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ScienceDirect
Journal of Sound and Vibration 307 (2007) 386–392
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

# Free vibration of a uniform beam with multiple elastically mounted two-degree-of-freedom systems

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Received 22 June 2006; received in revised form 7 October 2006; accepted 27 June 2007

**Abstract**

The free vibration of a beam with one or more elastically mounted two-degree-of-freedom systems that translate and rotate is considered in this note. The assumed-modes method is applied to formulate the equations of motion, and the natural frequencies of the system are found by solving for the roots of a given characteristic determinant. If the number of attached spring–mass systems is small, one can exploit the Sherman–Morrison–Woodbury determinant formula and reduce the characteristic determinant to one of smaller size, which will be easier to code and more computationally efficient to solve.


## 1. Introduction

In Ref. [1] the present author developed a general approach to formulate the frequency equation for a beam carrying miscellaneous attachments. Using the assumed-modes method [2] with $N$ component modes, the free vibration of such a combined dynamical system corresponds to the solution of a generalized eigenvalue problem of order $N \times N$, whose stiffness and mass matrices consist of diagonal matrices modified by the sum of $R$ rank-one matrices, where $R$ correspond to the number of constraints or lumped attachments. Manipulating this generalized eigenvalue problem, the free vibration can be calculated instead by solving a much smaller characteristic determinant of order $R \times R$, leading to considerable computational advantages.

The free vibration of a combined system consisting of a beam carrying an oscillator has received considerable interests, and many researchers have studied this problem over the years [3–14]. However, all the oscillators considered consisted of only a single degree-of-freedom, where the oscillator translates. In this technical note, the free vibration of a beam carrying a two-degree-of-freedom elastic system is first considered, where the system translates and rotates. The aforementioned elastic system can be used to model a vibration absorber that executes angular and up-and-down motions. Fig. 1 shows such a combined system. The oscillator is represented by a rigid rod of total mass $m$ with its center of mass $C$ at distances $l_1$ and $l_2$ from the springs $k_1$ and $k_2$, respectively. The springs are attached at $x_1$ and $x_2$ along the beam. The rod has a mass moment of inertia $J$ about its center of mass, and its vertical translation at $C$ and its rotation about $C$ are given

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0022-460X/$ - see front matter 
doi:10.1016/j.jsv.2007.06.063

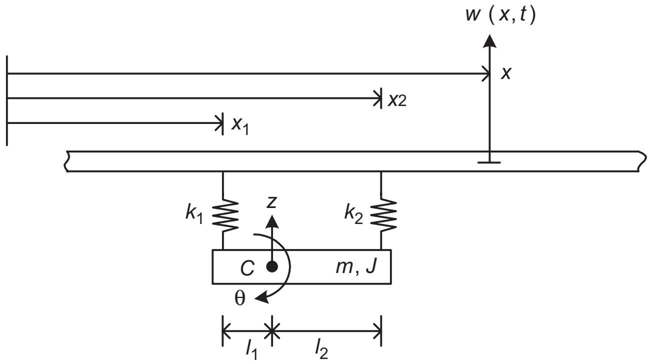


Fig. 1. An arbitrarily supported uniform beam carrying a two-degree-of-freedom spring–mass system.

by $z$ and $\theta$. The system of Fig. 1 can be analyzed by using various approaches. In Ref. [15], Wu and Whittaker used the analytical-and-numerical-combined method (ANCM) to determine the natural frequencies, and compared their results with those obtained by using the finite-element method. While ANCM is an effective technique to determine the natural frequencies of Fig. 1, the method is laborious to apply, and hence not frequently employed. Moreover, it is applicable only if each two-degree-of-freedom system is replaced by two equivalent springs using the method outlined in Ref. [15], which is rather complicated and nontrivial. In a later paper, Wu [16] proposed two finite-element methods to determine the natural frequencies and mode shapes of Fig. 1. More recently, Chen [17] applied the numerical assembly method (NAM) to determine the exact natural frequencies of a beam with arbitrary boundary conditions carrying multiple two-degree-of-freedom spring–mass systems. While the solution is exact, the approach is algebraically intensive to apply. In this technical note, the natural frequencies of Fig. 1 are obtained by using the method developed in Ref. [1], which is easy to apply and leads to a characteristic determinant that can be substantially reduced, as will soon be shown.

