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Radial vibration of piezoelectric/magnetostrictive composite hollow sphere

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Abstract

The radial vibration of a multilayered piezoelectric/magnetostrictive composite hollow sphere is investigated. In terms of the Gaussian equations, two unknown time functions are first introduced to complete the solution of electric displacement and magnetic induction. The solution for mechanical field involving two unknown time functions is obtained by means of the superposition method, the state space method as well as the separation of variables method. By means of the electric and magnetic boundary conditions and continuity conditions, two Volterra integral equations of the second kind with respect to two time functions are derived. The interpolation method is employed to solve the Volterra integral equations. The present solution is suitable for analyzing the transient responses of composite hollow sphere composed of piezoelectric and magnetostrictive layers with arbitrary composite sequence. Numerical results are presented and discussed.

1. Introduction

By inspecting, examining and analyzing the transient responses of the practical structures, the structural dynamic behaviors can be acquired more deeply and thoroughly. The transient responses of hollow sphere have long been an important topic in science and engineering. As an important dynamic parameter, the natural frequencies of spheres have been studied extensively. The study for this subject is also known as natural vibrations or free vibrations. Nelson [1] studied the natural vibrations of laminated orthotropic spheres in 1973, and Heyliger and Wu [2] further investigated the free vibration of layered piezoelectric spheres in 1999. Comparing to free vibrations, due to further involving the forced dynamic load and the initial conditions, the investigation for transient responses become more complicated. At the current stage, most achievements in analytical solutions are obtained only for one-dimensional problem.

For purely elastic materials, the transient responses for homogeneous hollow spheres have been investigated extensively [3–8]. The spherically symmetric transient responses for multilayered hollow spheres have also been studied in Refs. [9,10].

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For purely piezoelectric materials, the transient responses of homogeneous and multilayered piezoelectric hollow spheres under radial deformation have been investigated by Ding et al. [11] and Wang et al. [12], respectively.

Recently, the investigation for magneto-electro-elastic problem has been an increasing interested subject. The composites composed of piezoelectric and magnetostrictive layers will exhibit magnetoelectric coupling effect which is not present in each single-phase piezoelectric or magnetostrictive materials [13] and are predicted to have important applications in various devices [14]. The static analysis and the free vibration of magneto-electro-elastic rectangular plates have been carried out extensively [15–19]. The static analysis and the free vibration of magneto-electro-elastic cylinder have also been studied by Wang and Zhong [20] and Buchanan [21]. Wang and Zhong [22] also obtained the general solution of magneto-electro-elastic hollow sphere for static problem. For elastodynamic problem, Hou and Leung [23] studied the transient responses of single-layer magneto-electric-elastic hollow cylinder.

In this paper, the transient responses of multilayered magneto-electro-elastic hollow sphere composed of piezoelectric and magnetostrictive layers are investigated. The solution approach is performed directly in time domain.

2. Basic equations and their non-dimensional forms

Consider a composite hollow sphere composed of n layers. Suppose the inner and outer radii are, respectively, $r_0 = a$ and $r_n = b$. The radius of the interface from inner to outer is denoted $r_i (i = 1, 2, \ldots, n-1)$. In the theoretical analysis, at internal and external surfaces, respectively, we consider the hollow sphere is subjected to dynamic radial stresses $P_a(t)$ and $P_b(t)$ as well as time dependent electric potentials $\Phi_a(t)$ and $\Phi_b(t)$ and magnetic potentials $\Psi_a(t)$ and $\Psi_b(t)$. The model is shown in Fig. 1.

In the spherical coordinate system (r, θ, φ), for radial vibration problem of the hollow sphere, the nonzero components of displacement, electric and magnetic potentials in the ith layer $(r_{i-1} \le r \le r_i)$ are $u_r^{(i)} = u_r^{(i)}(r, t)$, $\Phi^{(i)} = \Phi^{(i)}(r, t)$ and $\Psi^{(i)} = \Psi^{(i)}(r, t)$. If each layer characterizes material spherical isotropy and is polarized in radial direction, then the constitutive relations of the ith layer are written as [24]

$$\begin{aligned}
\sigma_{\theta\theta}^{(i)} &= \sigma_{\varphi\varphi}^{(i)} = (c_{11}^{(i)} + c_{12}^{(i)})\frac{u_r^{(i)}}{r} + c_{13}^{(i)}\frac{\partial u_r^{(i)}}{\partial r} + e_{31}^{(i)}\frac{\partial \Phi^{(i)}}{\partial r} + q_{31}^{(i)}\frac{\partial \Psi^{(i)}}{\partial r},\\
\sigma_{rr}^{(i)} &= 2c_{13}^{(i)}\frac{u_r^{(i)}}{r} + c_{33}^{(i)}\frac{\partial u_r^{(i)}}{\partial r} + e_{33}^{(i)}\frac{\partial \Phi^{(i)}}{\partial r} + q_{33}^{(i)}\frac{\partial \Psi^{(i)}}{\partial r},\\
D_{rr}^{(i)} &= 2e_{31}^{(i)}\frac{u_r^{(i)}}{r} + e_{33}^{(i)}\frac{\partial u_r^{(i)}}{\partial r} - \varepsilon_{33}^{(i)}\frac{\partial \Phi^{(i)}}{\partial r} - g_{33}^{(i)}\frac{\partial \Psi^{(i)}}{\partial r},\\
B_{rr}^{(i)} &= 2q_{31}^{(i)}\frac{u_r^{(i)}}{r} + q_{33}^{(i)}\frac{\partial u_r^{(i)}}{\partial r} - g_{33}^{(i)}\frac{\partial \Phi^{(i)}}{\partial r} - m_{33}^{(i)}\frac{\partial \Psi^{(i)}}{\partial r},
\end{aligned} \tag{1}$$

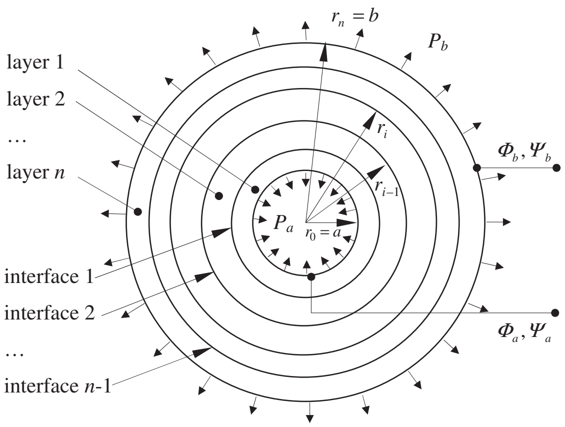

Fig. 1. Model of multilayered hollow sphere.

where $\sigma_{jj}^{(i)}(j=r,\theta,\varphi)$, $D_{rr}^{(i)}$ and $B_{rr}^{(i)}$ are the components of stresses, electric displacement and magnetic induction, $c_{jm}^{(i)}$, $e_{3j}^{(i)}$, $q_{3j}^{(i)}$, $\varepsilon_{33}^{(i)}$,$g_{33}^{(i)}$ and $m_{33}^{(i)}$ are respectively, the elastic, piezoelectric, piezomagnetic, dielectric, electromagnetic and magnetic constants of the ith layer. In the absence of body force, electric charge density and electric current density, the equations of motion of the ith layer are expressed as

$$\begin{aligned}&\frac{\partial\sigma_{rr}^{(i)}}{\partial r}+2\frac{\sigma_{rr}^{(i)}-\sigma_{\theta\theta}^{(i)}}{r}=\rho^{(i)}\frac{\partial^2 u_r^{(i)}}{\partial t^2},\\&\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2D_{rr}^{(i)}\right]=0,\\&\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2B_{rr}^{(i)}\right]=0,\end{aligned}\tag{2}$$

where $\rho^{(i)}$ is the mass density of the ith layer. The boundary conditions are expressed as

$$\sigma_{rr}^{(1)}(a,t)=P_a(t),\quad \sigma_{rr}^{(n)}(b,t)=P_b(t),\tag{3}$$

$$\Phi^{(1)}(a,t)=\Phi_a(t),\quad \Psi^{(1)}(a,t)=\Psi_a(t),\tag{4}$$

$$\Phi^{(n)}(b,t)=\Phi_b(t),\quad \Psi^{(n)}(b,t)=\Psi_b(t).\tag{5}$$

For perfectly bonded interfaces, the continuity conditions are written as

$$u_r^{(i+1)}(r_i,t)=u_r^{(i)}(r_i,t),\quad \sigma_{rr}^{(i+1)}(r_i,t)=\sigma_{rr}^{(i)}(r_i,t)\quad (i=1,2,\ldots,n-1),\tag{6}$$

$$D_{rr}^{(i+1)}(r_i,t)=D_{rr}^{(i)}(r_i,t),\quad B_{rr}^{(i+1)}(r_i,t)=B_{rr}^{(i)}(r_i,t)\quad (i=1,2,\ldots,n-1).\tag{7}$$

$$\Phi^{(i+1)}(r_i,t)=\Phi^{(i)}(r_i,t),\quad \Psi^{(i+1)}(r_i,t)=\Psi^{(i)}(r_i,t)\quad (i=1,2,\ldots,n-1).\tag{8}$$

For dynamic problem, the initial conditions ($t=0$) should be completed as

$$u_r^{(i)}(r,0)=U_0^{(i)}(r),\quad \dot{u}_r^{(i)}(r,0)=V_0^{(i)}(r)\quad (i=1,2,\ldots,n),\tag{9}$$

where $U_0^{(i)}(r)$ and $V_0^{(i)}(r)$ are known functions of the radial coordinate r and a dot over a quantity denotes its partial derivative with respect to time t.

