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A fast on-line frequency estimator of lightly damped vibrations in flexible structures

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Abstract

A significant research effort has been conducted in the past to achieve a reliable on-line vibration mode estimator. Frequently, pure sinusoidal models of the underlying mechanical system have been used to accomplish the estimation task. In this paper, an algebraic approach is proposed for the fast and reliable, on-line identification of the natural frequency of a flexible-link manipulator. The proposed method uses the pure sinusoidal model in combination with the algebraic derivative method for parameter identification. The method leads, in the time domain, to an exact computation formulae for the unknown frequency parameter. This formula is synthesized in terms of time-varying linear unstable filters in combination with classical low-pass filters of the Butterworth type. The computations are performed in a quite small-time interval which is only a small fraction of the first full cycle of the measured sinusoidal signal. The proposed method is verified to be robust with respect to un-modeled small attenuations, present in the flexible structure, and to measurement noise. An experimental setup has been developed as a benchmark to test and compare the proposed method with other recently developed parameter estimation techniques.

1. Introduction

Flexible arm manipulators span a wide range of applications: space robots, nuclear maintenance, microsurgery, collision control, contouring control, pattern recognition and many others. A literature survey of the many applications and of the challenging problems related to flexible manipulators may be found in Ref. [1]. Most robotic manipulators are designed in order to minimize vibration at the end effector. This normally implies high values of stiffness, i.e., heavy materials and bulky designs. These designs, however, lead to significant power consumption and to decreasing work speeds. To overcome these disadvantages, many researchers have studied manipulators with a low weight and a high payload to manipulator weight ratio. In other words, flexible manipulators may be a solution to increase productivity of these robots. Nevertheless,

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flexible arms exhibit vibrations. In order to control these vibrations, their frequency has to be estimated. This is the objective of the present work.

Generally speaking, vibration can be described by a single or a set of decoupled pure sinusoidal models, as it is customarily done in modal analysis. For a single-vibration mode, the angular frequency of the underlying sinusoidal behavior is the parameter to be estimated. A slightly more challenging problem is obtained from those cases in which the vibration frequency unexpectedly changes to a new constant unknown value [2].

The problem of reliably estimating the parameters in a sinusoidal signal has drawn considerable attention in the last decade. Although the problem is known to enjoy a simple technological solution in the analog electronics area, the control systems literature has only recently provided examples of different mathematical approaches dominated by the nonlinear adaptive control approach. Several interesting applications of this problem, ranging from bio-medics to power systems, are described by Ziarani and Konrad [3,4], Choi and Cho [5], Karini-Ghartemani and Iravan [6].

The methods proposed in the past for the solution of the sinusoid frequency estimation problem span a wide range of techniques. A non-parametric spectral power approach is presented in Kay and Marple [7]. Eigenvalue methods applied to non-parametric spectral techniques are exploited in Ref. [8] and also in Ref. [9]. An extended Kalman-filtering approach is used for the same purpose in Ref. [10]. Nevertheless, the spectral power-based estimation method is not precise enough. The eigenvalue method, on the other hand, is quite suitable for high signal-to-noise ratios cases and, hence, it has gained popularity in communications oriented applications. Kalman filtering gives good results in non-stationary situations. The computational demands are considerably high in either of the two last methods. A long-standing requirement is for an on-line, robust, sinusoid frequency identification method overcoming the limitation imposed by the noisy measurements. For active noise control applications, the notch-filter approach performs rather well. In this respect, one should highlight the work led by Kim and Park in Ref. [11] and in Ref. [12].

The natural oscillation frequency identification in flexible structures has also drawn considerable attention. In the work of Rew [13] several discrete-time frequency estimators were compared in the realm of smart structures. Here, the fact that any change in the structure, like a local failure or delamination of composite structures, is translated into changes in the system's natural frequency is suitably exploited. If one is capable of on-line monitoring these frequencies, one should be able to re-adjust the control parameters. Nevertheless, these estimators exhibited important difficulties in experiments performed on real structures. A most likely cause for the setback is that flexible structures present a damping phenomenon which is not modeled in a constant amplitude sinusoidal signal.

In this article, we propose a novel continuous-time frequency estimator which provides fast estimations of the natural frequency of a flexible-link manipulator. This estimator is fast enough to get an estimation before the damping phenomenon becomes important. Moreover, the estimation will not depend on any initial condition, or controller design parameter. This algorithm will be compared with other recently published methods which have not been tested on flexible structures so far.

The identification approach is entirely based on algebraic methods stemming from the algebraic derivative method, introduced by Fliess and Sira-Ramírez in Ref. [14] (see also Ref. [15]) for fast, reliable, constant parameter and state estimation in feedback control systems (see also Refs. [16–18]). Successful applications of this identification technique have been reported in Refs. [19–22].

Section 2 formulates the problem. Section 3 contains a set of composite linear time-varying dynamic filters and elementary nonlinear operations for the computation of the unknown characteristic parameter of the sinusoid. Section 4 presents some illustrative realistic digital computer simulations demonstrating the feasibility of the approach. Section 5 discusses the performance of the proposed method in an experimental setup. The results are compared to other methods that have been recently published in the literature. In particular, we compare some of these methods for the case when abrupt changes of the frequency take place due to mass changes at the flexible arm tip. The last section is devoted to conclusions.

2. Problem formulation

Given the noisy measurement signal, $y(t)$, of an uncertain sinusoidal signal of the form:

$$y(t) = A\sin(\omega t + \phi) + \xi(t), \tag{1}$$

where A, ω and ϕ are, respectively, the constant positive unknown amplitude, the angular frequency and the phase, it is desired to compute, as fast as possible, the unknown frequency ω. We assume, for the sake of simplicity, that ϕ is strictly positive and less than 2π. The additive signal $\xi(t)$ is a zero mean highly fluctuating process but, otherwise, of completely unknown statistics.