## 2. Theory

Consider the free vibration of Fig. 1. Using the assumed-modes method [2], the lateral displacement of the beam at point $x$ can be expressed in the form of a finite series as follows:

$$w(x,t)=\sum_{i=1}^{N}\phi_i(x)\eta_i(t), \tag{1}$$

where $N$ represents the number of modes used in the expansion, $\phi_i(x)$ are the eigenfunctions of the bare beam (or the beam without any attachment) that serve as the basis functions for this approximate solution, and $\eta_i(t)$ are the generalized coordinates of the beam. The total kinetic energy of the combined system is

$$T=\frac{1}{2}\sum_{i=1}^{N}M_i\dot{\eta}_i^2(t)+\frac{1}{2}m\dot{z}^2+\frac{1}{2}J\dot{\theta}^2, \tag{2}$$

where $M_i$ are the generalized masses of the unconstrained beam, and an overdot denotes a derivative with respect to $t$. The total potential energy of the system is

$$U=\frac{1}{2}\sum_{i=1}^{N}K_i\eta_i^2(t)+\frac{1}{2}k_1[w(x_1,t)-(z+l_1\theta)]^2+\frac{1}{2}k_2[w(x_2,t)-(z-l_2\theta)]^2, \tag{3}$$

where $K_i$ are the generalized stiffnesses of the unconstrained beam. Substituting Eq. (1) into Eqs. (2) and (3) and applying the Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\eta}_i}\right)-\frac{\partial T}{\partial\eta_i}+\frac{\partial U}{\partial\eta_i}=0\quad\text{for } i=1,2,\ldots,N, \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} + \frac{\partial U}{\partial z} = 0, \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = 0, \tag{6}$$

the following equations of motion are obtained:

$$[\mathbf{M_s}]\ddot{\mathbf{q}} + [\mathbf{K_s}]\mathbf{q} = \mathbf{0}, \tag{7}$$

where the system mass and stiffness matrices are

$$[\mathbf{M_s}] = \begin{bmatrix} [\mathbf{M^d}] & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix}, \tag{8}$$

$$[\mathbf{K_s}] = \begin{bmatrix} [\mathbf{K^d}] + k_1\boldsymbol{\phi}_1\boldsymbol{\phi}_1^{\mathrm{T}} + k_2\boldsymbol{\phi}_2\boldsymbol{\phi}_2^{\mathrm{T}} & -k_1\boldsymbol{\phi}_1 - k_2\boldsymbol{\phi}_2 & -k_1l_1\boldsymbol{\phi}_1 + k_2l_2\boldsymbol{\phi}_2 \\ -k_1\boldsymbol{\phi}_1^{\mathrm{T}} - k_2\boldsymbol{\phi}_2^{\mathrm{T}} & k_1 + k_2 & k_1l_1 - k_2l_2 \\ -k_1l_1\boldsymbol{\phi}_1^{\mathrm{T}} + k_2l_2\boldsymbol{\phi}_2^{\mathrm{T}} & k_1l_1 - k_2l_2 & k_1l_1^2 + k_2l_2^2 \end{bmatrix} \tag{9}$$

and the vector of generalized coordinates is given by

$$\mathbf{q} = \begin{bmatrix} \boldsymbol{\eta} \\ z \\ \theta \end{bmatrix}. \tag{10}$$

Vector $\boldsymbol{\phi}_i$ consists of the eigenfunctions evaluated at the attachment location, $x_i$, as follows:

$$\boldsymbol{\phi}_i = [\phi_1(x_i), \ldots, \phi_j(x_i), \ldots, \phi_N(x_i)]^{\mathrm{T}} \tag{11}$$

and matrices $[\mathbf{M^d}]$ and $[\mathbf{K^d}]$ are both diagonal, whose $i$th diagonal elements are given by $M_i$ and $K_i$ (the $i$th generalized mass and stiffness of the unconstrained beam), respectively. Assuming harmonic motion,

$$\mathbf{q} = \bar{\mathbf{q}}\,\mathrm{e}^{\mathrm{j}\omega t}, \tag{12}$$

where the vector of the amplitudes of the generalized coordinates is defined as

$$\bar{\mathbf{q}} = \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{z} \\ \bar{\theta} \end{bmatrix}. \tag{13}$$