For the sake of simplicity, the following non-dimensional variables and quantities are introduced as

$$\begin{aligned}&u^{(i)}=\frac{u_r^{(i)}}{b},\quad \phi^{(i)}=\frac{\Phi^{(i)}}{\Phi_0},\quad \psi^{(i)}=\frac{\Psi^{(i)}}{\Psi_0},\quad \xi=\frac{r}{b},\quad \xi_i=\frac{r_i}{b}\ \ (i=0,1,\ldots,n),\\&\sigma_r^{(i)}=\frac{\sigma_{rr}^{(i)}}{c_{33}^{(*)}},\quad \sigma_\theta^{(i)}=\frac{\sigma_{\theta\theta}^{(i)}}{c_{33}^{(*)}},\quad \sigma_\varphi^{(i)}=\frac{\sigma_{\varphi\varphi}^{(i)}}{c_{33}^{(*)}},\quad D_r^{(i)}=\frac{D_{rr}^{(i)}}{D_0},\quad B_r^{(i)}=\frac{B_{rr}^{(i)}}{B_0},\\&\phi_a=\frac{\Phi_a}{\Phi_0},\quad \phi_b=\frac{\Phi_b}{\Phi_0},\quad \Psi_a=\frac{\Psi_a}{\Psi_0},\quad \Psi_b=\frac{\Psi_b}{\Psi_0},\quad \tau=\frac{c_v}{b}t,\end{aligned}$$

$$\begin{aligned}&u_0^{(i)}=\frac{U_0^{(i)}}{b},\quad v_0^{(i)}=\frac{V_0^{(i)}}{c_v},\quad p_a=\frac{P_a}{c_{33}^{(*)}},\quad p_b=\frac{P_b}{c_{33}^{(*)}},\\&c_{11P}^{(i)}=\frac{c_{11}^{(i)}}{c_{33}^{(*)}},\quad c_{12P}^{(i)}=\frac{c_{12}^{(i)}}{c_{33}^{(*)}},\quad c_{13P}^{(i)}=\frac{c_{13}^{(i)}}{c_{33}^{(*)}},\quad c_{33P}^{(i)}=\frac{c_{33}^{(i)}}{c_{33}^{(*)}},\\&\varepsilon_3^{(i)}=\frac{\varepsilon_{33}^{(i)}}{\varepsilon_{33}^{(*)}},\quad m_3^{(i)}=\frac{m_{33}^{(i)}}{m_{33}^{(*)}},\quad g_3^{(i)}=\frac{g_{33}^{(i)}}{G_0},\quad \bar{\rho}^{(i)}=\frac{\rho^{(i)}}{\rho^{(*)}},\end{aligned}$$

$$e_1^{(i)} = \frac{e_{31}^{(i)}}{D_0}, \quad e_3^{(i)} = \frac{e_{33}^{(i)}}{D_0}, \quad q_1^{(i)} = \frac{q_{31}^{(i)}}{B_0}, \quad q_3^{(i)} = \frac{q_{33}^{(i)}}{B_0},$$
$$D_0 = \sqrt{c_{33}^{(*)}\varepsilon_{33}^{(*)}}, \quad B_0 = \sqrt{c_{33}^{(*)}m_{33}^{(*)}}, \quad G_0 = \sqrt{\varepsilon_{33}^{(*)}m_{33}^{(*)}},$$
$$\Phi_0 = b\sqrt{c_{33}^{(*)}\Big/\varepsilon_{33}^{(*)}}, \quad \Psi_0 = b\sqrt{c_{33}^{(*)}\Big/m_{33}^{(*)}}, \quad c_v = \sqrt{c_{33}^{(*)}\Big/\rho^{(*)}}. \tag{10}$$

In Eq. (10), the material constants $c_{33}^{(*)}, \varepsilon_{33}^{(*)}, m_{33}^{(*)}, \rho^{(*)}$ and the outer radius b are selected to normalize the variables and quantities. It should be noted that for both piezoelectric and magnetostrictive materials, the selected constants are not equal to zero. That is to say, the superscript "*" can be an arbitrary number between 1 and n. It should also be noted here that in the non-dimensional radial coordinates $\xi_i (i = 0, 1, \ldots, n)$, especially, we have $\xi_0 = a/b$ and $\xi_n = 1$. For the sake of convenience to understand by the readers, instead of 1, ξ_n is still reserved in all following equations. By virtue of Eq. (10), Eqs. (1) and (2) can be rewritten as

$$\sigma_\theta^{(i)} = \sigma_\varphi^{(i)} = (c_{11P}^{(i)} + c_{12P}^{(i)})\frac{u^{(i)}}{\xi} + c_{13p}^{(i)}\frac{\partial u^{(i)}}{\partial \xi} + e_1^{(i)}\frac{\partial \phi^{(i)}}{\partial \xi} + q_1^{(i)}\frac{\partial \psi^{(i)}}{\partial \xi},$$
$$\sigma_r^{(i)} = 2c_{13P}^{(i)}\frac{u^{(i)}}{\xi} + c_{33P}^{(i)}\frac{\partial u^{(i)}}{\partial \xi} + e_3^{(i)}\frac{\partial \phi^{(i)}}{\partial \xi} + q_3^{(i)}\frac{\partial \psi^{(i)}}{\partial \xi},$$
$$D_r^{(i)} = 2e_1^{(i)}\frac{u^{(i)}}{\xi} + e_3^{(i)}\frac{\partial u^{(i)}}{\partial \xi} - \varepsilon_3^{(i)}\frac{\partial \phi^{(i)}}{\partial \xi} - g_3^{(i)}\frac{\partial \psi^{(i)}}{\partial \xi},$$
$$B_r^{(i)} = 2q_1^{(i)}\frac{u^{(i)}}{\xi} + q_3^{(i)}\frac{\partial u^{(i)}}{\partial \xi} - g_3^{(i)}\frac{\partial \phi^{(i)}}{\partial \xi} - m_3^{(i)}\frac{\partial \psi^{(i)}}{\partial \xi}, \tag{11}$$

$$\frac{\partial \sigma_r^{(i)}}{\partial \xi} + 2\frac{\sigma_r^{(i)} - \sigma_\theta^{(i)}}{\xi} = \bar{\rho}^{(i)}\frac{\partial^2 u^{(i)}}{\partial \tau^2},$$
$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi^2}\left[\xi^2 D_r^{(i)}\right] = 0,$$
$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left[\xi^2 B_r^{(i)}\right] = 0. \tag{12}$$

The boundary conditions Eqs. (3)–(5) and the continuity conditions Eqs. (6)–(8) are rewritten as

$$\sigma_r^{(1)}(\xi_0, \tau) = p_a(\tau), \quad \sigma_r^{(n)}(\xi_n, \tau) = p_b(\tau), \tag{13}$$

$$\phi^{(1)}(\xi_0, \tau) = \phi_a(\tau), \quad \psi^{(1)}(\xi_0, \tau) = \psi_a(\tau), \tag{14}$$

$$\phi^{(n)}(\xi_n, \tau) = \phi_b(\tau), \quad \psi^{(n)}(\xi_n, \tau) = \psi_b(\tau), \tag{15}$$

$$u^{(i+1)}(\xi_i, \tau) = u^{(i)}(\xi_i, \tau), \quad \sigma_r^{(i+1)}(\xi_i, \tau) = \sigma_r^{(i)}(\xi_i, \tau) \quad (i = 1, 2, \ldots, n-1), \tag{16}$$

$$D_r^{(i+1)}(\xi_i, \tau) = D_r^{(i)}(\xi_i, \tau), \quad B_r^{(i+1)}(\xi_i, \tau) = B_r^{(i)}(\xi_i, \tau) \quad (i = 1, 2, \ldots, n-1), \tag{17}$$

$$\phi^{(i+1)}(\xi_i, \tau) = \phi^{(i)}(\xi_i, \tau), \quad \psi^{(i+1)}(\xi_i, \tau) = \psi^{(i)}(\xi_i, \tau) \quad (i = 1, 2, \ldots, n-1). \tag{18}$$

The initial conditions Eq. (9) are rewritten as

$$u^{(i)}(\xi, 0) = u_0^{(i)}(\xi), \quad \dot{u}^{(i)}(\xi, 0) = v_0^{(i)}(\xi) \quad (i = 1, 2, \ldots n). \tag{19}$$

In Eq. (19) and, hereafter, a dot over a quantity denotes its partial derivative with respect to non-dimensional time τ.

3. Solution for mechanical field

3.1. The superposition method

The solutions of electric displacement and magnetic induction can be obtained from the last two of Eq. (12) as

$$D_r^{(i)}(\xi,\tau)=\xi^{-2}\eta^{(i)}(\tau),\quad B_r^{(i)}(\xi,\tau)=\xi^{-2}\chi^{(i)}(\tau)\quad (i=1,2,\ldots,n), \tag{20}$$

where $\eta^{(i)}(\tau)$ and $\chi^{(i)}(\tau)$ are unknown functions of time. With the aid of Eq. (17), we have

$$\begin{aligned}\eta^{(1)}(\tau)=\eta^{(2)}(\tau)=\ldots=\eta^{(n)}(\tau)=\eta(\tau),\\ \chi^{(1)}(\tau)=\chi^{(2)}(\tau)=\ldots=\chi^{(n)}(\tau)=\chi(\tau).\end{aligned} \tag{21}$$

By means of Eqs. (20) and (21), the last two of Eq. (11) can be rewritten as

$$\begin{aligned}\frac{\partial\phi^{(i)}}{\partial\xi}&=2A_{11}^{(i)}\frac{u^{(i)}}{\xi}+A_{12}^{(i)}\frac{\partial u^{(i)}}{\partial\xi}+A_{13}^{(i)}\frac{\eta(\tau)}{\xi^2}+A_{14}^{(i)}\frac{\chi(\tau)}{\xi^2},\\ \frac{\partial\psi^{(i)}}{\partial\xi}&=2A_{21}^{(i)}\frac{u^{(i)}}{\xi}+A_{22}^{(i)}\frac{\partial u^{(i)}}{\partial\xi}+A_{23}^{(i)}\frac{\eta(\tau)}{\xi^2}+A_{24}^{(i)}\frac{\chi(\tau)}{\xi^2}.\end{aligned} \tag{22}$$

where

$$\begin{aligned}&A_{11}^{(i)}=(e_1^{(i)}m_3^{(i)}-q_1^{(i)}g_3^{(i)})\Big/C^{(i)},\quad A_{12}^{(i)}=(e_3^{(i)}m_3^{(i)}-q_3^{(i)}g_3^{(i)})\Big/C^{(i)},\\ &A_{21}^{(i)}=(q_1^{(i)}\varepsilon_3^{(i)}-e_1^{(i)}g_3^{(i)})\Big/C^{(i)},\quad A_{22}^{(i)}=(q_3^{(i)}\varepsilon_3^{(i)}-e_3^{(i)}g_3^{(i)})\Big/C^{(i)},\\ &A_{13}^{(i)}=-m_3^{(i)}\Big/C^{(i)},\quad A_{14}^{(i)}=A_{23}^{(i)}=g_3^{(i)}\Big/C^{(i)},\quad A_{24}^{(i)}=-\varepsilon_3^{(i)}\Big/C^{(i)},\\ &C^{(i)}=\varepsilon_3^{(i)}m_3^{(i)}-g_3^{(i)}g_3^{(i)}.\end{aligned} \tag{23}$$