3. An algebraic approach solution to the problem

Consider the noise free signal $y(t) = A\sin(\omega t + \phi)$. It is easy to verify that $y(t)$ satisfies:

$$\ddot{y}(t) = -\omega^2 y(t). \tag{2}$$

The Laplace transform of this signal is given by

$$s^2 y(s) - sy(0) - \dot{y}(0) + \omega^2 y(s) = 0, \tag{3}$$

where s is the complex variable, and the initial conditions are represented by $y(0)$ and $\dot{y}(0)$. The algebraic manipulations described in the following sections are aimed at obtaining an expression for ω, involving only integrations, or integral convolutions, of the signal $y(t)$ without intervention of the initial conditions nor of the rest of the signal parameters.

3.1. On-line computation of the frequency

If we differentiate two times the expression (3) with respect to the complex frequency s, we immediately obtain an elimination of the initial conditions: $y(0)$ and $\dot{y}(0)$. Indeed,

$$s^2 \frac{\mathrm{d}^2 y(s)}{\mathrm{d}s^2} + 4s\frac{\mathrm{d}y(s)}{\mathrm{d}s} + \left(2y(s) + \omega^2 \frac{\mathrm{d}y^2(s)}{\mathrm{d}s^2}\right) = 0. \tag{4}$$

Multiplication by s^{-2} eliminates all the derivations implicit in the multiplication by positive powers of s,

$$\frac{\mathrm{d}^2 y(s)}{\mathrm{d}s^2} + 4s^{-1}\frac{\mathrm{d}y(s)}{\mathrm{d}s} + s^{-2}\left(2y(s) + \omega^2 \frac{\mathrm{d}y^2(s)}{\mathrm{d}s^2}\right) = 0. \tag{5}$$

From Eq. (5) it is easy to obtain the unknown parameter ω^2:

$$\omega^2 = \frac{-\left[\dfrac{\mathrm{d}^2 y(s)}{\mathrm{d}s^2} + 4s^{-1}\dfrac{\mathrm{d}y(s)}{\mathrm{d}s} + s^{-2}2y(s)\right]}{s^{-2}\dfrac{\mathrm{d}^2 y(s)}{\mathrm{d}s^2}}. \tag{6}$$

We can express Eq. (6) in the time domain[1] in terms of the quotient of a linear combination of iterated convolution integrals over the signal $y(t)$. We write:

$$\omega^2 = \frac{n(t)}{\mathrm{d}(t)} = \frac{-[t^2 y(t) - 4\int_0^t \sigma y(\sigma)\mathrm{d}\sigma + 2\int_0^t\int_0^\sigma y(\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\sigma]}{\int_0^t\int_0^\sigma \lambda^2 y(\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\sigma}. \tag{7}$$

We establish a state space framework to express (7) via time-varying, unstable filters. The numerator and denominator functions being given in the following way:

$$\begin{aligned} &n(t) = -[x_1 + t^2 y(t)], \quad \mathrm{d}(t) = x_3,\\ &\dot{x}_1 = x_2 + 4ty(t), \quad \dot{x}_3 = x_4,\\ &\dot{x}_2 = 2y(t), \quad \dot{x}_4 = t^2 y(t). \end{aligned} \tag{8}$$

[1] Let $\mathscr{L}$ denote the usual operational calculus transform acting on exponentially bounded signals with bounded left support. Recall that $\mathscr{L}^{-1}s(\cdot) = \mathrm{d}/\mathrm{d}t(\cdot)$, $\mathscr{L}^{-1}\mathrm{d}^\nu/\mathrm{d}s^\nu(\cdot) = (-1)^\nu t^\nu(\cdot)$ and $\mathscr{L}^{-1}1/s(\cdot) = \int_0^t(\cdot)(\sigma)\mathrm{d}\sigma$.

Note that the quotient is ill defined at time $t = 0$. Nevertheless, the quotient is certainly well defined at the end of any interval of the form $[0, \varepsilon]$ with $\varepsilon > 0$ being a sufficiently small real number, chosen on the basis of the precision of the arithmetic processor on-line computing the numerator and denominator signals.

3.2. Frequency estimation and invariant filtering

Since the available signal $y(t)$ is noisy, we should rewrite $y(t)$ in the form given by Eq. (1). In order to minimize the noise effect, we further low-pass filter, independently, the numerator and denominator signals using the same low-pass filter. This idea was successfully used in Ref. [23]. It is clear, from the formula (6) for ω^2, that, when no measurement noise are present, using identical low-pass filters transfer functions for the numerator signal and the denominator signal does not affect the value of the constant quotient. In fact, in the noisy measurements case, the simultaneous filtering results in a clear enhancement of the signal to noise ratio in the numerator and denominator signals, thus providing an improved parameter estimation over the unfiltered alternative. In order to emphasize this invariance, we use the rather customary combination of time and frequency domain notations as follows:

$$\omega^2 = \frac{-\dfrac{\omega_c^2}{s^2 + 2\zeta\omega_c + \omega_c^2}\left[t^2 y(t) - 4\int_0^t \sigma y(\sigma)\mathrm{d}\sigma + 2\int_0^t\int_0^\sigma y(\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\sigma\right]}{\dfrac{\omega_c^2}{s^2 + 2\zeta\omega_c + \omega_c^2}\int_0^t\int_0^\sigma \lambda^2 y(\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\sigma}, \tag{9}$$

where we have used identical second-order low-pass filters with cut-off frequency, ω_c, and enhanced damping features. The filtered numerator and denominator, denoted by $n_f(t)$ and $d_f(t)$, respectively, are given by the solutions of linear second-order differential equations fed by the numerator and denominator signals. We have:

$$\begin{aligned} \ddot{n}_f &= -2\zeta\omega_c\dot{n}_f - \omega_c^2(n_f - n(t)),\\ \ddot{d}_f &= -2\zeta\omega_c\dot{d}_f - \omega_c^2(d_f - \mathrm{d}(t)). \end{aligned} \tag{10}$$