Eq. (7) becomes

$$([\mathbf{K_s}] - \omega^2[\mathbf{M_s}])\bar{\mathbf{q}} = \mathbf{0}, \tag{14}$$

whose bottom two equations are

$$(k_1 + k_2 - m\omega^2)\bar{z} + (k_1l_1 - k_2l_2)\bar{\theta} = (k_1\boldsymbol{\phi}_1^{\mathrm{T}} + k_2\boldsymbol{\phi}_2^{\mathrm{T}})\bar{\boldsymbol{\eta}} \tag{15}$$

and

$$(k_1l_1 - k_2l_2)\bar{z} + (k_1l_1^2 + k_2l_2^2 - J\omega^2)\bar{\theta} = (k_1l_1\boldsymbol{\phi}_1^{\mathrm{T}} - k_2l_2\boldsymbol{\phi}_2^{\mathrm{T}})\bar{\boldsymbol{\eta}}. \tag{16}$$

Solving Eqs. (15) and (16) simultaneously for $\bar{z}$ and $\bar{\theta}$ in terms of $\bar{\boldsymbol{\eta}}$, one obtains

$$\bar{z} = \frac{k_1(k_2l_2^2 + k_2l_1l_2 - J\omega^2)\boldsymbol{\phi}_1^{\mathrm{T}} + k_2(k_1l_1^2 + k_1l_1l_2 - J\omega^2)\boldsymbol{\phi}_2^{\mathrm{T}}}{d}\,\bar{\boldsymbol{\eta}} \tag{17}$$

and

$$\bar{\theta} = \frac{k_1(k_2l_1 + k_2l_2 - ml_1\omega^2)\boldsymbol{\phi}_1^{\mathrm{T}} - k_2(k_1l_1 + k_1l_2 - ml_2\omega^2)\boldsymbol{\phi}_2^{\mathrm{T}}}{d}\,\bar{\boldsymbol{\eta}}, \tag{18}$$

where

$$d = Jm\omega^4 - (k_1J + k_2J + k_1l_1^2m + k_2l_2^2m)\omega^2 + k_1k_2(l_1 + l_2)^2. \tag{19}$$

The top matrix equation of Eq. (14) is

$$([\mathbf{K^d}] + k_1\boldsymbol{\phi}_1\boldsymbol{\phi}_1^{\mathrm{T}} + k_2\boldsymbol{\phi}_2\boldsymbol{\phi}_2^{\mathrm{T}} - \omega^2[\mathbf{M^d}])\bar{\boldsymbol{\eta}} - (k_1\boldsymbol{\phi}_1 + k_2\boldsymbol{\phi}_2)\bar{z} + (k_2l_2\boldsymbol{\phi}_2 - k_1l_1\boldsymbol{\phi}_1)\bar{\theta} = \mathbf{0}, \tag{20}$$

which can be simplified to

$$([\mathbf{K^d}] - \omega^2[\mathbf{M^d}] + \alpha_1\boldsymbol{\phi}_1\boldsymbol{\phi}_1^{\mathrm{T}} + \alpha_2\boldsymbol{\phi}_1\boldsymbol{\phi}_2^{\mathrm{T}} + \alpha_3\boldsymbol{\phi}_2\boldsymbol{\phi}_1^{\mathrm{T}} + \alpha_4\boldsymbol{\phi}_2\boldsymbol{\phi}_2^{\mathrm{T}})\bar{\boldsymbol{\eta}} = \mathbf{0}, \tag{21}$$

where

$$\alpha_1 = k_1 - \frac{k_1^2(k_2l_2^2 + k_2l_1l_2 - J\omega^2) + k_1^2l_1(k_2l_1 + k_2l_2 - ml_1\omega^2)}{d}, \tag{22}$$

$$\alpha_2 = \frac{-k_1k_2(k_1l_1^2 + k_1l_1l_2 - J\omega^2) + k_1k_2l_1(k_1l_1 + k_1l_2 - ml_2\omega^2)}{d}, \tag{23}$$

$$\alpha_3 = \frac{-k_1k_2(k_2l_2^2 + k_2l_1l_2 - J\omega^2) + k_1k_2l_2(k_2l_1 + k_2l_2 - ml_1\omega^2)}{d}, \tag{24}$$

$$\alpha_4 = k_2 - \frac{k_2^2(k_1l_1^2 + k_1l_1l_2 - J\omega^2) + k_2^2l_2(k_1l_1 + k_1l_2 - ml_2\omega^2)}{d}. \tag{25}$$

