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Accurate approximate analytical solutions for nonlinear free vibration of systems with serial linear and nonlinear stiffness

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Abstract

This paper deals with free vibration of a nonlinear system having combined linear and nonlinear springs in series. The conservative oscillation system is formulated as a nonlinear ordinary differential equation having linear and nonlinear stiffness components. The governing equation is linearized and associated with the harmonic balance method to establish new and accurate higher-order analytical approximate solutions. Unlike the perturbation method which is restricted to nonlinear conservative systems with a small perturbed parameter and also unlike the classical harmonic balance method which results in a complicated set of algebraic equations, the new approach yields simple approximate analytical expressions valid for small as well as large amplitudes of oscillation. Some examples are solved and compared with numerical integration solutions and published results. New solutions to the nonlinear systems are also presented and discussed.

1. Introduction

The Lindstedt–Poincaré perturbation (LP) [1–6] and the classical harmonic balance (HB) methods [1,3,5–11] are some of the most commonly used asymptotic techniques for solving nonlinear oscillation systems. The perturbation method is, in principle, useful if there exist small parameters in the nonlinear systems for which the solution can be analytically expanded into power series of the parameters. The coefficients of the series are found as solutions of a set of linear problems. However, very frequently small parameters in many nonlinear problems in science and engineering are not available, for instance, the Klein–Gordon equation [12] and the Duffing-harmonic oscillator [13,14]. Even if such small parameters exist, the analytical solutions given by the perturbation methods have, in most cases, a small range of validity. Although a large nonlinear parameter can be treated by the classical HB method, it yields a set of complicated nonlinear algebraic equations upon harmonic balancing thus barring availability of simple analytical solutions. The new approach here extends both the perturbation and HB methods. Instead of nonlinear algebraic equations, a set of simple linear algebraic equations with accurate higher-order approximate analytical solutions valid beyond the scope of perturbation method can be obtained.

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In addition to the LP and classical HB methods, there have been many other methods developed for solving nonlinear oscillation systems, including the KBM method [1–4,6], the multiple scales method [1,3,4,6], which are applicable to nonlinear oscillation systems even for rather large amplitudes of oscillation. However, higher-order analytic approximations are usually difficult because they often result in a set of complicated coupled equations requiring further recursive numerical analysis. Recently, the weighted linearization method [15], the modified Lindstedt–Poincaré method [16], the power-series approach [17] and the homotopy analysis method [18] were proposed to solve approximate periods with large amplitude of oscillations. These methods involve tedious derivations and extensive computations.

Recently, Telli and Kopmaz [19] attempted to solve the motion of a mechanical system associated with linear and nonlinear properties using analytical and numerical techniques. It dealt with vibration of a conservative oscillation system with attached mass grounded by linear and nonlinear springs. The linkage of the linear and nonlinear springs in series has been derived with cubic nonlinear characteristics in the equations of motion [19]. As mentioned in Ref. [19], although there exists a vast literature on discrete systems including either linear or nonlinear springs/restoring forces [3,5], one does not encounter publications on mechanical systems with single-degree-of-freedom containing flexible component consisting of linear–nonlinear spring in series, which occurs in technical applications. One similar mechanical model is the conservative Duffing equation with linear and cubic characteristics governed by a second-order differential equation. The equation of motion can be formed by transforming intermediate variables into a set of differential algebraic equations and it may be further transformed into a nonlinear ordinary differential equation. The resulted nonlinear differential equation was separately solved by using the LP and HB methods and compared with numerical integration solutions using a built-in ODE-solver in MATLAB.

In an attempt to obtain accurate approximate analytical solutions for the system with combined linear and nonlinear stiffness, this paper presents an alternative extension of the harmonic balancing method by linearizing the governing equation prior to harmonic balancing, termed the LHB method [20]. There are numerous linearization techniques for linearizing nonlinear problems such as equivalent linearization [5,6,21] and phase space linearization [22,23]. Taylor-series [24–27] and Fourier-series expansions [28,29] have also been widely used for solving the nonlinear problems. In this context, the governing equation of motion in the paper is expanded in Fourier series because the elegant convergence of Fourier-series expansions significantly increases the accuracy of the higher-order analytical approximations. This approach has been proved effective for various conservative nonlinear oscillations with odd and even nonlinearities [30,31], inertia and static nonlinearities [32] and nonnatural systems [33]. A system is usually called a nonnatural system if the kinetic energy is not a quadratic function of the velocity [3,33,34], otherwise it is a natural system. Using this approach, accurate higher-order analytical approximate periodic solutions in simple linear algebraic expressions not restricted to a small perturbed parameter are presented. For comparison, numerical integration solutions are obtained by integrating directly the nonlinear ordinary differential equation numerically using the Runge–Kutta method, a built-in function in MATHEMATICA. It is concluded from this analysis that the LP method yields inaccurate results. Although sufficient number of terms for the perturbed parameter can be included, tedious derivations, computations and complicated implementation have to be involved and yet they are applicable only to nonlinear equations with the presence of small parameters. Similarly, the classical HB method with four-term and six-term approximations [19] yields inaccurate results. More harmonic terms can be included but a set of nonlinear algebraic equations has to be solved. Hence, numerical analysis is required because there no simple approximate analytical solution is available.