Substituting Eq. (22) into the first two of Eq. (11), we derive

$$\begin{aligned}\Sigma_\theta^{(i)}=\Sigma_\varphi^{(i)}&=(c_{11D}^{(i)}+c_{12D}^{(i)})u^{(i)}+c_{13D}^{(i)}\nabla u^{(i)}+e_{1D}^{(i)}\eta(\tau)+q_{1D}^{(i)}\chi(\tau),\\ \Sigma_r^{(i)}&=2c_{13D}^{(i)}u^{(i)}+c_{33D}^{(i)}\nabla u^{(i)}+e_{3D}^{(i)}\eta(\tau)+q_{3D}^{(i)}\chi(\tau),\end{aligned} \tag{24}$$

where

$$\begin{aligned}&c_{11D}^{(i)}=c_{11P}^{(i)}+(e_1^{(i)}A_{11}^{(i)}+q_1^{(i)}A_{21}^{(i)}),\quad c_{12D}^{(i)}=c_{12P}^{(i)}+(e_1^{(i)}A_{11}^{(i)}+q_1^{(i)}A_{21}^{(i)}),\\ &c_{13D}^{(i)}=c_{13P}^{(i)}+(e_3^{(i)}A_{11}^{(i)}+q_3^{(i)}A_{21}^{(i)}),\quad c_{33D}^{(i)}=c_{33P}^{(i)}+(e_3^{(i)}A_{12}^{(i)}+q_3^{(i)}A_{22}^{(i)}),\\ &e_{1D}^{(i)}=(e_1^{(i)}A_{13}^{(i)}+q_1^{(i)}A_{14}^{(i)}),\quad e_{3D}^{(i)}=(e_3^{(i)}A_{13}^{(i)}+q_3^{(i)}A_{14}^{(i)}),\\ &q_{1D}^{(i)}=(e_1^{(i)}A_{23}^{(i)}+q_1^{(i)}A_{24}^{(i)}),\quad q_{3D}^{(i)}=(e_3^{(i)}A_{23}^{(i)}+q_3^{(i)}A_{24}^{(i)}),\end{aligned} \tag{25a}$$

$$\Sigma_\theta^{(i)}=\xi\sigma_\theta^{(i)},\quad \Sigma_\varphi^{(i)}=\xi\sigma_\varphi^{(i)},\quad \Sigma_r^{(i)}=\xi\sigma_r^{(i)},\quad \nabla=\xi\frac{\partial}{\partial\xi}. \tag{25b}$$

By employing the newly introduced symbols Eq. (25b), the first of Eq. (12) can be rewritten as

$$\nabla\Sigma_r^{(i)}+\Sigma_r^{(i)}-2\Sigma_\theta^{(i)}=\bar{\rho}^{(i)}\xi^2\ddot{u}^{(i)}. \tag{26}$$

By means of the new variable Σ_r, mechanical boundary conditions Eq. (13) and the continuity conditions Eq. (16) are rewritten as

$$\Sigma_r^{(1)}(\xi_0,\tau)=\xi_0 p_a(\tau),\quad \Sigma_r^{(n)}(\xi_n,\tau)=\xi_n p_b(\tau), \tag{27}$$

$$u^{(i+1)}(\xi_i,\tau)=u^{(i)}(\xi_i,\tau),\quad \Sigma_r^{(i+1)}(\xi_i,\tau)=\Sigma_r^{(i)}(\xi_i,\tau)\quad (i=1,2,\ldots,n-1). \tag{28}$$

The solution for mechanical field is carried out by the superposition method. The displacement and stresses are divided into two parts as

$$u^{(i)} = u_s^{(i)} + u_d^{(i)}, \quad \Sigma_r^{(i)} = \Sigma_{rs}^{(i)} + \Sigma_{rd}^{(i)}, \quad \Sigma_\theta^{(i)} = \Sigma_{\theta s}^{(i)} + \Sigma_{\theta d}^{(i)}, \tag{29}$$

where $u_s^{(i)}$, $\Sigma_{rs}^{(i)}$, $\Sigma_{\theta s}^{(i)}$ and $u_d^{(i)}$, $\Sigma_{rd}^{(i)}$, $\Sigma_{\theta d}^{(i)}$ are named as the quasi-static and dynamic parts, respectively. The quasi-static part satisfies the following equations:

$$\begin{aligned} \Sigma_{\theta s}^{(i)} &= (c_{11D}^{(i)} + c_{12D}^{(i)})u_s^{(i)} + c_{13D}^{(i)}\nabla u_s^{(i)} + e_{1D}^{(i)}\eta(\tau) + q_{1D}^{(i)}\chi(\tau), \\ \Sigma_{rs}^{(i)} &= 2c_{13D}^{(i)}u_s^{(i)} + c_{33D}^{(i)}\nabla u_s^{(i)} + e_{3D}^{(i)}\eta(\tau) + q_{3D}^{(i)}\chi(\tau), \end{aligned} \tag{30}$$

$$\nabla\Sigma_{rs}^{(i)} + \Sigma_{rs}^{(i)} - 2\Sigma_{\theta s}^{(i)} = 0, \tag{31}$$

$$\Sigma_{rs}^{(1)}(\xi_0, \tau) = \xi_0 p_a(\tau), \quad \Sigma_{rs}^{(n)}(\xi_n, \tau) = \xi_n p_b(\tau), \tag{32}$$

$$u_s^{(i+1)}(\xi_i, \tau) = u_s^{(i)}(\xi_i, \tau), \quad \Sigma_{rs}^{(i+1)}(\xi_i, \tau) = \Sigma_{rs}^{(i)}(\xi_i, \tau) \quad (i = 1, 2, \ldots, n-1). \tag{33}$$

Substituting Eq. (29) into Eqs. (24), (26)–(28) and (19) and utilizing Eqs. (30)–(33) yields

$$\Sigma_{\theta d}^{(i)} = (c_{11D}^{(i)} + c_{12D}^{(i)})u_d^{(i)} + c_{13D}^{(i)}\nabla u_d^{(i)}, \quad \Sigma_{rd}^{(i)} = 2c_{13D}^{(i)}u_d^{(i)} + c_{33D}^{(i)}\nabla u_d^{(i)}, \tag{34}$$

$$\nabla\Sigma_{rd}^{(i)} + \Sigma_{rd}^{(i)} - 2\Sigma_{\theta d}^{(i)} = \bar{\rho}^{(i)}\xi^2(\ddot{u}_d^{(i)} + \ddot{u}_s^{(i)}), \tag{35}$$

$$\Sigma_{rd}^{(1)}(\xi_0, \tau) = 0, \quad \Sigma_{rd}^{(n)}(\xi_n, \tau) = 0, \tag{36}$$

$$u_d^{(i+1)}(\xi_i, \tau) = u_d^{(i)}(\xi_i, \tau), \quad \Sigma_{rd}^{(i+1)}(\xi_i, \tau) = \Sigma_{rd}^{(i)}(\xi_i, \tau) \quad (i = 1, 2, \ldots, n-1), \tag{37}$$

$$u_d^{(i)}(\xi, 0) = u_0^{(i)}(\xi) - u_s^{(i)}(\xi, 0), \quad \dot{u}_d^{(i)}(\xi, 0) = v_0^{(i)}(\xi) - \dot{u}_s^{(i)}(\xi, 0) \quad (i = 1, 2, \ldots n). \tag{38}$$

3.2. Quasi-static part

The second of Eq. (30) can be rewritten as

$$\nabla u_s^{(i)} = a_{11}^{(i)}u_s^{(i)} + a_{12}^{(i)}\Sigma_{rs}^{(i)} + a_{13}^{(i)}\eta(\tau) + a_{14}^{(i)}\chi(\tau), \tag{39}$$

where

$$a_{11}^{(i)} = -2\frac{c_{13D}^{(i)}}{c_{33D}^{(i)}}, \quad a_{12}^{(i)} = \frac{1}{c_{33D}^{(i)}}, \quad a_{13}^{(i)} = -\frac{e_{3D}^{(i)}}{c_{33D}^{(i)}}, \quad a_{14}^{(i)} = -\frac{q_{3D}^{(i)}}{c_{33D}^{(i)}}. \tag{40}$$

Substituting the first part of Eq. (30) into Eq. (31) and utilizing Eq. (39), we obtain

$$\nabla\Sigma_{rs}^{(i)} = a_{21}^{(i)}u_s^{(i)} + a_{22}^{(i)}\Sigma_{rs}^{(i)} + a_{23}^{(i)}\eta(\tau) + a_{24}^{(i)}\chi(\tau), \tag{41}$$

where

$$\begin{aligned} a_{21}^{(i)} &= 2(c_{11D}^{(i)} + c_{12D}^{(i)} + c_{13D}^{(i)}a_{11}^{(i)}), \quad a_{22}^{(i)} = 2c_{13D}^{(i)}a_{12}^{(i)} - 1, \\ a_{23}^{(i)} &= 2(e_{1D}^{(i)} + c_{13D}^{(i)}a_{13}^{(i)}), \quad a_{24}^{(i)} = 2(q_{1D}^{(i)} + c_{13D}^{(i)}a_{14}^{(i)}). \end{aligned} \tag{42}$$

Eqs. (39) and (41) can be rewritten in a matrix form as

$$\nabla\{X^{(i)}(\xi, \tau)\} = [N^{(i)}]\{X^{(i)}(\xi, \tau)\} + \{L^{(i)}\}\eta(\tau) + \{H^{(i)}\}\chi(\tau), \tag{43}$$

where

$$\{\mathrm{X}^{(i)}(\xi,\tau)\}=\left\{\begin{array}{c}u_s^{(i)}(\xi,\tau)\\ \Sigma_{rs}^{(i)}(\xi,\tau)\end{array}\right\},\quad [N^{(i)}]=\begin{bmatrix}a_{11}^{(i)} & a_{12}^{(i)}\\ a_{21}^{(i)} & a_{22}^{(i)}\end{bmatrix},\quad \{L^{(i)}\}=\left\{\begin{array}{c}a_{13}^{(i)}\\ a_{23}^{(i)}\end{array}\right\},\quad \{H^{(i)}\}=\left\{\begin{array}{c}a_{14}^{(i)}\\ a_{24}^{(i)}\end{array}\right\}. \tag{44}$$