The natural frequencies, $\omega$, of the system are obtained by setting the determinant of the coefficient matrix of Eq. (21) equal to zero

$$\det([\mathbf{K^d}] - \omega^2[\mathbf{M^d}] + [\mathbf{U}][\mathbf{V}]^{\mathrm{T}}) = 0, \tag{26}$$

where

$$[\mathbf{U}] = [\alpha_1\boldsymbol{\phi}_1 \quad \alpha_2\boldsymbol{\phi}_1 \quad \alpha_3\boldsymbol{\phi}_2 \quad \alpha_4\boldsymbol{\phi}_2] \tag{27}$$

and

$$[\mathbf{V}] = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2]. \tag{28}$$

Note that $[\mathbf{U}]$ and $[\mathbf{V}]$ are both of size $N \times 4$, and that the coefficient matrix of Eq. (21) consists of a matrix $[\mathbf{K^d}] - \omega^2[\mathbf{M^d}]$ modified by a finite sum of dyads.

The coefficient matrix of Eq. (21) is of size $N \times N$. To ensure sufficient accuracy, $N$ is generally large, and expanding the determinant of the coefficient matrix becomes computationally intensive. Fortunately, mathematicians have studied the determinant of a general matrix modified by a finite sum of dyads. In particular, if an invertible matrix $[\mathbf{A}]$ is modified by a matrix product $[\mathbf{U}][\mathbf{V}]^{\mathrm{T}}$, then the Sherman–Morrison–Woodbury determinant formula [18] can be used to compute the newly modified matrix as follows:

$$\det([\mathbf{A}] + [\mathbf{U}][\mathbf{V}]^{\mathrm{T}}) = \det[\mathbf{A}]\det([\mathbf{I}] + [\mathbf{V}]^{\mathrm{T}}[\mathbf{A}]^{-1}[\mathbf{U}]). \tag{29}$$

Incidentally, using a general determinant formula and after some algebraic manipulations, Gürgöze [19] also obtained the results of Eq. (29), which can be utilized to substantially reduce the computational effort needed to obtain the natural frequencies of the combined system. Thus, instead of expanding the coefficient matrix of Eq. (26) of size $N \times N$, one can exploit Eq. (29) and evaluate the determinant of a smaller matrix. Specifically, by setting $[\mathbf{A}] = [\mathbf{K^d}] - \omega^2[\mathbf{M^d}]$ (which is invertible), Eq. (26) becomes

$$\det[\mathbf{A}]\det([\mathbf{I}] + [\mathbf{V}]^{\mathrm{T}}[\mathbf{A}]^{-1}[\mathbf{U}]) = 0. \tag{30}$$

Note that since $[\mathbf{A}]$ is diagonal, its determinant is trivial to obtain. Moreover, the second determinant of Eq. (30) reduces to size $4 \times 4$. For $4 \ll N$, Eq. (30) proves more computationally efficient to solve. Finally, the natural frequencies, $\omega$, satisfy Eq. (30) and can be determined either graphically or numerically using any standard root solver routine such as *fsolve* in MATLAB or the subroutine *zeroin* in EISPACK.

## 3. Results

The eigenfunctions used in the assumed-modes method depend on the boundary conditions of the bare beam. For a uniform fixed–free beam, its normalized (with respect to the mass per unit length, $\rho$, of the beam) eigenfunctions are given by

$$\phi_i(x) = \frac{1}{\sqrt{\rho L}}\left(\cosh\beta_i x - \cos\beta_i x + \frac{\sin\beta_i L - \sinh\beta_i L}{\cos\beta_i L + \cosh\beta_i L}(\sinh\beta_i x - \sin\beta_i x)\right), \tag{31}$$

where $\beta_i L$ satisfies the following transcendental equation:

$$\cos\beta_i L\cosh\beta_i L = -1. \tag{32}$$

For a uniform fixed–fixed beam, its normalized eigenfunctions are given by

$$\phi_i(x) = \frac{1}{\sqrt{\rho L}}\left(\cosh\beta_i x - \cos\beta_i x + \frac{\cos\beta_i L - \cosh\beta_i L}{\sinh\beta_i L - \sin\beta_i L}(\sinh\beta_i x - \sin\beta_i x)\right), \tag{33}$$

where $\beta_i L$ satisfies the following transcendental equation:

$$\cos\beta_i L\cosh\beta_i L = 1. \tag{34}$$

The generalized masses and stiffnesses of either the fixed–free or fixed–fixed beam are

$$M_i = 1 \quad \text{and} \quad K_i = (\beta_i L)^4 EI/(\rho L^4), \tag{35}$$

where $E$ is the Young's modulus, $I$ is the moment of inertia of the cross-section of the beam, and $\beta_i L$ satisfies either Eq. (32) or (34), depending on if the beam is fixed–free or fixed–fixed.