2. Governing equation of motion and formulation

Consider free vibration of a conservative, single-degree-of-freedom system with a mass attached to linear and nonlinear springs in series as shown in Fig. 1. After transformation, the motion is governed by a nonlinear differential equation of motion [19] as

$$(1+3\varepsilon z v^2)\frac{\mathrm{d}^2 v}{\mathrm{d}t^2}+6\varepsilon z v\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2+\omega_e^2 v+\varepsilon\omega_e^2 v^3=0, \tag{1}$$

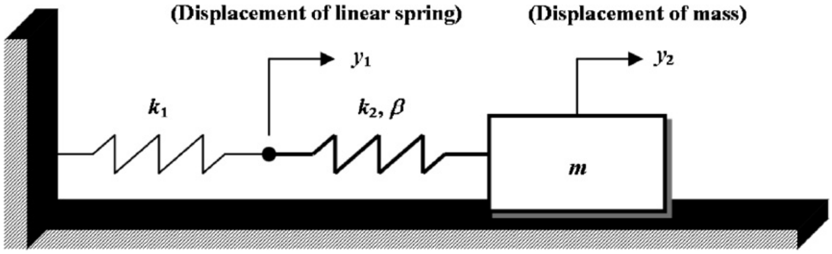

Fig. 1. Nonlinear free vibration of a system of mass with serial linear and nonlinear stiffness on a frictionless contact surface [19].

where

$$\varepsilon = \frac{\beta}{k_2}, \tag{2}$$

$$\xi = \frac{k_2}{k_1}, \tag{3}$$

$$z = \frac{\xi}{1+\xi}, \tag{4}$$

$$\omega_e = \sqrt{\frac{k_2}{m(1+\xi)}} \tag{5}$$

with the initial conditions

$$v(0) = A, \quad \frac{\mathrm{d}v}{\mathrm{d}t}(0) = 0 \tag{6}$$

in which ε, β, v, ω_e, m and ξ are perturbation parameter (not restricted to a small parameter), coefficient of nonlinear spring force, deflection of nonlinear spring, natural frequency, mass and the ratio of linear portion k_2 of the nonlinear spring constant to that of linear spring constant k_1, respectively. Note that the notations in Eqs. (1)–(5) follow those in Telli and Kopmaz [19].

The deflection of linear spring $y_1(t)$ and the displacement of attached mass $y_2(t)$ can be represented by the deflection of nonlinear spring v in simple relationships [19] as

$$y_1(t) = \xi v(t) + \varepsilon\xi[v(t)]^3 \tag{7}$$

and

$$y_2(t) = v(t) + y_1(t). \tag{8}$$

Introducing a new independent temporal variable, $\tau = \omega t$, Eqs. (1) and (6) become

$$\omega^2[(1+3\varepsilon z v^2)\ddot{v} + 6\varepsilon z v\dot{v}^2] + \omega_e^2 v + \varepsilon\omega_e^2 v^3 = 0 \tag{9}$$

and

$$v(0) = A, \quad \dot{v}(0) = 0, \tag{10}$$

where a dot denotes differentiation with respect to τ. The deflection of nonlinear spring v is a periodic function of τ of period 2π. Based on Eq. (9), the periodic solution $v(\tau)$ can be expanded in a Fourier series with only odd multiples of τ, as

$$v(\tau) = \sum_{n=0}^{\infty} h_{2n+1}\cos(2n+1)\tau. \tag{11}$$

To linearize the governing differential equation, we assume $v(\tau)$ as the sum of a principal term and a correction term as [32,33]

$$v(\tau) = v_1(\tau) + \Delta v_1(\tau). \tag{12}$$

Substituting Eq. (12) into Eq. (9) and neglecting nonlinear terms of $\Delta v_1(\tau)$ yields

$$\begin{aligned}&\omega^2[(1+3\varepsilon z v_1^2)\ddot{v}_1 + 6\varepsilon z v_1 \dot{v}_1^2] + \omega_e^2 v_1 + \varepsilon\omega_e^2 v_1^3 + \omega^2[(1+3\varepsilon z v_1^2)\Delta\ddot{v}_1 + 2(6\varepsilon z v_1\dot{v}_1)\Delta\dot{v}_1 \\ &\quad + (6\varepsilon z v_1\ddot{v}_1 + 6\varepsilon z\dot{v}_1^2)\Delta v_1] + (\omega_e^2 + 3\varepsilon\omega_e^2 v_1^2)\Delta v_1 = 0\end{aligned} \tag{13}$$

and

$$\Delta v_1(0) = 0, \quad \Delta\dot{v}_1(0) = 0, \tag{14}$$

where $v_1(\tau) = A\cos\tau$ is a periodic function of τ of period 2π.

Making use of $v_1(\tau) = A\cos\tau$, we have the following Fourier-series expansions:

$$(1+3\varepsilon z v_1^2)\ddot{v}_1 + 6\varepsilon z v_1\dot{v}_1^2 = \sum_{i=0}^{\infty} a_{2i+1}\cos(2i+1)\tau = -\frac{A(4+3A^2 z\varepsilon)}{4}\cos\tau - \frac{9A^3 z\varepsilon}{4}\cos 3\tau, \tag{15}$$

$$\omega_e^2 v_1 + \varepsilon\omega_e^2 v_1^3 = \sum_{i=0}^{\infty} b_{2i+1}\cos(2i+1)\tau = \frac{A\omega_e^2(4+3A^2\varepsilon)}{4}\cos\tau + \frac{A^3\varepsilon\omega_e^2}{4}\cos 3\tau, \tag{16}$$