The time realization of this formula can be accomplished by means of time-invariant linear filters. Consider $n(t)$ and $d(t)$ of Eq. (8) as inputs to the following system:

$$\begin{aligned} &n_f(t) = x_5, \quad d_f(t) = x_7,\\ &\dot{x}_5 = x_6, \quad \dot{x}_7 = x_8,\\ &\dot{x}_6 = -2\zeta\omega_c x_6 - \omega_c^2(x_5 - n(t)), \quad \dot{x}_8 = -2\zeta\omega_c x_7 - \omega_c^2(x_7 - \mathrm{d}(t)), \end{aligned} \tag{11}$$

Then, an estimate of ω^2 is obtained as follows:

$$\omega_e^2 = \begin{cases} \text{arbitrary} & t \in [0, \varepsilon] \\ \dfrac{n_f(t)}{d_f(t)} = \dfrac{x_5}{x_7} & t \in (\varepsilon, +\infty). \end{cases} \tag{12}$$

The invariant low-pass filter introduced above requires some a priori knowledge of the bandwidth of the system. If this knowledge is not available, we can use simultaneous iterated pure integrations of the numerator and denominator signals, using factors of the form $1/s^p$, $p \geqslant 1$. Our choice has been determined by our assumption of "high-frequency noise" in $\xi(t)$. This assumption is motivated and justified by recent developments pointing towards a brand new theory of realistic noise, avoiding the stochastic setting, with far reaching practical implications. Such a theory is based on *Non-standard Analysis*. Details may be found in Ref. [24].

4. Computer simulations

Computer simulations were carried out with the MATLAB-SIMULINK package. The perturbed sinusoidal signal chosen for the estimation of the frequency is

$$y(t) = A\sin(\omega t + \phi) + \xi(t),$$

where $A = 1$, $\omega = 10$ (rad/s); $\phi = 1$ (rad), and $\xi(t)$ is a zero mean stochastic process, generated by the instruction *rand* in the MATLAB package. This stochastic signal is built as a sequence of piece-wise constant random variables, uniformly distributed in the interval: $[-0.01, 0.01]$. The estimations have been filtered through a low-pass filter with the following features: $\omega_c = 15$ (rad/s) and $\zeta = 0.707$.

Fig. 1 depicts the actual signal $y(t)$, the frequency estimation with and without the invariant filtering. It shows the quality of the computation of the frequency ω for the sinusoidal signal, using the developed algorithm. In the first case, the computation of the frequency parameter only takes 0.1 s while the period T of the signal is $T \approx 0.63$ s. In the second case, we have not performed the invariant filtering and, as a consequence, the estimations yield less precise results. A complete robustness analysis of this technique has been developed in Ref. [22].

Despite of the high-quality results obtained above, in the case of flexible structures a damping phenomenon is unavoidably present. Therefore, it is required to study the effects caused by the un-modeled presence of the flexible structure damping. In order to do that, several computer simulations, based on a normalization of the signal, will be carried out.

It is assumed that the response of a damped second-order system may be modeled as:

$$y(t) = A\mathrm{e}^{-\zeta_n 2\pi f t} \sin(2\pi f \sqrt{1-\zeta_n^2} t + \phi), \tag{13}$$

where ζ_n is the damping coefficient and f is the natural frequency in Hertz.

The normalization is based on obtaining a sinusoidal signal which has a period of $T = 1$ s, as well as an unity amplitude if the damping coefficient is equal to zero. The following changes of time scale and amplitude are proposed:

$$\hat{t} = f\sqrt{1-\zeta_n^2} t, \quad \hat{y}(t) = \frac{y(t)}{A}. \tag{14}$$

As a result of this normalization, where $\hat{t}$ is the normalized time and $\hat{y}$ is the normalized amplitude, Eq. (13) is rewritten as:

$$\hat{y}(t) = \mathrm{e}^{-2\pi\zeta_n/\sqrt{1-\zeta_n^2}\hat{t}} \sin(2\pi\hat{t} + \phi). \tag{15}$$

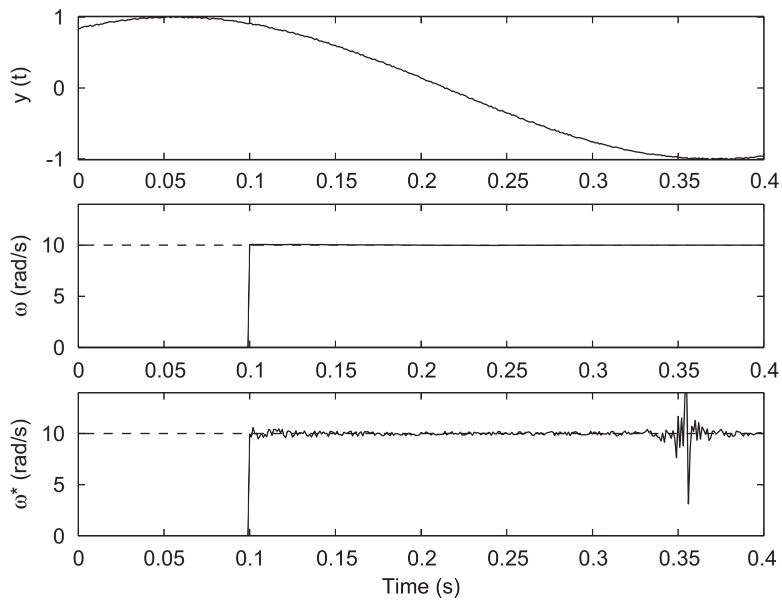

Fig. 1. From top to the bottom. Noisy sinusoidal signal, frequency estimate, (ω), with invariant filtering. Frequency estimate, (ω_*), without invariant filtering.