The solution of Eq. (43) is

$$\{X^{(i)}(\xi,\tau)\}=[T^{(i)}(\xi)](\{X^{(i)}(\xi_{i-1},\tau)\}+\{G^{(i)}(\xi)\}\eta(\tau)+\{Q^{(i)}(\xi)\}\chi(\tau)), \tag{45}$$

where

$$[T^{(i)}(\xi)]=\left(\frac{\xi}{\xi_{i-1}}\right)^{[N^{(i)}]},\quad \{G^{(i)}(\xi)\}=\int_{\xi_{i-1}}^{\xi}\zeta^{-1}[T^{(i)}(\zeta)]^{-1}\{L^{(i)}\}\mathrm{d}\zeta,$$
$$\{Q^{(i)}(\xi)\}=\int_{\xi_{i-1}}^{\xi}\zeta^{-1}[T^{(i)}(\zeta)]^{-1}\{H^{(i)}\}\mathrm{d}\zeta, \tag{46}$$

in which $[T^{(i)}(\xi)]$ is a 2×2 matrix. $\{G^{(i)}(\xi)\}$ and $\{Q^{(i)}(\xi)\}$ are two 2×1 column matrices. By means of the mechanical boundary conditions Eq. (32) and continuity conditions Eq. (33), $u_s^{(i)}(\xi,\tau)$ can be finally obtained in a form as

$$u_s^{(i)}(\xi,\tau)=f_1^{(i)}(\xi)p_a(\tau)+f_2^{(i)}(\xi)p_b(\tau)+f_3^{(i)}(\xi)\eta(\tau)+f_4^{(i)}(\xi)\chi(\tau), \tag{47}$$

where $f^{(i)}(\xi)$ $(i=1,2,3,4)$ are known functions of non-dimensional radial coordinate ξ. The detailed procedure for the determination of $u_s^{(i)}(\xi,\tau)$ is presented in Appendix A.

3.3. Dynamic solution

Substituting Eq. (34) into Eq. (35) and utilizing Eq. (47), we have

$$\frac{\partial^2 u_d^{(i)}}{\partial\xi^2}+\frac{2}{\xi}\frac{\partial u_d^{(i)}}{\partial\xi}-\frac{\bar{\mu}_i^2}{\xi^2}u_d^{(i)}=\frac{1}{c_i^2}\left(\frac{\partial^2 u_d^{(i)}}{\partial\tau^2}+f_1^{(i)}(\xi)\frac{\mathrm{d}^2p_a(\tau)}{\mathrm{d}\tau^2}+f_2^{(i)}(\xi)\frac{\mathrm{d}^2p_b(\tau)}{\mathrm{d}\tau^2}\right.$$
$$\left.+f_3^{(i)}(\xi)\frac{\mathrm{d}^2\eta(\tau)}{\mathrm{d}\tau^2}+f_4^{(i)}(\xi)\frac{\mathrm{d}^2\chi(\tau)}{\mathrm{d}\tau^2}\right), \tag{48}$$

where

$$\bar{\mu}_i=\sqrt{2\frac{c_{11D}^{(i)}+c_{12D}^{(i)}-c_{13D}^{(i)}}{c_{33D}^{(i)}}},\quad c_i=\sqrt{\frac{c_{33D}^{(i)}}{\bar{\rho}^{(i)}}}. \tag{49}$$

By means of separation of variables method, $u_d^{(i)}(\xi,\tau)$ can be assumed as

$$u_d^{(i)}(\xi,\tau)=\xi^{-1/2}\sum_{m=1}^{\infty}R_m^{(i)}(\xi)\Omega_m(\tau), \tag{50}$$

where $\Omega_m(\tau)$ is an undetermined function. $R_m^{(i)}(\xi)$ is a known function of ξ which can be determined by in initial parameter method [25]. And the following orthogonal property can be verified [26]:

$$\sum_{i=1}^{n}\bar{\rho}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\xi R_m^{(i)}(\xi)R_j^{(i)}(\xi)\mathrm{d}\xi=J_m\delta_{mj}, \tag{51}$$

where δ_{ml} is the Kronecker delta, and

$$J_m=\frac{1}{2}\sum_{i=1}^{n}\left\{\frac{c_{33D}^{(i)}}{\omega_m^2}\left[\xi\frac{\mathrm{d}R_m^{(i)}(\xi)}{\mathrm{d}\xi}\right]^2-\frac{\mu_i^2c_{33D}^{(i)}}{\omega_m^2}\left[R_m^{(i)}(\xi)\right]^2+\bar{\rho}^{(i)}\left[\xi R_m^{(i)}(\xi)\right]^2\right\}\Bigg|_{\xi_{i-1}}^{\xi_i}. \tag{52}$$

In Eq. (52), $\mu_i = \sqrt{\bar{\mu}_i^2 + (\frac{1}{2})^2}$ (also see the first of Eq. (B.7) in Appendix B). $\omega_m(m = 1, 2, \ldots, \infty)$ is a series of the positive real roots of an eigenequation. For the convenience to the readers, the derivation for $R_m^{(i)}(\xi)$ and the eigenequation is shown in Appendix B.

Substituting Eq. (50) into Eq. (48) and utilizing Eq. (51), we obtain

$$\frac{\mathrm{d}^2\Omega_m(\tau)}{\mathrm{d}\tau^2} + \omega_m^2\Omega_m(\tau) = \kappa_m(\tau) \quad (m = 1, 2, \ldots, \infty), \tag{53}$$

where

$$\kappa_m(\tau) = I_{1m}\ddot{p}_a(\tau) + I_{2m}\ddot{p}_b(\tau) + I_{3m}\ddot{\eta}(\tau) + I_{4m}\ddot{\chi}(\tau), \tag{54}$$

$$I_{jm} = -\frac{1}{J_m}\sum_{i=1}^{n}\bar{\rho}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\xi^{\frac{3}{2}}f_j^{(i)}(\xi)R_m^{(i)}(\xi)\mathrm{d}\xi \quad (j = 1, 2, 3, 4). \tag{55}$$

The solution of Eq. (53) is

$$\Omega_m(\tau) = \Omega_m(0)\cos\omega_m\tau + \frac{\dot{\Omega}_m(0)}{\omega_m}\sin\omega_m\tau + \frac{1}{\omega_m}\int_0^\tau \kappa_m(\zeta)\sin\omega_m(\tau - \zeta)\mathrm{d}\zeta. \tag{56}$$

The substitution of Eqs. (47) and (50) into Eq. (38) leads to

$$\begin{aligned} \xi^{-\frac{1}{2}}\sum_{m=1}^{\infty}R_m^{(i)}(\xi)\Omega_m(0) &= u_0^{(i)}(\xi) - f_1^{(i)}(\xi)p_a(0) - f_2^{(i)}(\xi)p_b(0) - f_3^{(i)}(\xi)\eta(0) - f_4^{(i)}(\xi)\chi(0), \\ \xi^{-\frac{1}{2}}\sum_{m=1}^{\infty}R_m^{(i)}(\xi)\dot{\Omega}_m(0) &= v_0^{(i)}(\xi) - f_1^{(i)}(\xi)\dot{p}_a(0) - f_2^{(i)}(\xi)\dot{p}_b(0) - f_3^{(i)}(\xi)\dot{\eta}(0) - f_4^{(i)}(\xi)\dot{\chi}(0). \end{aligned} \tag{57}$$

By the orthogonal property Eq. (52), the following equations can be derived from Eq. (57):

$$\begin{aligned} \Omega_m(0) &= I_{1m}p_a(0) + I_{2m}p_b(0) + I_{3m}\eta(0) + I_{4m}\chi(0) + I_{5m}, \\ \dot{\Omega}_m(0) &= I_{1m}\dot{p}_a(0) + I_{2m}\dot{p}_b(0) + I_{3m}\dot{\eta}(0) + I_{4m}\dot{\chi}(0) + I_{6m}, \end{aligned} \tag{58}$$

where

$$I_{5m} = \frac{1}{J_m}\sum_{i=1}^{n}\bar{\rho}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\xi^{3/2}u_0^{(i)}(\xi)R_m^{(i)}(\xi)\mathrm{d}\xi, \quad I_{6m} = \frac{1}{J_m}\sum_{i=1}^{n}\bar{\rho}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\xi^{3/2}v_0^{(i)}(\xi)R_m^{(i)}(\xi)\mathrm{d}\xi. \tag{59}$$

Noticing that $\ddot{\eta}(\tau)$ and $\ddot{\chi}(\tau)$ are involved in $\kappa_m(\tau)$ as shown in Eq. (54), the integration-by-parts formula is employed to perform the integration of the terms involving second derivative in Eq. (56) and finally we obtain

$$\begin{aligned} \Omega_m(\tau) = {} & \Omega_{1m}(\tau) + I_{3m}\eta(\tau) - I_{3m}\omega_m\int_0^\tau \eta(\zeta)\sin\omega_m(\tau - \zeta)\mathrm{d}\zeta \\ & + I_{4m}\chi(\tau) - I_{4m}\omega_m\int_0^\tau \chi(\zeta)\sin\omega_m(\tau - \zeta)\mathrm{d}\zeta, \end{aligned} \tag{60}$$

where

$$\begin{aligned} \Omega_{1m}(\tau) = {} & I_{5m}\cos\omega_m\tau + \frac{I_{6m}}{\omega_m}\sin\omega_m\tau + I_{1m}p_a(\tau) + I_{2m}p_b(\tau) \\ & - I_{1m}\omega_m\int_0^\tau p_a(\zeta)\sin\omega_m(\tau - \zeta)\mathrm{d}\zeta - I_{2m}\omega_m\int_0^\tau p_b(\zeta)\sin\omega_m(\tau - \zeta)\mathrm{d}\zeta. \end{aligned} \tag{61}$$

It should be noted here that $\Omega_{1m}(\tau)$ is a known function of time. Till this section, $\eta(\tau)$ and $\chi(\tau)$ in Eq. (60) are still unknown.