$$1 + 3\varepsilon z v_1^2 = \frac{1}{2}c_0 + \sum_{i=1}^{\infty} c_{2i}\cos 2i\tau = \frac{2+3zA^2\varepsilon}{2} + \frac{3zA^2\varepsilon}{2}\cos 2\tau, \tag{17}$$

$$2(6\varepsilon z v_1\dot{v}_1) = \sum_{i=0}^{\infty} d_{2(i+1)}\sin 2(i+1)\tau = -6zA^2\varepsilon\sin 2\tau, \tag{18}$$

$$6\varepsilon z v_1\ddot{v}_1 + 6\varepsilon z\dot{v}_1^2 = \frac{1}{2}e_0 + \sum_{i=1}^{\infty} e_{2i}\cos 2i\tau = -6zA^2\varepsilon\cos 2\tau, \tag{19}$$

$$\omega_e^2 + 3\varepsilon\omega_e^2 v_1^2 = \frac{1}{2}f_0 + \sum_{i=1}^{\infty} f_{2i}\cos 2i\tau = \frac{(2+3A^2\varepsilon)\omega_e^2}{2} + \frac{3A^2\varepsilon\omega_e^2}{2}\cos 2\tau, \tag{20}$$

where a_{2i+1}, b_{2i+1}, c_{2i}, $d_{2(i+1)}$, e_{2i} and f_{2i} for $i = 0,1,2,\ldots$ are Fourier-series coefficients.

2.1. *First-order analytical approximation*

For the first-order analytical approximation, we set

$$\Delta v_1(\tau) = 0 \tag{21}$$

and therefore

$$v(\tau) = v_1(\tau) = A\cos\tau. \tag{22}$$

Substituting Eqs. (15)–(21) into Eq. (13), expanding the resulting expression in a trigonometric series and setting the coefficient of $\cos\tau$ to zero yield the solution of the angular frequency ω_1 where subscript 1 indicates the first-order analytical approximation. The analytical approximation of ω_1 can be expressed as

$$\omega_1(A) = \sqrt{\frac{(4+3A^2\varepsilon)\omega_e^2}{4+3A^2 z\varepsilon}} \tag{23}$$

and the periodic solution is

$$v_1(t) = A\cos[\omega_1(A)t]. \tag{24}$$

2.2. Second-order analytical approximation

For the second analytical approximation, we set

$$\Delta v_1(\tau) = x_1(\cos\,\tau - \cos\,3\tau). \tag{25}$$

Substituting Eqs. (15)–(20) and (25) into Eq. (13), expanding the resulting expression in a trigonometric series and setting the coefficients of $\cos\tau$ and $\cos 3\tau$ to zero result in a quadratic equation of ω_2^2 where subscript 2 indicates the second-order analytical approximation. The angular frequency ω_2 can be expressed as

$$\omega_2(A) = \sqrt{\frac{-b - \sqrt{b^2 - 4ac}}{2a}}, \tag{26}$$

where

$$a = -144A - 252zA^3\varepsilon - 135z^2A^5\varepsilon^2, \tag{27}$$

$$b = 160A\omega_e^2 + 124A^3\varepsilon\omega_e^2 + 156A^3z\varepsilon\omega_e^2 + 150zA^5\varepsilon^2\omega_e^2, \tag{28}$$

$$c = -16A\omega_e^4 - 28A^3\varepsilon\omega_e^4 - 15A^5\varepsilon^2\omega_e^4 \tag{29}$$

and a, b and c are the coefficients of the quadratic equation of ω_2^2. The solution of ω_2 in Eq. (26) with respect to $+\sqrt{b^2 - 4ac}$ is omitted so that $\omega_2/\omega_1 \approx 1$. The periodic solution is

$$v_2(t) = [A + x_1(A)]\cos[\omega_2(A)t] - x_1(A)\cos[3\omega_2(A)t], \tag{30}$$

where

$$\begin{aligned} x_1(A) = & -[32A\omega_e^2 + 25A^3\varepsilon\omega_e^2 + 15A^3z\varepsilon\omega_e^2 + 6A^5z\varepsilon^2\omega_e^2 - (1024A^2\omega_e^4 \\ & + 1472A^4\varepsilon\omega_e^4 + 2112A^4z\varepsilon\omega_e^4 + 421A^6\varepsilon^2\omega_e^4 + 3654A^6z\varepsilon^2\omega_e^4 \\ & + 981A^6z^2\varepsilon^2\omega_e^4 + 1380A^8z\varepsilon^3\omega_e^4 + 1980A^8z^2\varepsilon^3\omega_e^4 \\ & + 900A^{10}z^2\varepsilon^4\omega_e^4)^{1/2}]/[2\omega_e^2(32 + 51A^2\varepsilon + 21A^2z\varepsilon + 36A^4z\varepsilon^2)]. \end{aligned} \tag{31}$$

2.3. Third-order analytical approximation

Although the first- and the second-order analytical approximations are expected to agree with other solutions, the agreement deteriorates as τ progresses during the steady-state response. Therefore, the third-order analytical approximation is derived for more accurate steady-state response. To construct the hird-order analytical approximation, the previous related expressions must be adjusted due to interaction between lower-order and higher-order harmonics. Here, $\Delta v_1(\tau)$ and $v_1(\tau)$ in Eqs. (12), (13) and (15)–(20) are replaced by $\Delta v_2(\tau)$ and $v_2(\tau)$, respectively, and Eq. (13) is modified as

$$\begin{aligned} & \omega^2[(1 + 3\varepsilon z v_2^2)\ddot{v}_2 + 6\varepsilon z v_2\dot{v}_2^2] + \omega_e^2 v_2 + \varepsilon\omega_e^2 v_2^3 + \omega^2[(1 + 3\varepsilon z v_2^2)\Delta\ddot{v}_2 + 2(6\varepsilon z v_2\dot{v}_2)\Delta\dot{v}_2 \\ & + (6\varepsilon z v_2\ddot{v}_2 + 6\varepsilon z\dot{v}_2^2)\Delta v_2] + (\omega_e^2 + 3\varepsilon\omega_e^2 v_2^2)\Delta v_2 = 0. \end{aligned} \tag{32}$$