The simulation exercise consists of estimating the frequency of the damped sinusoidal signal varying the parameter ζ_n. The phase of the signal (13) is set to be zero. If we had a phase different from zero, we would get better results than when the phase is zero, because the signal-to-noise ratio content would be larger at times close to zero. It is intended to measure the convergence time and the precision of the estimation.

The convergence time is given by the following convergence criterion:

$$\frac{MA(k) - MA(k-M)}{MA(k)} < \text{tolerance}, \tag{16}$$

where

$$MA(k) = \frac{\sum_{i=0}^{M} \hat{f}(k-i)}{M+1}, \quad k = 1, 2, \ldots, N \tag{17}$$

is the moving average of the frequency estimate $\hat{f}(k)$ at time $t = kT_s$, and T_s is the sampling time. M is the length of the window, i.e., the number of delays that we have taken into account and N is the number of samples. We have chosen: $T_s = 2 \times 10^{-3}$ s and have set a tolerance of 10^{-3}.

On the other hand, the precision of the algorithm is measured by the Mean Absolute Error (MAE) which is given by:

$$\text{MAE} = |\hat{f}(T^*) - 1|, \tag{18}$$

where T^* is the time when the convergence is achieved according to (16).

Fig. 2 represents the effect of the damping coefficient over the convergence time and the MAE. Note that the convergence time decreases when ζ_n rises. Moreover, the MAE depends exponentially on the damping parameter ζ_n. The tolerance of the convergence criterion (10^{-3}) is shown using a dashed line. In particular, when ζ_n is equal to zero the algebraic technique achieves the given tolerance. The size of the window according to Eq. (16) is $M = 10$. Note that large values of M provide more accurate estimations, while small values provide estimations which are more sensitive to abrupt changes in the frequency.

Fig. 2 depicts the effects on the convergence time of a set of small damping coefficient values. Fortunately, flexible structures exhibit a low value of ζ_n, so we did not have any difficulties in applying the proposed algorithm to quite flexible structures.

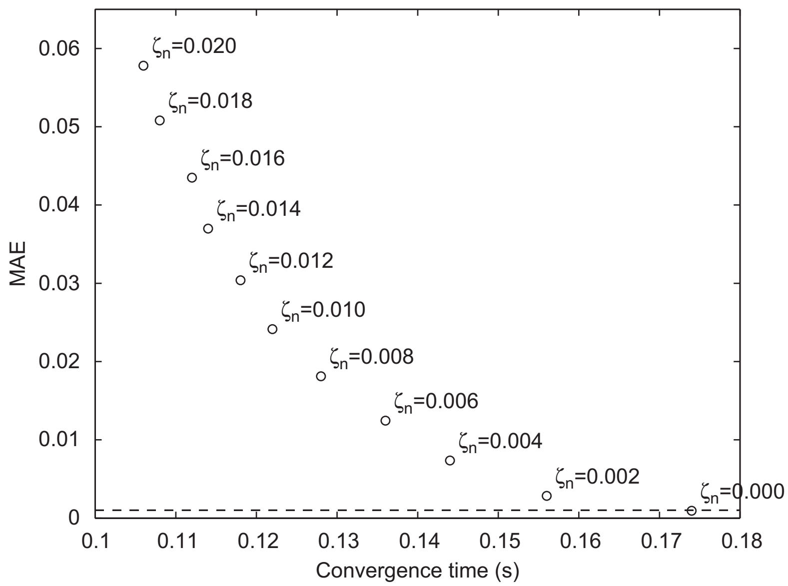

Fig. 2. Effect of the damping coefficient over the convergence time (s) and MAE. The criterion convergence tolerance is depicted via a dashed line (–).

5. Case study

An immediate application of the frequency-estimation algorithm is the determination of the vibration modes of a flexible structure. In Ref. [13], several multi-frequency estimators were proposed for the analysis and control of smart structures. In that reference, it was shown that some well-known frequency-estimation techniques do not achieve reasonable estimates under severe working conditions. In this section, we carry out an experimental implementation of the proposed algorithm on a flexible-link manipulator.

5.1. Model description

Consider the following simplified model of a lightweight flexible link (see Ref. [2]) with its mass concentrated at the tip and actuated by a DC motor as shown in Fig. 3. The dynamics of the system is given by:

$$mL^2\ddot{\theta}_t = c(\theta_m - \theta_t), \tag{19}$$

$$ku = J\ddot{\hat{\theta}}_m + v\dot{\hat{\theta}}_m + \hat{\Gamma}_c + \hat{\Gamma}_{\text{coup}}, \tag{20}$$

$$\hat{\Gamma}_{\text{coup}} = \frac{c}{n}(\theta_m - \theta_t), \tag{21}$$

where m is an unknown mass at the tip. L and c are, respectively, the length of the flexible arm and the stiffness of the bar assumed to be perfectly known. J is the inertia of the motor, v is the viscous friction coefficient, $\hat{\Gamma}_c$ is the unknown Coulomb friction torque, $\hat{\Gamma}_{\text{coup}}$ is the measured coupling torque between the motor and the link, k is a constant which involves the amplifier and electromechanical constant of the motor, u is the voltage signal which controls the motor, $\ddot{\hat{\theta}}_m$ stands for the acceleration of the motor gear, $\dot{\hat{\theta}}_m$ is the velocity of the motor gear. The constant factor n is the reduction ratio of the motor gear; thus $\theta_m = \hat{\theta}_m/n$. We denote by $\hat{\theta}_m$ the angular position of the motor while θ_t is the unmeasured angular position of the tip.