3.4. Determination for $\eta(\tau)$ and $\chi(\tau)$

In the following, boundary conditions and continuity conditions of the electric and magnetic fields will be applied to determine $\eta(\tau)$ and $\chi(\tau)$. Substitution of Eqs. (47) and (50) into the first of Eq. (29) yields

$$u^{(i)}(\xi,\tau)=\xi^{-\frac{1}{2}}\sum_{m=1}^{\infty}R_m^{(i)}(\xi)\Omega_m(\tau)+f_1^{(i)}(\xi)p_a(\tau)+f_2^{(i)}(\xi)p_b(\tau)+f_3^{(i)}(\xi)\eta(\tau)+f_4^{(i)}(\xi)\chi(\tau). \tag{62}$$

Substituting Eq. (62) into Eq. (22) and then integrating it at each spatial interval $[\xi_{i-1},\xi_i](i=1,2\ldots n)$, we obtain n equations. By utilizing Eqs. (14), (15) and (18), the summation of the obtained n equations leads to

$$\begin{aligned}\phi_b(\tau)-\phi_a(\tau)&=K_{11}p_a(\tau)+K_{12}p_b(\tau)+K_{13}\eta(\tau)+K_{14}\chi(\tau)+\sum_{m=1}^{\infty}K_{15m}\Omega_m(\tau),\\ \psi_b(\tau)-\psi_a(\tau)&=K_{21}p_a(\tau)+K_{22}p_b(\tau)+K_{23}\eta(\tau)+K_{24}\chi(\tau)+\sum_{m=1}^{\infty}K_{25m}\Omega_m(\tau),\end{aligned} \tag{63}$$

where

$$\begin{aligned}K_{j1}&=\sum_{i=1}^{n}\left(2A_{j1}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\zeta^{-1}f_1^{(i)}(\zeta)\mathrm{d}\zeta+A_{j2}^{(i)}\left[f_1^{(i)}(\xi_i)-f_1^{(i)}(\xi_{i-1})\right]\right),\\ K_{j2}&=\sum_{i=1}^{n}\left(2A_{j1}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\zeta^{-1}f_2^{(i)}(\zeta)\mathrm{d}\zeta+A_{j2}^{(i)}\left[f_2^{(i)}(\xi_i)-f_2^{(i)}(\xi_{i-1})\right]\right),\\ K_{j3}&=\sum_{i=1}^{n}\left(2A_{j1}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\zeta^{-1}f_3^{(i)}(\zeta)\mathrm{d}\zeta+A_{j2}^{(i)}\left[f_3^{(i)}(\xi_i)-f_3^{(i)}(\xi_{i-1})\right]+A_{j3}^{(i)}\left(\frac{1}{\xi_{i-1}}-\frac{1}{\xi_i}\right)\right),\\ K_{j4}&=\sum_{i=1}^{n}\left(2A_{j1}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\zeta^{-1}f_4^{(i)}(\zeta)\mathrm{d}\zeta+A_{j2}^{(i)}\left[f_4^{(i)}(\xi_i)-f_4^{(i)}(\xi_{i-1})\right]+A_{j4}^{(i)}\left(\frac{1}{\xi_{i-1}}-\frac{1}{\xi_i}\right)\right),\\ K_{jm}&=\sum_{i=1}^{n}\left(2A_{j1}^{(i)}\int_{\xi_{i-1}}^{\xi_i}\zeta^{-3/2}R_m^{(i)}(\zeta)\mathrm{d}\zeta+A_{j2}^{(i)}\left[\xi_i^{-\frac{1}{2}}R_m^{(i)}(\xi_i)-\xi_{i-1}^{-1/2}R_m^{(i)}(\xi_{i-1})\right]\right)\quad(j=1,2).\end{aligned} \tag{64}$$

Substituting Eq. (60) into Eq (63), we have

$$\begin{aligned}B_{11}\eta(\tau)+B_{12}\chi(\tau)+\sum_{m=1}^{\infty}\int_0^{\tau}[B_{13m}\eta(\zeta)+B_{14m}\chi(\zeta)]\sin\omega_m(\tau-\zeta)\mathrm{d}\zeta&=Y_1(\tau),\\ B_{21}\eta(\tau)+B_{22}\chi(\tau)+\sum_{m=1}^{\infty}\int_0^{\tau}[B_{23m}\eta(\zeta)+B_{24m}\chi(\zeta)]\sin\omega_m(\tau-\zeta)\mathrm{d}\zeta&=Y_2(\tau),\end{aligned} \tag{65}$$

where

$$\begin{aligned}Y_1(\tau)&=\phi_b(\tau)-\phi_a(\tau)-K_{11}p_a(\tau)-K_{12}p_b(\tau)-\sum_{m=1}^{\infty}K_{15m}\Omega_{1m}(\tau),\\ Y_2(\tau)&=\psi_b(\tau)-\psi_a(\tau)-K_{21}p_a(\tau)-K_{22}p_b(\tau)-\sum_{m=1}^{\infty}K_{25m}\Omega_{1m}(\tau),\\ B_{11}&=K_{13}+\sum_{m=1}^{\infty}K_{15m}I_{3m},\quad B_{12}=K_{14}+\sum_{m=1}^{\infty}K_{15m}I_{4m},\\ B_{21}&=K_{23}+\sum_{m=1}^{\infty}K_{25m}I_{3m},\quad B_{22}=K_{24}+\sum_{m=1}^{\infty}K_{25m}I_{4m},\\ B_{13m}&=-\omega_mK_{15m}I_{3m},\quad B_{14m}=-\omega_mK_{15m}I_{4m},\\ B_{23m}&=-\omega_mK_{25m}I_{3m},\quad B_{24m}=-\omega_mK_{25m}I_{4m}.\end{aligned} \tag{66}$$

Eq. (65) can be solved efficiently and quickly by the recently developed recursive formula [27,28]. After $\eta(\tau)$ and $\chi(\tau)$ are obtained, the mechanical, electric and magnetic fields can then be determined completely at the end.

4. Numerical results and analysis

4.1. Free vibration

It should be particularly noted here that the positive real roots $\omega_m(m = 1, 2, \ldots, \infty)$ obtained from eigenequation (B.18) (Appendix B) are just the non-dimensional natural frequencies of the radial mode of the piezoelectric/magnetostrictive composite hollow sphere for open-circuit boundary conditions. Especially, replacing the magnetostrictive layer with piezoelectric layer, we then obtain the natural frequencies of the radial mode of the multilayered piezoelectric composite hollow sphere. If we further set the piezoelectric coefficients equal to zero, the eigenequation then become that for multilayered elastic composite hollow sphere. By means of the presented eigenequation (B.18), we demonstrated the natural frequencies of the radial mode for hollow elastic and piezoelectric spheres. As expected, the results are in excellent agreement with achievements of Refs. [2,29].

In the following, the free vibration of a three-layer piezoelectric/magnetostrictive composite hollow sphere is considered. The material constants are listed in Table 1 [19] and the stacking sequence is taken as $BaTiO_3$[inner]/ $CoFe_2O_4$[middle]/$BaTiO_3$[outer] (called BFB). In the computation, the geometric parameters of the three-layer composite hollow sphere are assumed as $\xi_0 = 0.5$, $\xi_1 = 0.7$, $\xi_2 = 0.8$ and $\xi_3 = 1.0$. For the sake of convenience for the reader to trace the work, the first forty radial modes are shown in Table 2. We should further mention here that the natural frequencies in Table 2 are normalized by the material constants of $BaTiO_3$ layer. The normalization task can be easily performed by replacing the superscript "*" in Eq. (10) with "1".

Table 1
Material constants

Parameter	Unit	$BaTiO_3$	$CoFe_2O_4$
c_{11}	GPa	166.0	286.0
c_{12}	GPa	77.0	173.0
c_{13}	GPa	78.0	170.5
c_{33}	GPa	162.0	269.5
e_{31}	C/m^2	−4.4	0.0
e_{33}	C/m^2	18.6	0.0
q_{31}	N/(Am)	0.0	580.3
q_{33}	N/(Am)	0.0	699.7
ε_{33}	$C^2/(Nm^2)$	12.6×10^{-9}	0.093×10^{-9}
g_{33}	Ns/(VC)	0.0	0.0
m_{33}	Ns^2/C^2	10.0×10^{-6}	157×10^{-6}
ρ	kg/m^3	5.8×10^{-3}	5.3×10^{-3}

Table 2
Nondimensional frequencies of radial mode of BFB composite hollow sphere

Radial mode	ω				
1–5	2.21752	7.61030	14.12029	21.68019	28.17973
6–10	35.65199	42.56315	49.47783	56.96941	63.48143
11–15	71.13973	77.79372	85.02930	92.24738	98.91485
16–20	106.57212	113.08169	120.59392	127.49995	134.43715
21–25	141.92621	148.44424	156.11163	162.75817	170.00976
26–30	177.21681	183.89465	191.55001	198.05731	205.58007
31–35	212.47468	219.42494	226.90524	233.42717	241.09819
36–40	247.73682	255.00107	262.19613	268.88318	276.53498

4.2. Dynamic response

The dynamic responses of a three-layer piezoelectric/magnetostrictive composite hollow sphere subjected to dynamic pressure at the internal surface are performed in this section. As an illustrative example, the stacking sequence and the configuration as well as the geometric parameters of the composite hollow sphere in free vibration case will be reused. Furthermore, the boundary conditions are prescribed as

$$\begin{aligned} p_a(\tau) &= \mathrm{e}^{-\alpha\tau} - 1, \quad p_b(\tau) = 0, \\ \phi_a(\tau) &= 0, \quad \phi_b(\tau) = 0, \\ \psi_a(\tau) &= 0, \quad \psi_b(\tau) = 0, \end{aligned} \tag{67}$$

where α is angular frequency of dynamic load. The histories of dynamic pressure load acting at the internal surface $p_a(\tau)$ for different αs are shown in Fig. 2. Apparently, as time increases, dynamic load approaches to -1. Particularly, the dynamic pressure load for the limit case $\alpha \to \infty$ just denotes the constant pressure shock load $p_a(\tau) = -H(\tau)$, where $H(\tau)$ is Heaviside function of time.