The right-hand sides of Eqs. (15)–(20) in the third-order analytical approximation are completely different from the first- and second-order analytical approximations because $v_1(\tau)$ is replaced by $v_2(\tau)$ of Eq. (30). The coefficients of Fourier series in Eqs. (15)–(20) are presented in Appendix A.

For the third-order analytical approximation, we set

$$\Delta v_2(\tau) = x_2(\cos\,\tau - \cos\,3\tau) + x_3(\cos\,3\tau - \cos\,5\tau). \tag{33}$$

Substituting the modified Eqs. (15)–(20) with $v_1(\tau)$ replaced by $v_2(\tau)$ and Eq. (33) into Eq. (32), expanding the resulting expression in a trigonometric series and setting the coefficients of $\cos\tau$, $\cos 3\tau$ and $\cos 5\tau$ to zero yield ω_3 as a function of A. The corresponding approximate analytical periodic solution can then be solved, as

$$v_3(t) = [A + x_1(A) + x_2(A)]\cos[\omega_3(A)t] + [x_3(A) - x_2(A) - x_1(A)]\cos[3\omega_3(A)t] - x_3(A)\cos[5\omega_3(A)t]. \tag{34}$$

The angular frequency ω_3 is the squared-roots of roots of a quartic equation of ω_3^2 in the form of

$$a'(\omega_3^2)^4 + b'(\omega_3^2)^3 + c'(\omega_3^2)^2 + d'(\omega_3^2) + e' = 0, \tag{35}$$

where subscript 3 indicates the third-order analytical approximation and a', b', c', d' and e' are coefficients of the quartic equation of ω_3^2. There is a total of eight roots and the particular root which is closest to ω_2 is identified as the most appropriate solution because ω_3 is a more accurate, higher-order approximation to ω_2. Comparison of ω_3 in the following section shows that it is in excellent agreement with numerical integration solution for small as well as large amplitudes of oscillation. The quartic equation is presented in a simplified form in Appendix A where coefficients a', b', c', d' and e' in Eq. (35) can be obtained by rearranging Eq. (A.1) in the form of Eq. (35). It can be subsequently solved by any symbolic software such as MATHEMATICA for ω_3. The constants x_2 and x_3 in Eq. (34) derived in terms of the coefficients of Fourier series are also presented in Appendix A.

2.4. *The limiting case for infinite amplitude $A^2 \to \infty$*

Since the proposed approach is suitable for oscillation amplitudes of any order approximations for the limiting case of infinite amplitude as $A^2 \to \infty$ can be derived. For the first-order analytical approximation, we have from Eq. (23),

$$\lim_{A^2\to\infty} \omega_1(A) = \lim_{A^2\to\infty} \sqrt{\frac{(4 + 3A^2\varepsilon)\omega_e^2}{4 + 3A^2 z\varepsilon}} = \frac{\omega_e}{\sqrt{z}}. \tag{36}$$

For the second-order analytical approximation, we have from Eq. (26),

$$\lim_{A^2\to\infty} \omega_2(A) = \lim_{A^2\to\infty} \sqrt{\frac{-b - \sqrt{b^2 - 4ac}}{2a}} = \frac{\omega_e}{\sqrt{z}}, \tag{37}$$

where a, b and c are given in Eqs. (27)–(29).

For the third-order analytical approximation, we first solve

$$\lim_{A^2\to\infty} x_1(A) = \frac{A}{3} \tag{38}$$

from Eq. (31). Using Eq. (38), the coefficients of Fourier series a_{2i+1}, b_{2i+1}, c_{2i}, $d_{2(i+1)}$, e_{2i} and f_{2i} for $i = 0,1,2,\ldots$ can be found from Eqs. (A.4) to (A.28) which are then substituted into Eq. (A.1). The limiting case as $A^2 \to \infty$ for the resulting equation can be rearranged to form

$$\lim_{A^2\to\infty} A^3[\tilde{a}(\omega_3) + \tilde{b}(\omega_3)A^2 + \tilde{c}(\omega_3)A^4 + \tilde{d}(\omega_3)A^6] = 0, \tag{39}$$

where

$$\lim_{A^2\to\infty} \tilde{a}(\omega_3) = \frac{28}{3}\varepsilon(z\omega_3^2 - \omega_e^2)(225\omega_3^6 - 259\omega_3^4\omega_e^2 + 35\omega_3^2\omega_e^4 - \omega_e^6), \tag{40}$$

$$\lim_{A^2\to\infty} \tilde{b}(\omega_3) = \frac{49}{81}\varepsilon^2(z\omega_3^2 - \omega_e^2)(25,875z\omega_3^6 - 3211\omega_3^4\omega_e^2 - 26574z\omega_3^4\omega_e^2 + 2174\omega_3^2\omega_e^4 + 1851z\omega_3^2\omega_e^4 - 115\omega_e^6), \tag{41}$$

$$\lim_{A^2\to\infty} \tilde{c}(\omega_3) = \frac{7}{972}\varepsilon^3(z\omega_3^2-\omega_e^2)(4,605,975z^2\omega_3^6 - 2,081,174z\omega_3^4\omega_e^2 - 3,220,815z^2\omega_3^4\omega_e^2 + 143,615\omega_3^2\omega_e^4 + 572,870z\omega_3^2\omega_e^4 - 20,471\omega_e^6), \tag{42}$$

$$\lim_{A^2\to\infty} \tilde{d}(\omega_3) = \frac{319,039}{3888}\varepsilon^4(z\omega_3^2-\omega_e^2)^2(9z\omega_3^2-\omega_e^2)(25z\omega_3^2-\omega_e^2). \tag{43}$$