In Laplace transforms notation, the flexible bar transfer function, obtained from Eq. (19), can be written as follows:

$$Gb(s) = \frac{\theta_t(s)}{\theta_m(s)} = \frac{\omega_0^2}{s^2 + \omega_0^2}, \tag{22}$$

where

$$\omega_0 = (c/mL^2)^{1/2} = 2\pi f_0 \tag{23}$$

is the natural frequency of the bar clamped by its base. From Eq. (22), it is readily seen that the system is completely determined by the parameter ω_0. Notice that because we do not know the value of the tip mass m, we neither know the value of ω_0.

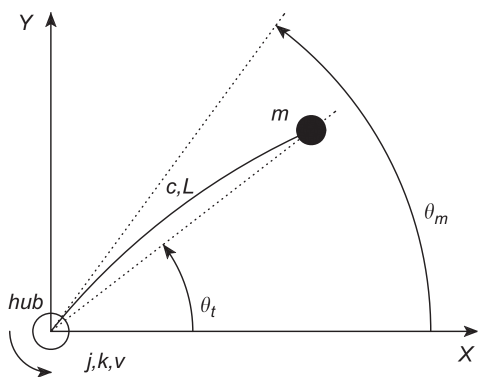

Fig. 3. Diagram of a single link flexible arm.

5.2. Experimental platform

The flexible-link manipulator here studied is a carbon fiber flexible beam, whose features are described in Table 1. This beam is embedded in the motor shaft (see Fig. 4). At the tip of the beam there is a load with the shape of a disc. The load freely rotates with respect to its vertical axis thus nullifying at the tip the effects of the torque produced by the load inertia. The load also floats on the surface of an air table, so the gravity effects and the frictions on the load are negligible.

The DC motor is fed by a servo amplifier that accepts control inputs from the computer in the dc range of $[-10, 10]$ V.

The sensor system consists of an encoder and a set of strain gauges. The encoder is embedded in the motor and it allows us to know the motor position with a precision of 7×10^{-5} (rad). The pair of strain gauges exhibit a gauge factor 2.16 and a resistance of $120.2\,\Omega$. The sample time for the processing of the signals was set to be of 2×10^{-3} s.

The control of the motor is based on a proportional derivative (PD) scheme, that uses feedback of the motor angle as described in [2]. No controller to cancel the arm vibrations has been implemented, so after any motor movement the flexible link vibrates as an one-end clamped beam with frequency ω_0.

Table 1
Flexible bar parameters

Stiffness (nm)	Diameter (mm)	Length (mm)
1.56	3	700

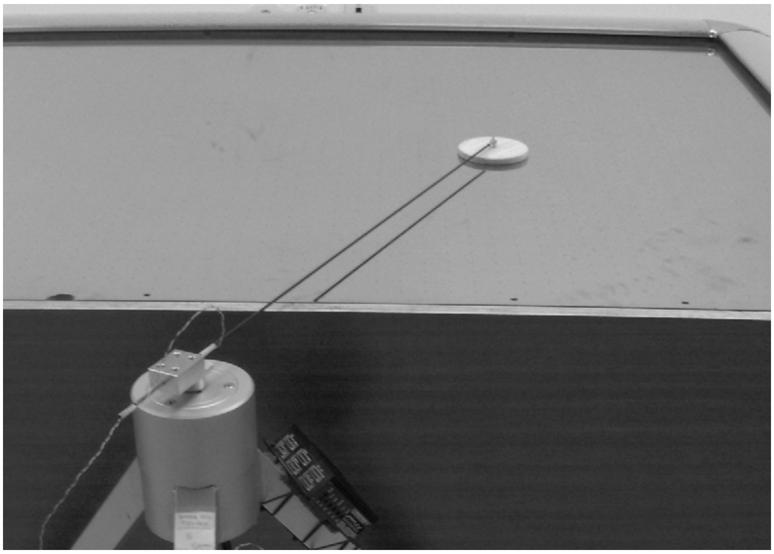

Fig. 4. Platform.

Table 2
Estimation bias and variance for the algebraic method

	f (Hz)		
	1.64	1.33	1.15
Bias	7.0×10^{-2}	3.0×10^{-2}	1.5×10^{-2}
Variance	1.2×10^{-4}	3.1×10^{-5}	7.1×10^{-5}

5.3. On-line algebraic estimation

Since our estimation equations (8), (11) and (12) yield accurate results in a quite fast manner, it is then possible to use the proposed sinusoidal frequency estimation algorithm to obtain a reliable enough frequency estimate in spite of the un-modeled damping.

Fig. 5 presents the measured coupling torque ($\Gamma_{coup} = \hat{\Gamma}_{coup} n$) from the flexible robot arm shown in Fig. 4. The command to the motor control system θ_m^* is a ramp of slope, and final value, equal to one. In flexible structures, such ramp response experiments, as well as other kind of trajectories, yield periodic signals with a non-constant amplitude, i.e., these signals exhibit the influence of an un-modeled damping coefficient. In the case of our flexible structure, this damping is relatively low. In addition, if T_1 and T_2 are peak times of the signal shown in the upper part of Fig. 5, its damping coefficient may be approximated by:

$$\zeta_0 = \frac{ln\left(\frac{y(T_2)}{y(T_1)}\right)}{2\pi f_0 (T_2 - T_1)}. \tag{24}$$

Substituting $y(T_1) = 0.0969$, $y(T_2) = 0.0945$, $T_1 = 0.394$ and $T_2 = 1.136$, it results in $\zeta_0 \approx 0.004$. With this value of damping and considering Fig. 2, we are able to estimate the natural frequency ω_0.

