

where $\theta = \omega t$.

Application of the artificial parameter-decomposition method to Eq. (66) yields

$$u_0(\theta) = A\cos(\theta) \tag{67}$$

and

$$\frac{d^2u_1}{d\theta^2} + u_1 = \left(-\frac{A}{\omega^2} + \frac{3\varepsilon A^3}{4}\right)\cos(\theta) + \frac{\varepsilon A^3}{4}\cos(3\theta), \quad u_1(0) = 0, \quad \frac{du_1}{d\theta}(0) = 0. \tag{68}$$

The absence of secular terms in u_1 requires that

$$\omega^2 = \frac{4}{3\varepsilon A^2}, \tag{69}$$

which is the same frequency as that resulting from the application of the harmonic balance method to (cf. Eq. (65)) $\varepsilon u^2 u'' = -u$ upon assuming that $u(t) = A\cos(\omega t)$. It is also the same value as the one that results upon writing Eq. (65) as $0u'' + 1u + \varepsilon u^2 u'' = 0$ and expanding u and the coefficients 0 and 1 in series of ε [53].

Upon employing Eq. (69), Eq. (68) becomes

$$\frac{d^2u_1}{d\theta^2} + u_1 = \frac{\varepsilon A^3}{4}\cos(3\theta), \quad u_1(0) = 0, \quad \frac{du_1}{d\theta}(0) = 0, \tag{70}$$

whose solution is

$$u_1(\theta) = -\frac{\varepsilon A^3}{32}(\cos(3\theta) - \cos(\theta)). \tag{71}$$

Example 6. This example corresponds to

$$u'' = -\frac{u}{1+\varepsilon u^2}, \quad u(0) = A, \quad u'(0) = 0, \tag{72}$$

which is first written as

$$\frac{d^2u}{d\theta^2} + u = u - \frac{1}{\omega^2}u - \varepsilon u^2\frac{d^2u}{d\theta^2}, \quad u(0) = A, \quad \frac{du}{d\theta}(0) = 0, \tag{73}$$

where $\theta = \omega t$.

Application of the artificial parameter-decomposition method to Eq. (73) yields

$$u_0(\theta) = A\cos(\theta) \tag{74}$$

and

$$\frac{d^2u_1}{d\theta^2} + u_1 = \left(A - \frac{A}{\omega^2} + \frac{3\varepsilon A^3}{4}\right)\cos(\theta) + \frac{\varepsilon A^3}{4}\cos(3\theta), \quad u_1(0) = 0, \quad \frac{du_1}{d\theta}(0) = 0. \tag{75}$$

The absence of secular terms in u_1 requires that

$$\omega^2 = \frac{4}{4 + 3\varepsilon A^2}, \tag{76}$$

which is the same frequency as that resulting from the application of the harmonic balance method to Eq. (72) upon assuming that $u(t) = A\cos(\omega t)$. It is also exactly the same result as that of a modified Linstedt–Poincaré method based on writing Eq. (72) as $u'' + 1u + \varepsilon u^2 u'' = 0$ and expanding the solution and the coefficient 1 in series of ε [3]. By way of contrast, the asymptotic method presented in this paper expands the solution in terms of a book-keeping parameter p which is set to unity at the end of the calculation as indicated in Section 3.

Using Eq. (76), Eq. (75) becomes

$$\frac{d^2u_1}{d\theta^2} + u_1 = \frac{\varepsilon A^3}{4}\cos(3\theta), \quad u_1(0) = 0, \quad \frac{du_1}{d\theta}(0) = 0, \tag{77}$$

whose solution is

$$u_1(\theta) = -\frac{\varepsilon A^3}{32}(\cos(3\theta) - \cos(\theta)). \tag{78}$$

Example 7. This example corresponds to

$$u'' = -u^{\frac{1}{3}}, \quad u(0) = A, \quad u'(0) = 0, \tag{79}$$

which is first written as

$$\frac{d^2u}{d\theta^2} + u = u - \frac{1}{\omega^2}u^{\frac{1}{3}}, \quad u(0) = A, \quad \frac{du}{d\theta}(0) = 0, \tag{80}$$

where $\theta = \omega t$.

Application of the artificial parameter-decomposition method to Eq. (80) yields

$$u_0(\theta) = A\cos(\theta) \tag{81}$$

and

$$\frac{d^2u_1}{d\theta^2} + u_1 = u_0 - \frac{1}{\omega^2}u_0^{\frac{1}{3}}, \quad u_1(0) = 0, \quad \frac{du_1}{d\theta}(0) = 0. \tag{82}$$

The solution $u_1(\theta)$ is free from secular terms provided that the right-hand side of Eq. (82) is orthogonal to the corresponding solutions of the homogeneous equation in $[0, 2\pi]$. This condition has been applied after expanding $(\cos(\theta))^{\frac{1}{3}}$ in Fourier series as

$$(\cos(\theta))^{\frac{1}{3}} = a_1\left(\cos(\theta) + \sum_{k=1}^{\infty} a_{2k+1}\cos((2k+1)\theta)\right), \tag{83}$$

where $a_1 a_{2n+1} = 3\Gamma(\frac{7}{3})/(2^{\frac{4}{3}}\Gamma(n+\frac{5}{3})\Gamma(\frac{2}{3}-n))$, $n \geqslant 1$, or $a_1 = 1.159595266963929\ldots$, $a_3 = -\frac{1}{5}$, $a_5 = \frac{1}{10}$, $a_7 = -\frac{7}{110}$, $a_9 = \frac{1}{22}$, $a_{11} = -\frac{13}{374}, \ldots$, and yields

$$\omega^2 = \frac{a_1}{A^{\frac{2}{3}}}, \tag{84}$$

i.e., $\omega A^{1/3} = 1.076845$, which is the same frequency as that resulting from the application of the harmonic balance method to Eq. (79) upon assuming that $u(t) = A\cos(\omega t)$ and using Eq. (83).