For the limit in Eq. (39) to vanish as $A^2\to\infty$, the respective coefficients must be zero or $\lim_{A^2\to\infty}\tilde{a}(\omega_3) = \lim_{A^2\to\infty}\tilde{b}(\omega_3) = \lim_{A^2\to\infty}\tilde{c}(\omega_3) = \lim_{A^2\to\infty}\tilde{d}(\omega_3) = 0$. Setting Eqs. (40)–(43) to zero yield multiple roots and the particular root closest to ω_2 in Eq. (37) is the limit for ω_3. Following the procedure, we obtain the limiting third-order analytical approximation as

$$\lim_{A^2\to\infty} \omega_3(A) = \frac{\omega_e}{\sqrt{z}}. \tag{44}$$

Comparing Eqs. (36), (37) and (44), we observe that the various approximate analytical solutions yield identical limiting frequency. Hence, we can also proof that the limiting period for the respective cases is identical or

$$\lim_{A^2\to\infty} T(A) = \frac{2\pi\sqrt{z}}{\omega_e}. \tag{45}$$

3. Approximate results and discussion

To illustrate and verify accuracy of the new approximate analytical approach, a comparison of angular frequencies via different approaches is presented in Table 1 for different m, A, ε, k_1 and k_2. The parameter ε is linearly dependent on the coefficient of nonlinear spring force β as given in Eq. (2). The latter can be positive or negative depending on whether the nonlinear spring has hard or soft-spring properties. As discussed in Ref. [19], for a quasi-harmonic or periodic motion the terms $(1+3\varepsilon z v^2)\mathrm{d}^2v/\mathrm{d}t^2$ and $\omega_e^2 v + \varepsilon\omega_e^2 v^3$ in Eq. (1) are guaranteed to be neither non-positive nor zero. The notations LP, HB, 1, 2, 3, n denoted as subscripts of v,ω and superscripts of y_1,y_2 refer to the LP method, the HB method using six-term approximation [19], the first-, second- and third-order analytical approximations of the LHB method and the numerical integration solution, respectively. The results for ω_n is numerically obtained from Eq. (1) using the Runge–Kutta numerical

Table 1
Comparison of various approximate angular frequencies with respect to the numerical integration solution

m	A	ε	k_1	k_2	ω_{LP} Eq. (46)	ω_{HB} Eq. (41d) [19]	ω_1 Eq. (23)	ω_2 Eq. (26)	ω_3 Eq. (35)	ω_n
1	0.5	0.5	50	5	2.220197	2.220239	2.220265	2.220231	2.220231	2.220231
1	2	0.5	50	5	3.134986	3.257248	3.162278	3.177242	3.175209	3.175501
1	2	0.5	5	5	1.838180	1.726619	1.889822	1.908164	1.900724	1.903569
1	2	0.5	5	50	2.144360	2.145708	2.192645	2.196361	2.194560	2.195284
3	5	1	8	16	**	1.176927	1.612707	1.616354	1.614287	1.615107
3	5	1	10	5	**	1.052717	1.739776	1.753819	1.745984	1.749115
5	10	2	12	16	**	**	1.545360	1.546115	1.545682	1.545853
5	30	5	15	5	**	**	1.731282	1.731435	1.731347	1.731382
10	200	5	5	250	**	**	0.707107	0.707107	0.707107	0.707107
10	100	10	5	25	**	**	0.707106	0.707106	0.707106	0.707106
1	0.5	−0.5	50	5	2.038254	2.038207	2.038315	2.038210	2.038209	2.038209
2	2	−0.1	10	10	1.444007	1.458194	1.434860	1.443962	1.445356	1.446389
3	5	−0.02	30	10	1.320867	1.336111	1.313064	1.317663	1.318255	1.318370
4	10	−0.008	6	3	0.519615	0.555358	0.500000	0.510678	0.514250	0.517327
10	5	−0.01	8	16	0.705078	0.706817	0.703732	0.705144	0.705312	0.705412