To validate the present approach, the natural frequencies of a uniform fixed–free or fixed–fixed beam with an elastically mounted two-degree-of-freedom system are determined, and the results are compared to those obtained by using the ANCM and the finite-element method published in Ref. [15]. The dimensions and material constants for the beam are identical to those used in Ref. [15], namely: Young's modulus $E = 2.069\times10^{11}\,\mathrm{N\,m^{-2}}$, mass per unit length $\rho = 15.3875\,\mathrm{kg\,m^{-1}}$, moment of inertia of the cross-section $I = 3.06796\times10^{-7}\,\mathrm{m^4}$, and length $L = 1.0\,\mathrm{m}$. The system parameters for the elastically mounted two-degree-of-freedom system are: mass $m = 1.53875\,\mathrm{kg}$, mass moment of inertia $J = 1.53875\,\mathrm{kg\,m^2}$, spring stiffnesses $k_1 = k_2 = 6.34761\times10^6\,\mathrm{N\,m^{-1}}$, distances of $k_1$ and $k_2$ from the center of mass $l_1 = 0.06667\,\mathrm{m}$ and $l_2 = 0.13333\,\mathrm{m}$.

Consider first a fixed–fixed beam carrying a two-degree-of-freedom spring–mass system located at $x_1 = 0.2\,\mathrm{m}$ and $x_2 = 0.4\,\mathrm{m}$. Table 1 shows the first five natural frequencies of the combined system, obtained using ANCM, the finite-element method [15], and the scheme outlined in this note. Note the excellent agreement among the different approaches. Consider next a fixed–free beam with an elastically mounted two-degree-of-freedom system located at $x_1 = 0.8\,\mathrm{m}$ and $x_2 = 1.0\,\mathrm{m}$. Table 2 shows the first five natural frequencies obtained using the various methods. Note again how well the approaches track one another.

Table 1
The first five natural frequencies of Fig. 1

| $\omega_i$ (in rad s$^{-1}$) | ANCM [15] | FEM [15] | Proposed method |
|---|---|---|---|
| $\omega_1$ | 273.8904 | 273.8565 | 273.8892 |
| $\omega_2$ | 1388.6244 | 1388.5937 | 1388.6073 |
| $\omega_3$ | 2880.5511 | 2879.7694 | 2880.0323 |
| $\omega_4$ | 4222.2172 | 4221.9181 | 4221.9610 |
| $\omega_5$ | 7837.1068 | 7837.4548 | 7836.9696 |

The beam is fixed–fixed, and the attachment locations for the elastically mounted system are $x_1 = 0.2\,\mathrm{m}$ and $x_2 = 0.4\,\mathrm{m}$. For the proposed method, 10 modes are used ($N = 10$).

Table 2
The first five natural frequencies of Fig. 1

| $\omega_i$ (in rad $s^{-1}$) | ANCM [15] | FEM [15] | Proposed method |
|---|---|---|---|
| $\omega_1$ | 143.4354 | 143.4206 | 143.4308 |
| $\omega_2$ | 324.3061 | 324.2268 | 324.2986 |
| $\omega_3$ | 1526.9812 | 1526.8963 | 1526.9820 |
| $\omega_4$ | 3330.0140 | 3326.6748 | 3327.5687 |
| $\omega_5$ | 4281.0266 | 4279.7603 | 4280.3173 |

The beam is fixed–free, and the attachment locations for the elastically mounted system are $x_1 = 0.8$ m and $x_2 = 1.0$ m. For the proposed method, 10 modes are used ($N = 10$).

Table 3
The locations and system parameters of the three two-degree-of-freedom spring–mass systems

| Physical properties | $i = 1$ | $i = 2$ | $i = 3$ |
|---|---|---|---|
| $x_1^i$ (m) | 0.1 | 0.4 | 0.8 |
| $x_2^i$ (m) | 0.3 | 0.6 | 1.0 |
| $l_1^i$ (m) | 0.07 | 0.06 | 0.12 |
| $l_2^i$ (m) | 0.13 | 0.14 | 0.08 |
| $k_1^i = k_2^i$ ($Nm^{-1}$) | 60 | 600 | 6000 |
| $m^i$ (kg) | 1.6 | 1.6 | 1.6 |
| $J^i$ (kg $m^2$) | 1.6 | 3.2 | 4.8 |

The superscript $i$ denotes the $i$th spring–mass system.