In the demonstration, we suppose the composite hollow sphere is initially at rest, i.e., $u_0^{(i)}(\xi) = 0$ and $v_0^{(i)}(\xi) = 0 (i = 1, 2, 3)$. Also, in the computation, the first 40 terms in the series Eq. (50) are adopted. In the following figures and Table 3, all the numerical results are normalized by the material constants of $BaTiO_3$ layer.

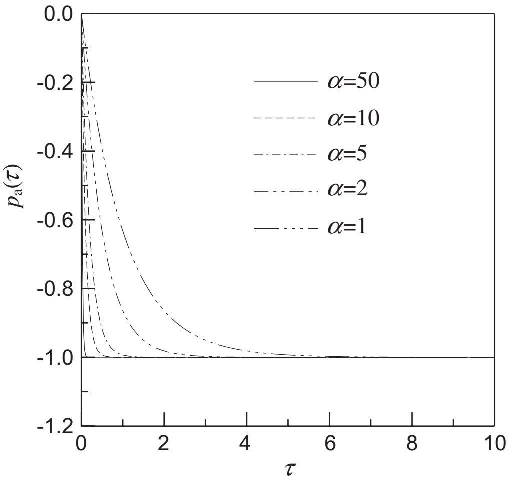

Fig. 2. Histories of dynamic pressure at the internal surface for different ωs.

Table 3
Peak values for different dynamic loads

Dynamic load	$u(\xi, \tau)$		$\sigma_r(\xi, \tau)$	$\sigma_\theta(\xi, \tau)$	
	$u(0.5, 1.3)$	$u(0.5, 7.0)$	$\sigma_r(0.7, 2.9)$	$\sigma_\theta(0.5, 1.3)$	$\sigma_\theta(0.5, 7.0)$
$p_a(\tau) = \mathrm{e}^{-2\tau}-1$	0.5449	0.5691	−0.3616	0.9370	0.9661
$p_a(\tau) = \mathrm{e}^{-5\tau}-1$	0.7231	0.7185	−0.5161	1.3303	1.3192
$p_a(\tau) = \mathrm{e}^{-10\tau}-1$	0.7943	0.7916	−0.7157	1.4979	1.4918
$p_a(\tau) = \mathrm{e}^{-50\tau}-1$	0.8253	0.8432	−1.0978	1.5708	1.6135
$p_a(\tau) = \mathrm{e}^{-100\tau}-1$	0.8240	0.8471	−1.1027	1.5678	1.6226
$p_a(\tau) = -H(\tau)$	0.8216	0.8491	−1.0633	1.5621	1.6274

Fig. 3 shows the histories of radial displacement u at the internal surface ($\xi = 0.5$) for $\alpha = 2$, 5 and 50, respectively. Clearly, although the dynamic pressure load grows smoothly, the responses of u peaks periodically. Also, the amplitude of the peaks increases gradually with increase in α.

Histories of σ_r at the two interfaces ($\xi = 0.7$ and 0.8) for different α's are depicted in Figs. 4 and 5. From the curves, we find that the responses of radial stresses at the two interfaces vary dramatically. Such phenomena are known as caused by the stress wave propagation in the radial direction.

Figs. 6 and 7 present the transient responses of σ_θ at the internal surface ($\xi = 0.5$) and the external surface ($\xi = 1.0$). Obviously, for the same α, peak values of the hoop tensile stresses at the internal surface are much lager than those at the external surface. Also, at internal and external surfaces, respectively, amplitude of peaks increases gradually with increase in α.

Table 3 lists some peak values in the composite hollow sphere subjected to different dynamic loads. As mentioned above, physically, with increase in α, dynamic pressure load $p_a(\tau) = \mathrm{e}^{-\alpha\tau} - 1$ will approach gradually to $p_a(\tau) = -H(\tau)$. This is clearly supported by numerical results in Table 3.

Figs. 8 and 9 illustrate the distributions of non-dimensional electric potential ϕ and magnetic potential ψ at $\tau = 1.0$ and 5.0, respectively. The calculated electric and magnetic potentials for each time keep zero stably at the

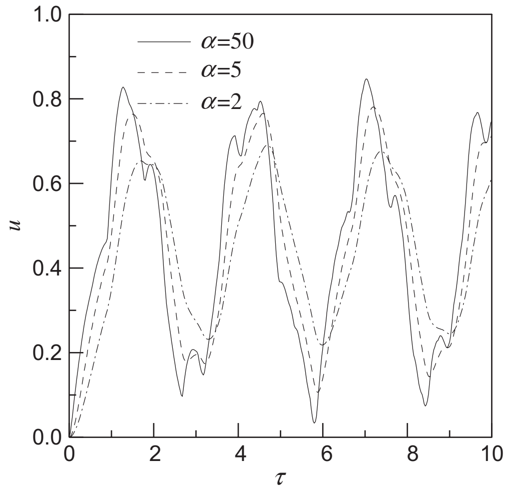

Fig. 3. Histories of radial displacement u at the internal surface $\xi = 0.5$.

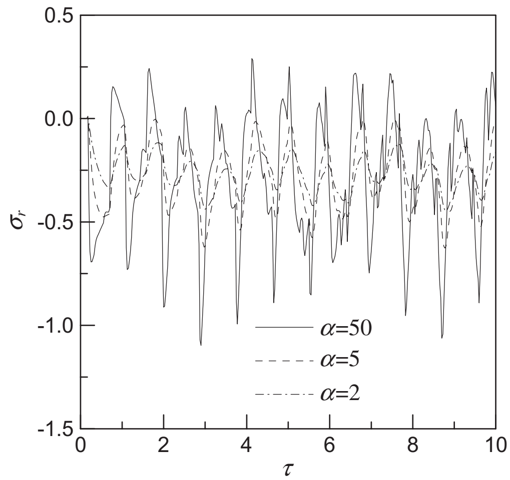

Fig. 4. Histories of radial stress σ_r at the interface $\xi = 0.7$.

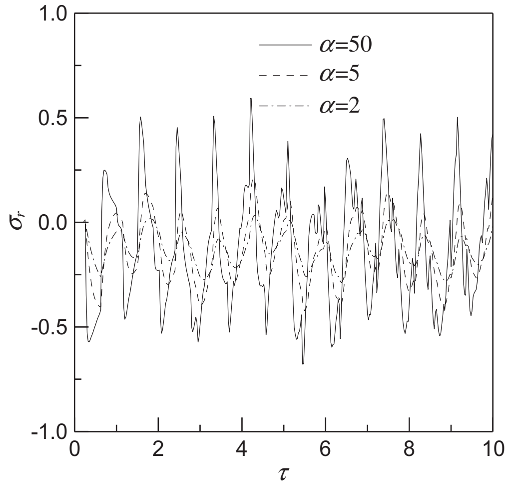

Fig. 5. Histories of radial stress σ_r at the interface $\xi = 0.8$.

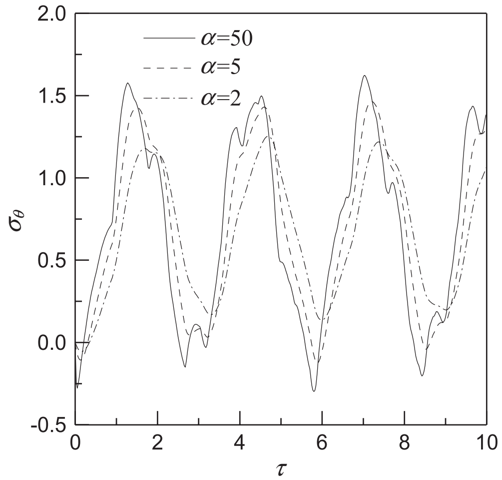

Fig. 6. Histories of hoop stress σ_θ at the internal surfaces $\xi = 0.5$.

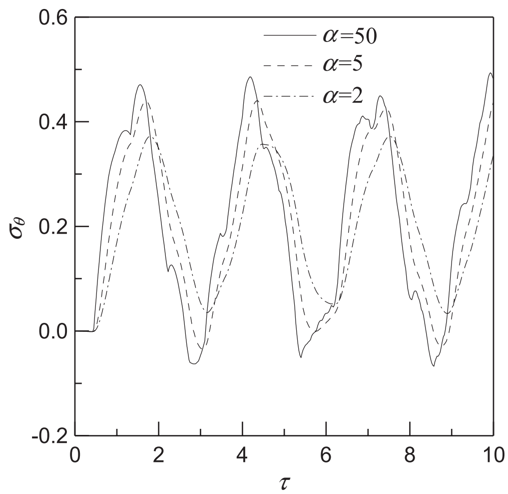

Fig. 7. Histories of hoop stress σ_θ at the external surfaces $\xi = 1$.

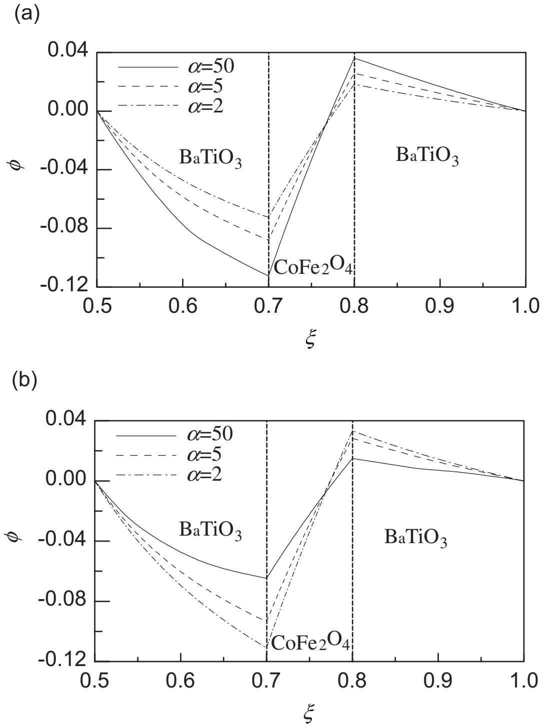

Fig. 8. Distributions of electric potential ϕ at different times (*a*) $\tau = 1.0$; (*b*) $\tau = 5.0$.