**Invalid numerical solutions in complex values.

integration method in combination with the bisection method. The solutions of Eq. (1) using the second-order LP perturbation method [2] are briefly derived here. By expanding the frequency $\omega^2 = \omega_{\mathrm{LP}}^2$ and the periodic solution $v(\tau) = v_{\mathrm{LP}}(\tau)$ of Eq. (9) into a power series as a function of ε,

$$\omega_{\mathrm{LP}}^2 = \omega_e^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \tag{46}$$

$$v_{\mathrm{LP}}(\tau) = v_0(\tau) + \varepsilon v_1(\tau) + \varepsilon^2 v_2(\tau) + \cdots, \quad \tau = \omega_{\mathrm{LP}} t \tag{47}$$

and setting the coefficients of ε^0, ε^1 and ε^2 as zero yields

$$\ddot{v}_0 + v_0 = 0, \quad v_0(0) = A, \quad \dot{v}_0(0) = 0, \tag{48}$$

$$\ddot{v}_1 + v_1 = -v_0^3 - 6zv_0\dot{v}_0^2 - 3zv_0^2\ddot{v}_0 - \frac{\omega_1\ddot{v}_0}{\omega_e^2}, \quad v_1(0) = 0, \quad \dot{v}_1(0) = 0, \tag{49}$$

$$\begin{aligned}\ddot{v}_2 + v_2 = & -3v_0^2v_1 - 6zv_1\dot{v}_0^2 - \frac{6z\omega_1 v_0\dot{v}_0^2}{\omega_e^2} - 12zv_0\dot{v}_0\dot{v}_1 - 6zv_0v_1\ddot{v}_0 - \frac{3z\omega_1 v_0^2\ddot{v}_0}{\omega_e^2} \\ & - \frac{\omega_2\ddot{v}_0}{\omega_e^2} - 3zv_0^2\ddot{v}_1 - \frac{\omega_1\ddot{v}_1}{\omega_e^2}, \quad v_2(0) = 0, \quad \dot{v}_2(0) = 0.\end{aligned} \tag{50}$$

Solving the linear second-order differential equations (48)–(50) with the corresponding initial conditions, we obtain,

$$\omega_1 = -\frac{3}{4}A^2\omega_e^2(z-1), \quad \omega_2 = \frac{3}{128}A^4\omega_e^2(15z^2 - 14z - 1), \tag{51}$$

$$\begin{aligned} v_0 &= A\cos\omega_{\mathrm{LP}}t, v_1 = \frac{A^3}{32}(9z-1)(\cos\omega_{\mathrm{LP}}t - \cos 3\omega_{\mathrm{LP}}t), \\ v_2 &= -\frac{A^5(441z^2 - 34z - 23)}{1024}\cos\omega_{\mathrm{LP}}t + \frac{3A^5(9z^2-1)}{128}\cos 3\omega_{\mathrm{LP}}t \\ &\quad + \frac{A^5(225z^2 - 34z + 1)}{1024}\cos 5\omega_{\mathrm{LP}}t. \end{aligned} \tag{52}$$

In Table 1, the percentage errors for different analytical approximations with respect to ω_n as a reference are very small for small parameters $m = 1$, $A = 0.5$, $\varepsilon = 0.5$, $k_1 = 50$ and $k_2 = 5$. The relative errors of ω_{LP}, ω_{HB}, ω_1, ω_2 and ω_3 are -0.0015%, 0.0004%, 0.0015%, 0% and 0%, respectively. For larger parameters $m = 1$, $A = 2$, $\varepsilon = 0.5$, $k_1 = 5$ and $k_2 = 50$, the relative errors for the LP and HB methods increase significantly to -2.3197% and -2.2583%, respectively, but the first-, second- and third-order analytical approximations still maintain excellent agreement with respect to ω_n. The relative errors for these three analytical approximations are only -0.1202%, 0.0491% and -0.0330%, respectively. For extremely large parameters in Table 1, invalid complex solutions are obtained using the LP and HB methods implying inapplicability of the methods in these cases. Even for such extremely large parameters, the three different approximations ω_1, ω_2 and ω_3 are still in excellent agreement with respect to the numerical integration solution ω_n. Although numerical integration methods for Eq. (1) are not restricted to small parameters, it is virtually impossible to predict the limiting frequency $\lim_{A^2\to\infty}\omega_n$ because an explicit function is not available. For large parameters in the examples for $m = 10$, $A = 200$, $\varepsilon = 5$, $k_1 = 5$, $k_2 = 250$ and $m = 10$, $A = 100$, $\varepsilon = 10$, $k_1 = 5$, $k_2 = 25$, the results ω_1, ω_2, ω_3 and ω_n are identical. Theoretically speaking, the numerical integration solutions for $A^2 \to \infty$ should be equal to $\omega_e/\sqrt{z}$ derived in Eqs. (36), (37) and (44). It is also clearly observed that the relative errors for ω_1, ω_2, ω_3 and ω_n in all cases are stable for different parameters m, A, ε, k_1 and k_2 indicating that numerical values of the parameters do not affect the accuracy of analysis, even for soft-spring systems (i.e. $\varepsilon < 0$). Hence, the LHB approach as proposed here has clear-cut advantage over the LP and HB methods, which have various restrictions as discussed in Section 1.

To obtain further approximation although it is frequently not necessary, the fourth-order analytical approximation can be constructed based on the third-order analytical approximation. For illustrative