The exact value of $\omega A^{\frac{1}{3}}$ is $\sqrt{\pi}\Gamma(\frac{1}{4})/2\sqrt{6}\Gamma(\frac{3}{4}) \approx 1.070451$ [8].

The value of $\omega A^{1/3}$ predicted by first- [63] and second-order [64] harmonic balance methods, the first iteration of two- [8] and three-level [12] iterative techniques, the second iteration of a three-level iterative method [12], and an equivalent linearization method [55] are 1.049115, 1.06349, 1.07685, 1.0704, 1.07685 and 1.076, respectively.

The value of the leading-order frequency reported above is also identical to that obtained by introducing $\omega^2 u$ in both sides of Eq. (79), applying a convolution technique to the resulting equation, approximating the solution by the term $u_0(t) = A\cos(\omega t)$ and imposing the non-secularity condition in the resulting equation [9].

Example 8. This example corresponds to

$$u'' + u(1 + u'^2) = 0, \quad u(0) = A, \quad u'(0) = 0, \tag{85}$$

which is first written as

$$\frac{d^2u}{d\theta^2} + u = u - \frac{1}{\omega^2}u - u\left(\frac{du}{d\theta}\right)^2, \quad u(0) = A, \quad \frac{du}{d\theta}(0) = 0, \tag{86}$$

where $\theta = \omega t$.

Application of the artificial parameter-decomposition method to Eq. (86) yields

$$u_0(\theta) = A\cos(\theta), \tag{87}$$

and

$$\frac{d^2u_1}{d\theta^2} + u_1 = \left(A - \frac{A}{\omega^2} - \frac{A^3}{4}\right)\cos(\theta) + \frac{A^3}{4}\cos(3\theta), \quad u_1(0) = 0, \quad \frac{du_1}{d\theta}(0) = 0. \tag{88}$$

The absence of secular terms in u_1 requires that

$$\omega^2 = \frac{4}{4 - A^2}, \tag{89}$$

which coincides with the result obtained from the use of a first-order harmonic balance method [65,66], a harmonic balance technique based on averaging [67] and a modified Linstedt–Poincaré method based on writing Eq. (85) as $u'' + 1u + 1u^2u'' = 0$ and expanding u and the coefficients 1 in terms of an artificial parameter [53].

Using Eq. (89), Eq. (88) becomes

$$\frac{d^2u_1}{d\theta^2} + u_1 = \frac{A^3}{4}\cos(3\theta), \quad u_1(0) = 0, \quad u_1'(0) = 0, \tag{90}$$

whose solution can be written as

$$u_1 = -\frac{A^3}{32}(\cos(3\theta) - \cos(\theta)). \tag{91}$$

Eq. (89) indicates that the frequency is not defined for $A \geqslant 2$, whereas the exact frequency can be shown to be an increasing positive function of A for $0 \leqslant A < \infty$ and the initial conditions considered above [66,68,69]. This discrepancy may be an indication of the limitations of perturbation methods for this example.

5. Conclusions

Series solutions of nonlinear ordinary differential equations have been obtained by writing these equations as nonlinear Volterra integral equations and introducing an artificial parameter in the resulting integrals. By expanding the solution in terms of this parameter, it has been shown that the resulting method is identical to the decomposition technique. It has also been shown that the artificial parameter may be introduced directly into the differential equation.

For oscillators with odd nonlinearities, three formulations have been presented for the determination of the frequency of the limit cycles or periodic solutions. The three formulations are based on the introduction of a linear stiffness term proportional to the unknown frequency of oscillation and an artificial parameter, and two of them are based on differential equations in terms of either the original independent variable or a new independent variable that depends on the unknown frequency of the oscillator. The third technique uses an integral formulation, and the three formulations have been shown to be equivalent.

Application of the artificial parameter method presented in this paper to eight oscillators with odd nonlinearities in the displacement and the avoidance of secular terms in the second-order approximation to the solution provide frequencies which are in full accord with those determined by means of the harmonic balance, iterative and standard and modified Linstedt–Poincaré methods. However, the method presented in the paper is based on the introduction of an unknown frequency and an artificial parameter and expansion of the solution in terms of this parameter and the requirement of non-secular terms, and not on the presence of a small perturbation parameter in the governing differential equation and the expansion of the solution in terms of this parameter. The artificial parameter-decomposition method does not require that constants appearing in the governing differential equation be expanded in terms of either a small parameter or an artificial one. However, it only provides an approximation to the frequency of oscillation.

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