It should be noted that the mass is unknown, but we use the expression (23) in order to compare the results obtained by the estimator. If we substitute the values of Table 1 and use a value of the mass: $m = 45.55$ g, we obtain $f_0 = 1.33$ Hz. The frequency estimation is achieved applying the proposed algorithm until its convergence according to Eq. (16). After the estimator has converged we reset it and the estimation process is restarted. This procedure provides consecutive estimations (something like an aperiodic sampling of the vibration frequency value) at times determined by the convergence criterion (16).

Employing the proposed algorithm, a sequence of frequency estimates are obtained represented by asterisks ($*$) in the center of Fig. 5, where the nominal frequency f_0 is represented by a dashed line. The sample mean of the frequency estimates is $\hat{f} = 1.31$ Hz. Therefore, the sinusoidal period is $\hat{T} \approx \frac{1}{1.31} \approx 0.76$ s. In the estimations, a second-order filter with $\omega_c = 6\pi$ rad/s and $\zeta = 0.707$ was used. The size of the window was $M = 3$ (see Eq. (16)). Since the frequency estimates are obtained rather quickly, the distortion on the pure sinusoidal signal caused by the un-modeled damping affects minimally the accuracy of the estimation. The bottom part

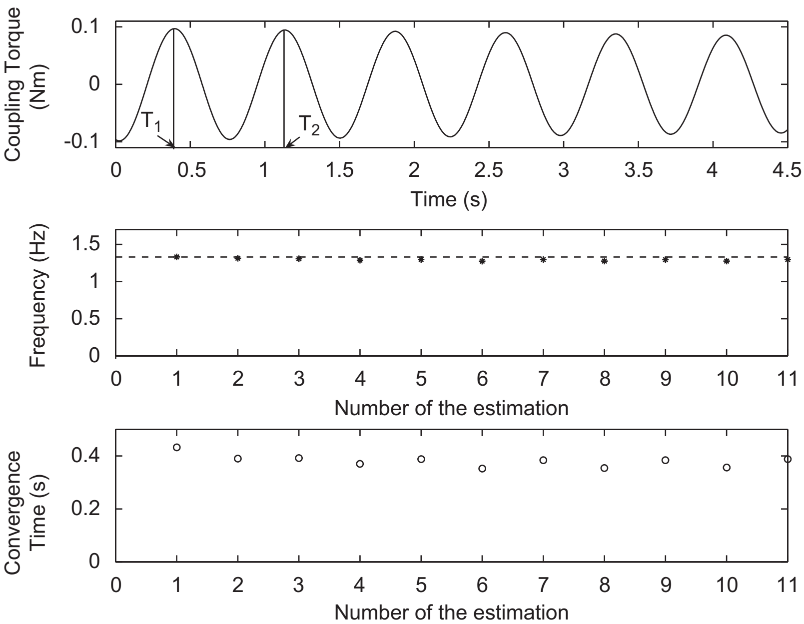

Fig. 5. From top to bottom. Vibration signal measured via Coupling Torque. Consecutive frequency estimates ($*$), f_0 (- -). Required time for each frequency estimate (o).

of this figure depicts the convergence times achieved for each frequency estimate. The convergence time is close to one-half-period.

The consecutive estimations, obtained by the algebraic method, are also represented by asterisks ($*$) in Fig. 6, where other estimation methods are used for comparison. This is explained in detail in the following section.

5.4. Comparison with other recently published methods

Discrete-time sinusoidal frequency identifiers have been proposed in several articles. See, for instance, the articles by Li and Kedem [25], Xiao and Tadokoro in Ref. [26]. The discrete-time approach results are considerably faster than continuous-time approach advocated by Hsu et al. [27] or that proposed in Ref. [28]. However, the discrete-time methods are quite sensitive to sampling frequencies and to initial conditions. Indeed, if the sampling period is very small compared to the natural time constant, these algorithms have convergence problems, see Ref. [29]. For that reason, we compare our proposed method only against methods designed in the continuous time domain.

In order to test the performance of the algebraic frequency estimator, here proposed, we compare it with the algorithm proposed by Ziarani and Konrad in [4]. This last estimator is known to be superior to some other recent well-known frequency estimators as, for instance, that developed in Refs. [27,28]. Moreover, Ziarani and Konrad have applied their method to eliminate power line interference in Ref. [3]. It should be noted that there are not many papers which apply their estimation methods to actual experimental signals. This is surprising in view of the large number of potential practical applications.

A brief note about Ziarani and Konrad method will be carried out next. In reference to Eq. (1), consider the following set of differential equations:

$$\begin{aligned}\frac{\mathrm{d}A(t)}{\mathrm{d}t} &= 2\mu_1 e(t)\sin(\gamma(t)),\\ \frac{\mathrm{d}\omega(t)}{\mathrm{d}t} &= 2\mu_2 e(t)A(t)\cos(\gamma(t)),\\ \frac{\gamma(t)}{\mathrm{d}t} &= \omega(t) + \mu_3\frac{\mathrm{d}\omega(t)}{\mathrm{d}t},\end{aligned} \tag{25}$$

where

$$\begin{aligned}e(t) &= y(t) - A(t)\sin(\gamma(t)),\\ \gamma(t) &= \omega t + \phi\end{aligned} \tag{26}$$

and μ_1, μ_2, μ_3 are design parameters that have to be tuned.[2] Furthermore, in this method, an initial frequency estimation is required. The natural frequency estimate is depicted in Fig. 6, in solid line (–), where the values of the design parameters are equal to $\mu_1 = 10, \mu_2 = 3000$, $\mu_3 = 0.1$ and $\omega_i = 6\pi\,\mathrm{rad/s}$. The figure shows the performance of this nonlinear adaptive technique, which indeed achieves an estimate of the parameter with a convergence time much larger than that of the algebraic method. The algebraic method needs less than 0.5 s to report a reliable estimate. The estimation portrayed in Fig. 6 can be smoothed with a suitable selection of μ_1, μ_2, μ_3, however the smoothed version will require a larger time for convergence.