The formulation outlined in Section 3 can be easily modified to analyze the free vibration of a beam carrying multiple, say $S$, two-degree-of-freedom systems. In this case, Eq. (26) still remains valid, except now

$$[\mathbf{U}] = [\alpha_1^1\boldsymbol{\phi}_1^1 \quad \alpha_2^1\boldsymbol{\phi}_1^1 \quad \alpha_3^1\boldsymbol{\phi}_2^1 \quad \alpha_4^1\boldsymbol{\phi}_2^1 \ldots \alpha_1^S\boldsymbol{\phi}_1^S \quad \alpha_2^S\boldsymbol{\phi}_1^S \quad \alpha_3^S\boldsymbol{\phi}_2^S \quad \alpha_4^S\boldsymbol{\phi}_2^S] \tag{36}$$

and

$$[\mathbf{V}] = [\boldsymbol{\phi}_1^1 \quad \boldsymbol{\phi}_2^1 \quad \boldsymbol{\phi}_1^1 \quad \boldsymbol{\phi}_2^1 \ldots \boldsymbol{\phi}_1^S \quad \boldsymbol{\phi}_2^S \quad \boldsymbol{\phi}_1^S \quad \boldsymbol{\phi}_2^S], \tag{37}$$

where the superscript designates the spring–mass system number, and $\boldsymbol{\phi}_1^j$ and $\boldsymbol{\phi}_2^j$ are the vector of eigenfunctions evaluated at $x_1^j$ and $x_2^j$, the attachment locations for the $j$th spring–mass. The $\alpha_i^j$ are given by Eqs. (22)–(25), where the physical parameters correspond to those of the $j$th elastically mounted two-degree-of-freedom system, i.e., for the $j$th spring–mass, $m$, $J$, $k_1$, $k_2$, $l_1$ and $l_2$ of Eqs. (22) to (25) are replaced by $m^j$, $J^j$, $k_1^j$, $k_2^j$, $l_1^j$ and $l_2^j$. Note that [**U**] and [**V**] are both of size $N \times 4S$. Depending on $S$, the natural frequencies can be obtained by solving either Eq. (26) (of size $N \times N$) or Eq. (30) (of size $4S \times 4S$), whichever size is smaller. Consider a fixed–free beam carrying three two-degree-of-freedom spring–mass systems. The locations and physical properties of the spring–mass systems are listed in Table 3, and they are identical to those of Table 3 in Ref. [15]. Table 4 shows the first five natural frequencies obtained by using the various schemes. Note how well the natural frequencies agree with each other, validating that the proposed technique can be extended to study the free vibration of a beam carrying multiple two-degree-of-freedom spring–mass systems.

The approach described in Ref. [1] is general and can be extended to analyze the free vibration of an arbitrarily supported beam carrying one more elastically mounted two-degree-of-freedom systems. Unlike ANCM, it does not require one to replace the each spring–mass system by two equivalent springs, the technique of which is non-trivial. The method outlined in this note is straightforward to apply, simple to code, and leads to a characteristic determinant whose roots can be solved either graphically or numerically. It can also be easily extended to accommodate a beam with arbitrary boundary conditions.

Table 4
The first five natural frequencies of a beam carrying three two-degree-of-freedom spring–mass systems

| $\omega_i$ (in rad s$^{-1}$) | ANCM [15] | FEM [15] | Proposed method |
|---|---|---|---|
| $\omega_1$ | 231.9466 | 231.9355 | 231.9355 |
| $\omega_2$ | 1415.7972 | 1415.8251 | 1415.8221 |
| $\omega_3$ | 3962.8895 | 3962.9617 | 3962.8968 |
| $\omega_4$ | 7765.4563 | 7765.8580 | 7765.3729 |
| $\omega_5$ | 12836.626 | 12838.743 | 12836.568 |

The beam is fixed–free, and the spring–mass system parameters are shown in Table 3. For the proposed method, 10 modes are used ($N = 10$).

## 4. Conclusions

This technical note deals with the free vibration of an arbitrarily supported beam with one or more elastically mounted two-degree-of-freedom systems. The assumed modes method and the Lagrange's equations lead to a characteristic determinant whose roots correspond to the natural frequencies of the system. Exploiting the Sherman–Morrison–Woodbury determinant formula, one can reduce the size of the characteristic determinant that needs to the solved. The proposed approach is systematic to apply and easy to code. Numerical experiments validated the proposed scheme, and excellent agreements were found between the proposed method and those published in the literature.

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