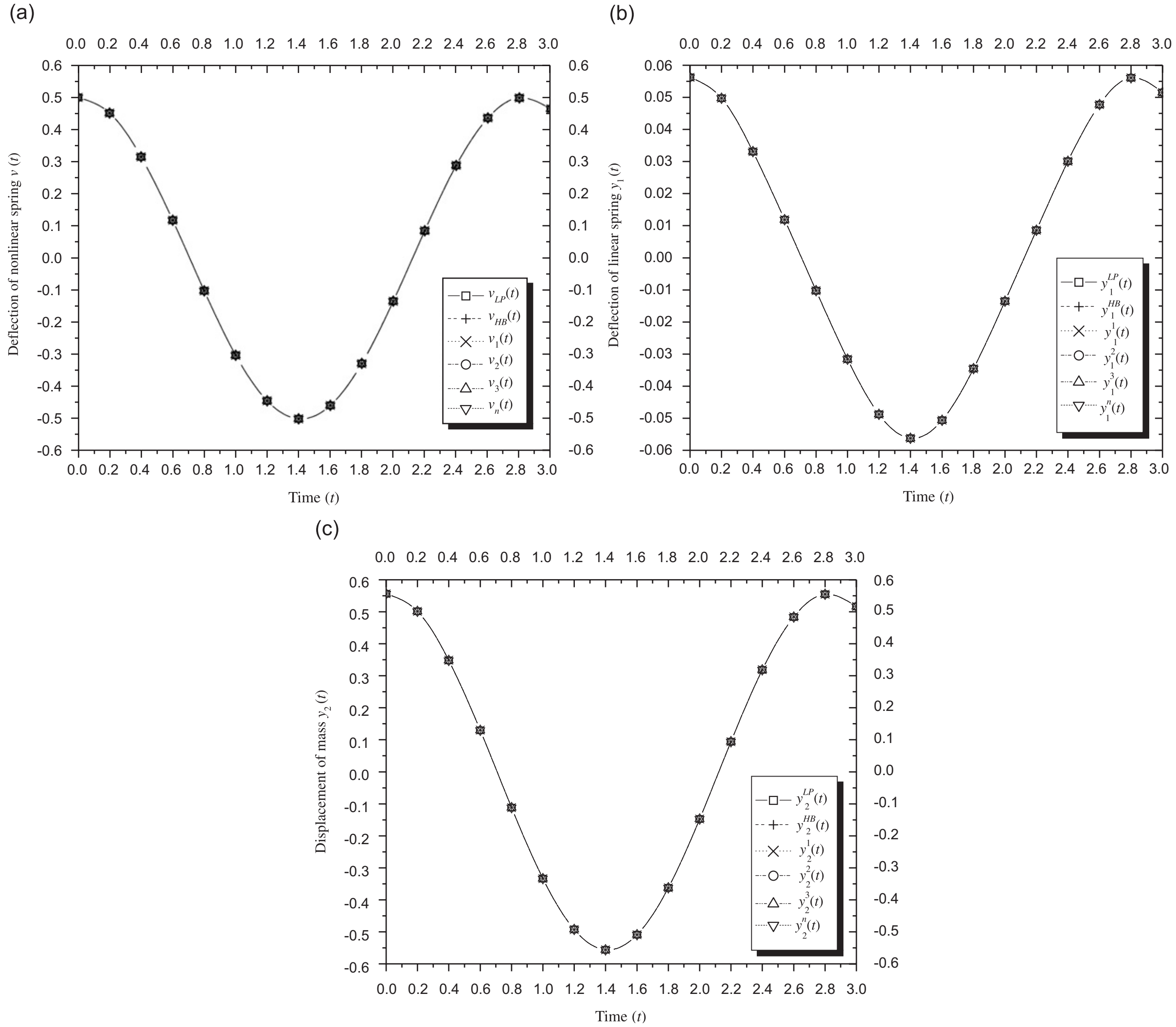

Fig. 2. (a) Comparison of deflection of nonlinear spring $v(t)$ for various analytical approximations and the numerical integration solution for $m=1$, $A=0.5$, $\varepsilon=0.5$ and $\xi=0.1$ ($k_1=50$, $k_2=5$). (b) Comparison of the deflection of linear spring $y_1(t)$ for various analytical approximations and the numerical integration solutions for $m=1$, $\varepsilon=0.5$ and $\xi=0.1$ ($k_1=50$, $k_2=5$). (c) Comparison of the displacement of mass $y_2(t)$ for various analytical approximations and the numerical integration solutions for $m=1$, $\varepsilon=0.5$ and $\xi=0.1$ ($k_1=50$, $k_2=5$).

purposes, $\Delta v_2(\tau)$ and $v_2(\tau)$ in Eq. (32) can be replaced by $\Delta v_3(\tau)$ and $v_3(\tau)$ and the equation becomes

$$\begin{aligned}&\omega^2[(1+3\varepsilon z v_3^2)\ddot{v}_3+6\varepsilon z v_3\dot{v}_3^2]+\omega_e^2 v_3+\varepsilon\omega_e^2 v_3^3+\omega^2[(1+3\varepsilon z v_3^2)\Delta\ddot{v}_3+2(6\varepsilon z v_3\dot{v}_3)\Delta\dot{v}_3\\&\quad+(6\varepsilon z v_3\ddot{v}_3+6\varepsilon z\dot{v}_3^2)\Delta v_3]+(\omega_e^2+3\varepsilon\omega_e^2 v_3^2)\Delta v_3=0.\end{aligned}\tag{53}$$

Setting

$$\Delta v_3(\tau)=x_4(\cos\tau-\cos 3\tau)+x_5(\cos 3\tau-\cos 5\tau)+x_6(\cos 5\tau-\cos 7\tau)\tag{54}$$

and substituting the modified Eqs. (15)–(20), with $v_1(\tau)$ replaced by $v_3(\tau)$ given in Eq. (34), and Eq. (54) into Eq. (53), then expanding the resulting expression in a trigonometric series and setting the coefficients of $\cos\tau$, $\cos 3\tau$, $\cos 5\tau$ and $\cos 7\tau$ to zero yield the results for $\omega=\omega_4$, x_4, x_5 and x_6 as a function of A. The