The algebraic method is also capable of estimating some other important unknown parameters of a sinusoid, such as the amplitude and the phase. The work proposed by Hou in Ref. [30] corresponds to a dynamic frequency estimator with global convergence, which can also be extended to estimate amplitude and phase parameters. This algorithm is defined by a second-order system as follows:

$$\dot{z} = -\alpha_1 z + (\alpha_2 y^2/2 - \eta - \alpha_1^2)y, \tag{27}$$

$$\dot{\eta} = \alpha_2 y(z + \alpha_1 y), \tag{28}$$

$$\hat{\theta} = \eta - \alpha_2 y^2/2, \tag{29}$$

[2]As in other frequency estimators the tuning of design parameters define a trade-off between tracking and precision of the estimations.

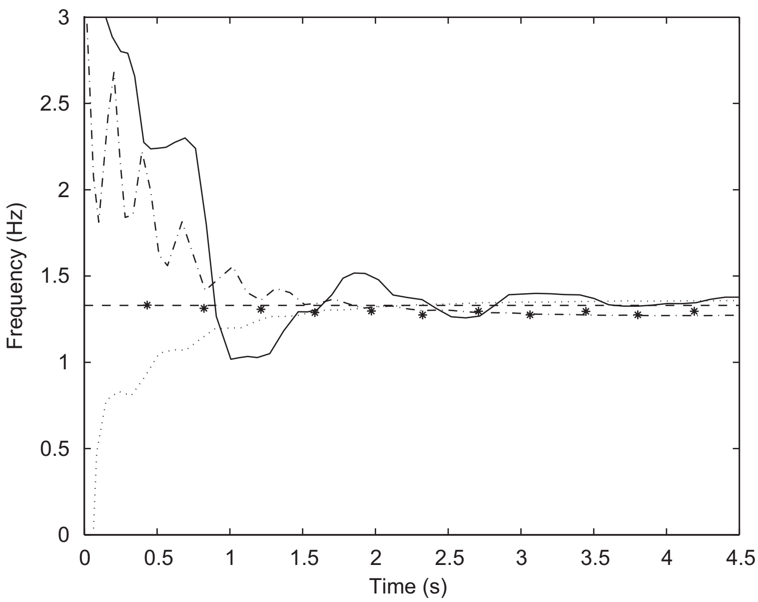

Fig. 6. Frequency estimation of the actual signal. Algebraic (*), Ziarani(-), Hou (:), Mojiri–Bakshai(−·), f_0 (- -).

where α_1 and α_2 are positive real numbers which influence the frequency estimator behavior regarding tracking and precision features. The quantity $\hat{\theta}$ converges to the square value of the frequency when time tends to infinity. The global convergence property is analyzed in Ref. [30].

Fig. 6 depicts the frequency estimate of the method (27)–(29), in dotted line (:), $\alpha_1 = 10$ and $\alpha_2 = 5000$ have been chosen as parameter values. In this case, the estimation of the unknown frequency is achieved in 1.5 s approximately.

Mojiri and Bakhshai in Ref. [31] proposed a method which accomplishes frequency estimates of periodic, but not necessarily, sinusoidal signals. The modified structure is based on the work of Hsu in Ref. [27] where the sinusoidal signal $y(t)$ is scaled by the factor $2\xi_1\theta^2$. The relevant set of differential equations are given by:

$$\ddot{x} + 2\xi_1\theta\dot{x} + \theta^2 x = 2\xi_1\theta^2 y(t), \tag{30}$$

$$\dot{\theta} = -\xi_2 x(2\xi_1\theta^2 y(t) - 2\xi_1\theta\dot{x}), \tag{31}$$

where ξ_1 and ξ_2 are design parameters, and the signal θ is found to converge to ω. For a pure sinusoidal signal, the periodic orbit is placed at:

$$O = \begin{pmatrix} \bar{x} \\ \dot{\bar{x}} \\ \bar{\theta} \end{pmatrix} = \begin{pmatrix} -A\cos\omega t \\ A\omega\sin\omega t \\ \omega \end{pmatrix}. \tag{32}$$

If the signal $y(t)$ is not a pure sinusoid the estimator will converge to a vicinity of the limit cycle of fundamental frequency approximately. The convergence precision will depend on the design parameter values and the signal distortion. In this estimator, a stability condition is provided to tune the design parameters:

$$\frac{A^2\xi_2}{2} < 1. \tag{33}$$

The frequency estimate, provided by this last method, is displayed by a dash-dotted line, (−·), in Fig. 6. The convergence of this method is achieved in approximately 1.5 s. Here the values chosen for the design parameters are $\xi_1 = 0.1$ and $\xi_2 = 1.25$. The vector of initial conditions is set to: $[x, \dot{x}, \theta_i] = [1, 1, 6\pi]$, where θ_i is given in rad/s, so $f_i = 3$ Hz. It should be noted that better estimations are achieved if the initial frequency is chosen greater than the actual frequency.

We can conclude that all other methods studied here need several cycles for the convergence of the estimate. We note that the proposed algebraic method converges in about a half of period of the unknown sinusoidal signal. Furthermore, all those methods require a previous tuning of the design parameters, or a guess of the initial conditions. These selections define important features such as: convergence ratio and precision. Such tunings are not present in our algebraic method.