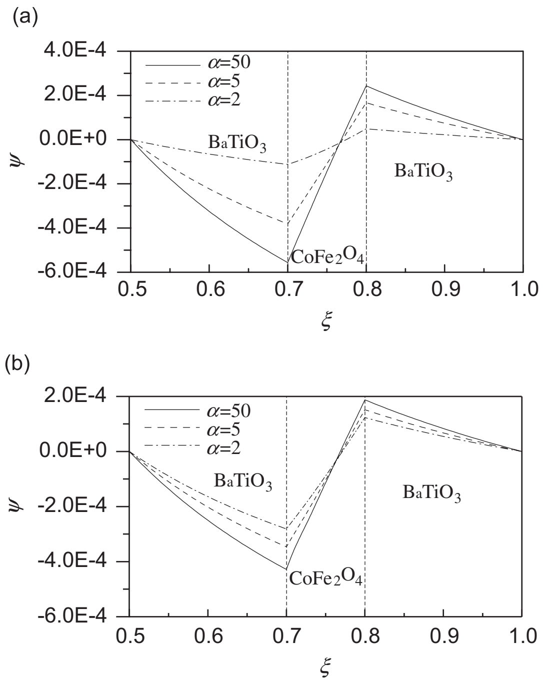

Fig. 9. Distributions of magnetic potential ψ at different times (*a*) $\tau = 1.0$; (*b*) $\tau = 5.0$.

both internal and external surfaces, which agree with the prescribed electric and magnetic boundary conditions. The correctness of the numerical results is thus clarified in this respect. By inspecting the curves, we notice that in dynamic pressure disturbed composites, the electric and magnetic potentials in both $BaTiO_3$ and $CoFe_2O_4$ layers are not equal to zero. The magneto-electro-elastic coupling effect is thus exhibited directly in this respect.

5. Summary

The radial vibration of a piezoelectric/magnetostrictive composite hollow sphere is successfully transformed to two Volterra integral equations with respect to two functions of time. The present approach method is suitable for piezoelectric/magnetostrictive composites with arbitrary stacking sequence. Numerical experiments of an internal dynamic pressure disturbed three-layer $BaTiO_3/CoFe_2O_4/BaTiO_3$ composite hollow sphere are presented and some general dynamic behaviors are illustrated. The achievements should be useful in predicting and analyzing the dynamic behaviors of the piezoelectric/magnetostrictive composites.

Acknowledgments

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Appendix A

The determination for $u_s^{(i)}(\xi,\tau)$ is presented in this Appendix.

By means of the new introduced column vector $\{X^{(i)}(\xi,\tau)\}$ in the first of Eq. (44), the continuity conditions Eq. (33) can be rewritten as

$$\{X^{(i+1)}(\xi_i,\tau)\} = \{X^{(i)}(\xi_i,\tau)\} \quad (i = 1, 2, \ldots, n-1) \tag{A.1}$$

Repeatedly utilizing Eqs. (45) and (A.1), we obtain

$$\{X^{(i)}(\xi_i,\tau)\} = [H^{(i)}]\{X^{(1)}(\xi_0,\tau)\} + \{M^{(i)}\}\eta(\tau) + \{W^{(i)}\}\chi(\tau) \quad (i = 1, 2, \ldots, n), \tag{A.2}$$

where

$$\begin{gathered}
[H^{(i)}] = [\tilde{T}_1^{(i)}], \quad [\tilde{T}_m^{(i)}] = \prod_{j=i}^{m}[T^{(j)}(\xi_j)] \quad (m = 1, 2, \ldots, i), \\
\{M^{(i)}\} = \sum_{m=1}^{i}[\tilde{T}_m^{(i)}]\{G^{(m)}(\xi_m)\}, \quad \{W^{(i)}\} = \sum_{m=1}^{i}[\tilde{T}_m^{(i)}]\{Q^{(m)}(\xi_m)\},
\end{gathered} \tag{A.3}$$

in which $\Pi_{j=1}^{1}()$ denotes continued multiplication symbol. $[H^{(i)}]$ is a 2×2 matrix. $\{M^{(i)}\}$ and $\{W^{(i)}\}$ are two 2×1 column matrices. Setting $i=n$ in Eq. (A.2) and utilizing the boundary conditions Eq. (32), we have

$$\begin{Bmatrix} u_s^{(n)}(\xi_n,\tau) \\ \xi_n p_b(\tau) \end{Bmatrix} = \begin{bmatrix} H_{11}^{(n)} & H_{12}^{(n)} \\ H_{21}^{(n)} & H_{22}^{(n)} \end{bmatrix} \begin{Bmatrix} u_s^{(1)}(\xi_0,\tau) \\ \xi_0 p_a(\tau) \end{Bmatrix} + \begin{Bmatrix} M_1^{(n)} \\ M_2^{(n)} \end{Bmatrix}\eta(\tau) + \begin{Bmatrix} W_1^{(n)} \\ W_2^{(n)} \end{Bmatrix}\chi(\tau). \tag{A.4}$$

From the second of Eq. (A.4), we obtain

$$u_s^{(1)}(\xi_0,\tau) = [\xi_n p_b(\tau) - H_{22}^{(n)}\xi_0 p_a(\tau) - M_2^{(n)}\eta(\tau) - W_2^{(n)}\chi(\tau)]\Big/H_{21}^{(n)}. \tag{A.5}$$

By means of Eq. (A.1), Eq. (45) can be rewritten as

$$\{X^{(i)}(\xi,\tau)\} = [T^{(i)}(\xi)](\{X^{(i-1)}(\xi_{i-1},\tau)\} + \{G^{(i)}(\xi)\}\eta(\tau) + \{Q^{(i)}(\xi)\}\chi(\tau)), \tag{A.6}$$

By utilizing Eq. (A.2), we have

$$\begin{aligned}
\{X^{(i)}(\xi,\tau)\} = [T^{(i)}(\xi)]([H^{(i-1)}]\{X^{(1)}(\xi_0,\tau)\} &+ \{M^{(i-1)}\}\eta(\tau) + \{W^{(i-1)}\}\chi(\tau) \\
&+ \{G^{(i)}(\xi)\}\eta(\tau) + \{Q^{(i)}(\xi)\}\chi(\tau)),
\end{aligned} \tag{A.7}$$

Eq. (A.7) can be rewritten in detailed form as

$$\begin{Bmatrix} u_s^{(i)}(\xi,\tau) \\ \Sigma_{rs}^{(i)}(\xi,\tau) \end{Bmatrix} = \begin{bmatrix} T_{11}^{(i)}(\xi) & T_{12}^{(i)}(\xi) \\ T_{21}^{(i)}(\xi) & T_{22}^{(i)}(\xi) \end{bmatrix} \left(\begin{bmatrix} H_{11}^{(i-1)} & H_{12}^{(i-1)} \\ H_{21}^{(i-1)} & H_{22}^{(i-1)} \end{bmatrix} \begin{Bmatrix} u_s^{(1)}(\xi_0,\tau) \\ \xi_0 p_a(\tau) \end{Bmatrix} + \begin{Bmatrix} M_1^{(i-1)} \\ M_2^{(i-1)} \end{Bmatrix} \eta(\tau) + \begin{Bmatrix} W_1^{(i-1)} \\ W_2^{(i-1)} \end{Bmatrix} \chi(\tau) + \begin{Bmatrix} G_1^{(i)}(\xi) \\ G_2^{(i)}(\xi) \end{Bmatrix} \eta(\tau) + \begin{Bmatrix} Q_1^{(i)}(\xi) \\ Q_2^{(i)}(\xi) \end{Bmatrix} \chi(\tau) \right). \tag{A.8}$$

Then $u_s^{(i)}(\xi,\tau)$ can be obtained from the first of Eq. (A.8). By means of Eq. (A.5), $u_s^{(i)}(\xi,\tau)$ is finally obtained as

$$u_s^{(i)}(\xi,\tau) = f_1^{(i)}(\xi)p_a(\tau) + f_2^{(i)}(\xi)p_b(\tau) + f_3^{(i)}(\xi)\eta(\tau) + f_4^{(i)}(\xi)\chi(\tau), \tag{A.9}$$

where

$$\begin{aligned} f_1^{(i)}(\xi) &= \xi_0 \left\{ \left[H_{12}^{(i-1)} - \frac{H_{22}^{(n)}}{H_{21}^{(n)}} H_{11}^{(i-1)} \right] T_{11}^{(i)}(\xi) + \left[H_{22}^{(i-1)} - \frac{H_{22}^{(n)}}{H_{21}^{(n)}} H_{21}^{(i-1)} \right] T_{12}^{(i)}(\xi) \right\}, \\ f_2^{(i)}(\xi) &= \frac{\xi_n}{H_{21}^{(n)}} [H_{11}^{(i-1)} T_{11}^{(i)}(\xi) + H_{21}^{(i-1)} T_{12}^{(i)}(\xi)], \\ f_3^{(i)}(\xi) &= T_{11}^{(i)}(\xi) \left[G_1^{(i)}(\xi) + M_1^{(i-1)} - \frac{M_2^{(n)}}{H_{21}^{(n)}} H_{11}^{(i-1)} \right] + T_{12}^{(i)}(\xi) \left[G_2^{(i)}(\xi) + M_2^{(i-1)} - \frac{M_2^{(n)}}{H_{21}^{(n)}} H_{21}^{(i-1)} \right], \\ f_4^{(i)}(\xi) &= T_{11}^{(i)}(\xi) \left[Q_1^{(i)}(\xi) + W_1^{(i-1)} - \frac{W_2^{(n)}}{H_{21}^{(n)}} H_{11}^{(i-1)} \right] + T_{12}^{(i)}(\xi) \left[Q_2^{(i)}(\xi) + W_2^{(i-1)} - \frac{W_2^{(n)}}{H_{21}^{(n)}} H_{21}^{(i-1)} \right]. \end{aligned} \tag{A.10}$$

It should be mentioned here that all the elements in matrix $[T^{(i)}(\xi)]$ can be obtained in explicit form by applying the Cayley–Hamilton theorem. Thus, $f^{(i)}(\xi)$ $(i = 1, 2, 3, 4)$ are known functions of ξ.

Appendix B

Derivation for $R_m^{(i)}(\xi)$ and the eigenequation is shown in this appendix.