5.5. *Tracking abrupt changes of frequency*

In order to asses the robustness of the proposed algorithm, we have designed an experiment where changes of mass at the tip of the robot shown in Fig. 4 are produced. The mass changes are translated into frequency changes. We have chosen as masses the following values, $m_1 = 30$ g, $m_2 = 45.55$ g and $m_3 = 61.17$ g, which correspond, according to expression (23), to a set of frequencies $f_1 = 1.64$, $f_2 = 1.33$ and $f_3 = 1.15$ Hz. Furthermore, we have varied the final values of the ramp input signals to get different vibrations with different signal-to-noise ratios. It should be noted that signals studied here yield severe conditions of performance for most algorithms.

The upper part of Fig. 7 depicts the signal analyzed which corresponds to several records of vibration with different tip masses and inputs. We observe that in the vibrations produced by the first mass we have a low signal-to-noise ratio, and in the third mass we have a high-nonlinear attenuation of the signal. The second and third graph of Fig. 7 represents the frequency estimations of the different methods. The set of parameter chosen are: $u_1 = 10$, $u_2 = 3000$, $u_3 = 0.1$; $\alpha_1 = 10$, $\alpha_2 = 5000$; $\xi_1 = 0.1$, $\xi_2 = 1.25$, $f_i = 3$ Hz for Ziarani, Hou and Mojiri–Bakshai methods, respectively, where f_i is the initial condition. The nominal frequencies $f_1 = 1.64$, $f_2 = 1.33$ and $f_3 = 1.15$ Hz are also depicted in dashed line in the second and third graph of Fig. 7 in order to assess the evolution of the algorithms. In spite of starting near the actual frequency, none of the other methods can estimate the three natural frequencies of the bar, except for the algebraic algorithm, whose estimates are seen to closely coincide with the nominal frequencies in the Fig. 7. In this case, the invariant filter used is determined by $\omega_c = 6\pi$ and $\xi = 0.707$. Furthermore, the size of the convergence criterion window is

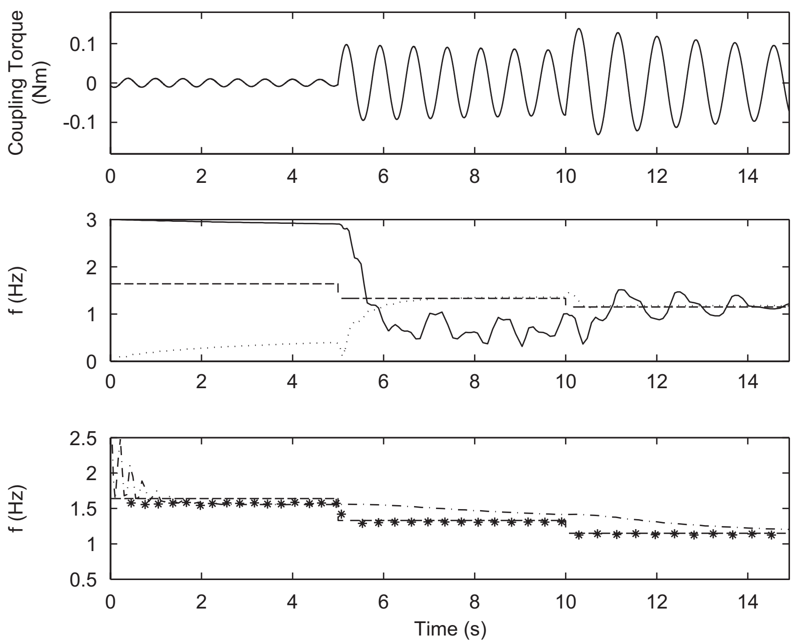

Fig. 7. Frequency estimation of the actual signal. Nominal frequencies f_1, f_2, f_3(- -), Ziarani(-), Hou (:), Algebraic ($*$), Mojiri–Bakshai (–·).

$M = 3$. A statistical analysis of the algebraic technique estimations is shown in Table 2, where bias and variance are given by the decomposition of the mean square error ($\text{MSE} = \text{bias}^2 + \text{Variance}$).

Since the first experiment is characterized by a low signal-to-noise ratio, the frequency estimation $\hat{f}_1$ has a variance greater than the estimates produced by the second and third experiment. This is in contrast to the frequencies obtained via from expression (23) based on the knowledge of the arm and its tip mass. We can thus verify that the algebraic method is more robust and much faster than the rest of the studied approaches. Aside from this, the proposed technique does not require any design parameter and it is completely independent of initial conditions.

These advantages place the proposed method as one well suited to face the vibration estimation problem. In particular, when we know that vibrations may experience abrupt frequency changes. The application of this algorithm to mechanical structures health monitoring may bring important improvements.

6. Conclusions

In this article, we have presented an algebraic approach for the fast and reliable estimation of the vibration mode of a flexible link manipulator including:

(1) zero mean, additive measurement noise.
(2) small damping of the vibration.

The algebraic method entitles devising a set of linear time-varying, unstable, filters in combination with some classical low-pass filters on the expressions resulting from algebraic manipulations carried out on the Laplace transform expression of the signal. These algebraic manipulations include elimination of unknown constants and initial conditions through derivation with respect to the complex frequency variable s.

The algebraic identification method can be successfully applied to the computation of amplitude, frequency and phase of a constant-biased sinusoidal signal as it was demonstrated in Ref. [19]. Furthermore, in Ref. [20] the frequency estimation problem has been extended to the determination of the frequencies in the noisy sum of two sinusoids. The algebraic method is also capable of estimating more than two frequencies but its real-time implementation for actual signals could be computationally difficult.

In order to test the performance of the frequency estimator under experimental conditions, we developed a benchmark prototype. The experimental on-line determination of the vibration frequency was devised using a flexible-link manipulator. The performance of the proposed frequency estimator was compared against several adaptive nonlinear frequency estimators. The proposed algebraic method stands out as the fastest one, and the only whose accuracy and convergence does not depend on initial conditions nor on the selection of design parameters.

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