The substitution of Eq. (50) into the second of Eq. (34) leads to

$$\Sigma_{rd}^{(i)}(\xi,\tau) = \xi^{-1/2} \sum_{m=1}^{\infty} \sigma_m^{(i)}(\xi)\Omega_m(\tau), \tag{B.1}$$

where

$$\sigma_m^{(i)}(\xi) = P^{(i)}(\nabla)R_m^{(i)}(\xi), \tag{B.2}$$

$$P^{(i)}(\nabla) = c_{33D}^{(i)}\nabla + 2c_{13D}^{(i)} - c_{33D}^{(i)}\Big/2. \tag{B.3}$$

Then by means of Eq. (B.1), the following equations can be derived from Eqs. (36) and (37):

$$\sigma_m^{(1)}(\xi_0) = 0, \quad \sigma_m^{(n)}(\xi_n) = 0 \quad (m = 1, 2, \ldots, \infty), \tag{B.4}$$

$$R_m^{(i+1)}(\xi_i) = R_m^{(i)}(\xi_i), \quad \sigma_m^{(i+1)}(\xi_i) = \sigma_m^{(i)}(\xi_i) \quad (m = 1,\ 2,\ \ldots, \infty;\ i = 1, 2, \ldots, n). \tag{B.5}$$

By observing the differential form of Eq. (48), we know that $R_m^{(i)}(\xi)$ must be a linear combination of $J_{\mu_i}(k_{im}\xi)$ and $Y_{\mu_i}(k_{im}\xi)$ and we assume

$$R_m^{(i)}(\xi) = D_1^{(i)} J_{\mu_i}(k_{im}\xi) + D_2^{(i)} Y_{\mu_i}(k_{im}\xi), \tag{B.6}$$

where $D_1^{(i)}$ and $D_2^{(i)}$ are undetermined constants, $J_{\mu_i}()$ and $Y_{\mu_i}()$ are Bessel functions of the first and second kinds of order μ_i, and

$$\mu_i = \sqrt{\bar{\mu}_i^2 + (\tfrac{1}{2})^2}, \quad k_{im} = \frac{\omega_m}{c_i}, \tag{B.7}$$

where ω_m is a series of positive real numbers. Substitution of Eq. (B.6) into Eq. (B.2) yields

$$\sigma_m^{(i)}(\xi) = D_1^{(i)} P^{(i)}(\nabla) J_{\mu_i}(k_{im}\xi) + D_2^{(i)} P^{(i)}(\nabla) Y_{\mu_i}(k_{im}\xi). \tag{B.8}$$

Setting $\xi = \xi_{i-1}$ in Eqs. (B.6) and (B.8), we have

$$\begin{aligned} R_m^{(i)}(\xi_{i-1}) &= D_1^{(i)} J_{\mu_i}(k_{im}\xi_{i-1}) + D_2^{(i)} Y_{\mu_i}(k_{im}\xi_{i-1}), \\ \sigma_m^{(i)}(\xi_{i-1}) &= D_1^{(i)} P^{(i)}(\nabla) J_{\mu_i}(k_{im}\xi_{i-1}) + D_2^{(i)} P^{(i)}(\nabla) Y_{\mu_i}(k_{im}\xi_{i-1}). \end{aligned} \tag{B.9}$$

Then $D_1^{(i)}$ and $D_2^{(i)}$ can be determined from Eq. (B.9). Substitution of the obtained $D_1^{(i)}$ and $D_2^{(i)}$ into Eqs. (B.6) and (B.8) yields

$$\{Z_m^{(i)}(\xi)\} = [S^{(i)}(k_{im}, \xi)]\{Z_m^{(i)}(\xi_{i-1})\}, \tag{B.10}$$

where

$$\{Z_m^{(i)}(\xi)\} = \begin{Bmatrix} R_m^{(i)}(\xi) \\ \sigma_m^{(i)}(\xi) \end{Bmatrix}, \quad [S^{(i)}(k_{im}, \xi)] = \begin{bmatrix} S_{11}^{(i)}(k_{im}, \xi) & S_{12}^{(i)}(k_{im}, \xi) \\ S_{21}^{(i)}(k_{im}, \xi) & S_{22}^{(i)}(k_{im}, \xi) \end{bmatrix} \tag{B.11}$$

and

$$\begin{aligned} S_{11}^{(i)}(k_{im}, \xi) &= [P_Y(i, k_{im}, \xi_{i-1}) J_{\mu_i}(k_{im}\xi) - P_J(i, k_{im}, \xi_{i-1}) Y_{\mu_i}(k_{im}\xi)]/\Lambda_i, \\ S_{12}^{(i)}(k_{im}, \xi) &= [J_{\mu_i}(k_{im}\xi_{i-1}) Y_{\mu_i}(k_{im}\xi) - Y_{\mu_i}(k_{im}\xi_{i-1}) J_{\mu_i}(k_{im}\xi)]/\Lambda_i, \\ S_{21}^{(i)}(k_{im}, \xi) &= [P_Y(i, k_{im}, \xi_{i-1}) P_J(i, k_{im}, \xi) - P_J(i, k_{im}, \xi_{i-1}) P_Y(i, k_{im}, \xi)]/\Lambda_i, \\ S_{22}^{(i)}(k_{im}, \xi) &= [J_{\mu_i}(k_{im}\xi_{i-1}) P_Y(i, k_{im}, \xi) - Y_{\mu_i}(k_{im}\xi_{i-1}) P_J(i, k_{im}, \xi)]/\Lambda_i, \\ \Lambda_i &= P_Y(i, k_{im}, \xi_{i-1}) J_{\mu_i}(k_{im}\xi_{i-1}) - P_J(i, k_{im}, \xi_{i-1}) Y_{\mu_i}(k_{im}\xi_{i-1}), \\ P_J(i, k_{im}, \xi) &= P^{(i)}(\nabla) J_{\mu_i}(k_{im}\xi), \quad P_Y(i, k_{im}, \xi) = P^{(i)}(\nabla) Y_{\mu_i}(k_{im}\xi). \end{aligned} \tag{B.12}$$

In Eq. (B.10), $\{Z_m^{(i)}(\xi_{i-1})\}$ is just the so-called initial parameter. By means of the newly introduced symbol $\{Z_m^{(i)}(\xi)\}$, Eq. (B.5) can be rewritten as

$$\{Z_m^{(i+1)}(\xi_i)\} = \{Z_m^{(i)}(\xi_i)\} \quad (i = 1, 2, \ldots, n-1). \tag{B.13}$$

Setting $\xi = \xi_i$ in Eq. (B.10) and repeatedly applying Eq. (B.13), we obtain

$$\{Z_m^{(i)}(\xi_i)\} = \prod_{j=i}^{1} [S^{(j)}(k_{jm}, \xi_j)]\{Z_m^{(1)}(\xi_0)\} \quad (i = 1, 2, \ldots, n), \tag{B.14}$$

where $\Pi_{j=1}^{1}()$ denotes the continued multiplication symbol. With the aid of the second of Eq. (B.7), we know that all the elements in $\Pi_{j=1}^{1}[S^{(j)}(k_{jm}, \xi_j)]$ must be the functions of ω_m and we define

$$\prod_{j=i}^{1} [S^{(j)}(k_{jm}, \xi_j)] = [\Theta^{(i)}(\omega_m)]. \tag{B.15}$$

Here $[\Theta^{(i)}(k_{im})]$ is a 2×2 matrix. Then Eq. (B.14) is rewritten as

$$\{Z_m^{(i)}(\xi_i)\} = [\Theta^{(i)}(\omega_m)]\{Z_m^{(1)}(\xi_0)\} \quad (i = 1, 2, \ldots, n). \tag{B.16}$$

Setting $i = n$ in Eq. (B.16) and utilizing Eq. (B.4), we obtain

$$\left\{ \begin{array}{c} R_m^{(n)}(\xi_n) \\ 0 \end{array} \right\} = \begin{bmatrix} \Theta_{11}^{(n)}(\omega_m) & \Theta_{12}^{(n)}(\omega_m) \\ \Theta_{21}^{(n)}(\omega_m) & \Theta_{22}^{(n)}(\omega_m) \end{bmatrix} \left\{ \begin{array}{c} R_m^{(1)}(\xi_0) \\ 0 \end{array} \right\}. \tag{B.17}$$

From the second of Eq. (B.17), the existence of nonzero solution leads to

$$\Theta_{21}^{(n)}(\omega_m) = 0. \tag{B.18}$$

Eq. (B.18), a transcendental equation, is the eigenequation from which a series of positive real roots $\omega_m (m = 1, 2, \ldots, \infty)$ can be obtained. After $\omega_m (m = 1, 2, \ldots, \infty)$, arranged in an ascending order, have been obtained, Eq. (B.10) can then be rewritten in the following form with the aid of Eqs. (B.13) and (B.16):

$$\left\{ \begin{array}{c} R_m^{(i)}(\xi) \\ \sigma_m^{(i)}(\xi) \end{array} \right\} = \begin{bmatrix} S_{11}^{(i)}(k_{im}, \xi) & S_{12}^{(i)}(k_{im}, \xi) \\ S_{21}^{(i)}(k_{im}, \xi) & S_{22}^{(i)}(k_{im}, \xi) \end{bmatrix} \begin{bmatrix} \Theta_{11}^{(i-1)}(\omega_m) & \Theta_{12}^{(i-1)}(\omega_m) \\ \Theta_{21}^{(i-1)}(\omega_m) & \Theta_{22}^{(i-1)}(\omega_m) \end{bmatrix} \left\{ \begin{array}{c} R_m^{(1)}(\xi_0) \\ 0 \end{array} \right\}. \tag{B.19}$$

From the first of Eq. (B.19), $R_m^{(i)}(\xi)$ is then obtained as

$$R_m^{(i)}(\xi) = [\Theta_{11}^{(i-1)}(\omega_m) S_{11}^{(i)}(k_{im}, \xi) + \Theta_{21}^{(i-1)}(\omega_m) S_{12}^{(i)}(k_{im}, \xi)] R_m^{(1)}(\xi_0). \tag{B.20}$$

It should be noted here that $R_m^{(1)}(\xi_0)$ in Eq. (B.20) is a common constant for every layer and can be taken as $R_m^{(1)}(\xi_0) = 1$ in the numerical calculation. Thus $R_m^{(i)}(\xi)$ is determined completely.

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