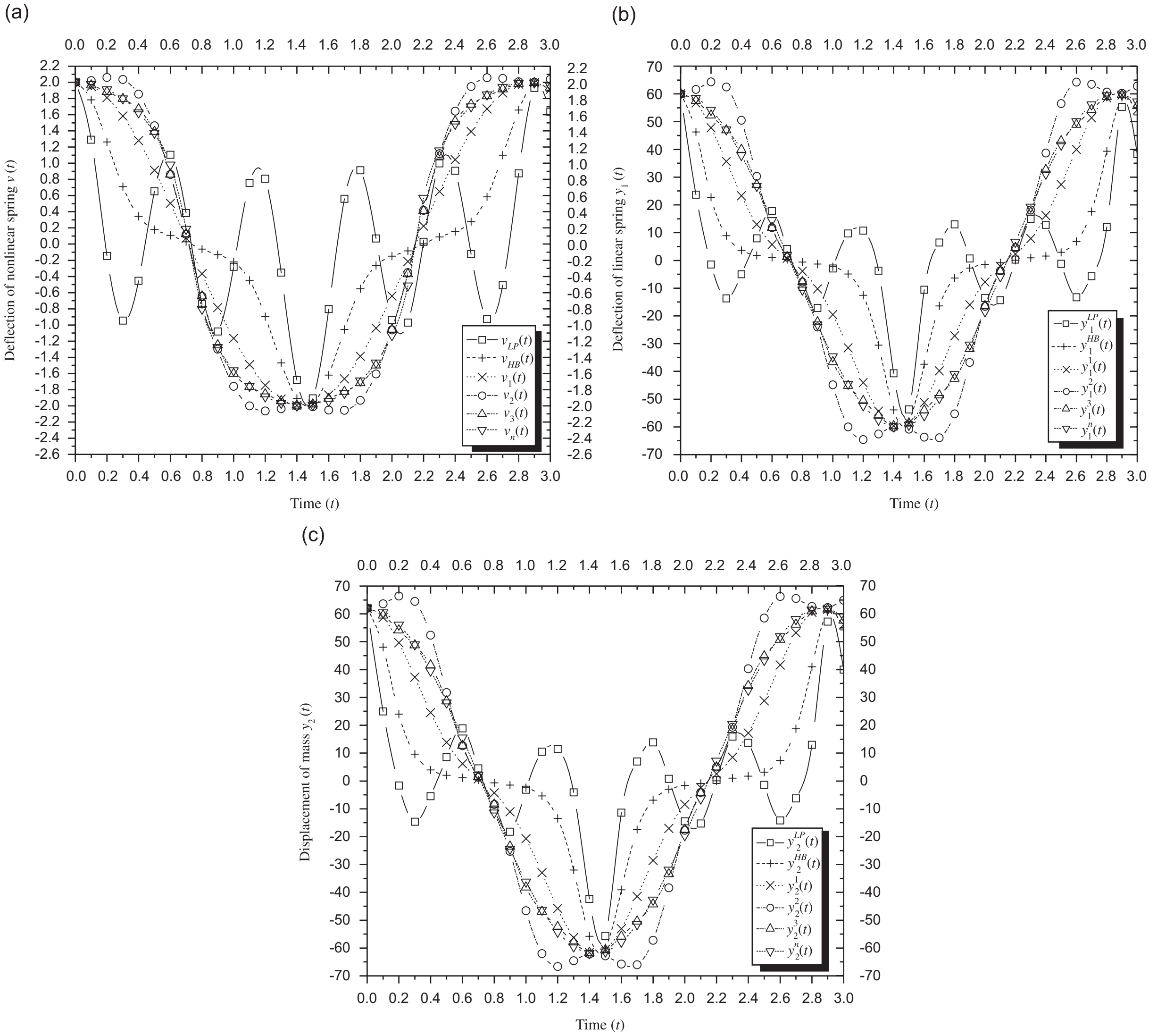

Fig. 3. (a) Comparison of the deflection of nonlinear spring $v(t)$ for various analytical approximations and the numerical integration solutions for $m = 1$, $A = 2$, $\varepsilon = 0.5$ and $\xi = 10$ ($k_1 = 5$, $k_2 = 50$). (b) Comparison of the deflection of linear spring $y_1(t)$ for various analytical approximations and the numerical integration solutions for $m = 1$, $\varepsilon = 0.5$ and $\xi = 10$ ($k_1 = 5$, $k_2 = 50$). (c) Comparison of the displacement of mass $y_2(t)$ for various analytical approximations and the numerical integration solutions for $m = 1$, $\varepsilon = 0.5$ and $\xi = 10$ ($k_1 = 5$, $k_2 = 50$).

corresponding approximate analytical periodic solution can be expressed as

$$\begin{aligned} v_4(t) &= v_3(t) + \Delta v_3(t) \\ &= [A + x_1(A) + x_2(A) + x_4(A)]\cos[\omega_4(A)t] \\ &\quad + [x_5(A) - x_4(A) + x_3(A) - x_2(A) - x_1(A)]\cos[3\omega_4(A)t] \\ &\quad + [x_6(A) - x_5(A) - x_3(A)]\cos[5\omega_4(A)t] - x_6(A)\cos[7\omega_4(A)t]. \end{aligned} \tag{55}$$

The accuracy of the fourth-order analytical approximation is better than the previous three analytical approximations, but a set of more complicated and lengthy linear algebraic equations are involved.

To further illustrate and verify accuracy of this new approximate analytical approach, a comparison of the time history response of nonlinear spring deflection $v(t)$, linear spring deflection $y_1(t)$ and mass displacement

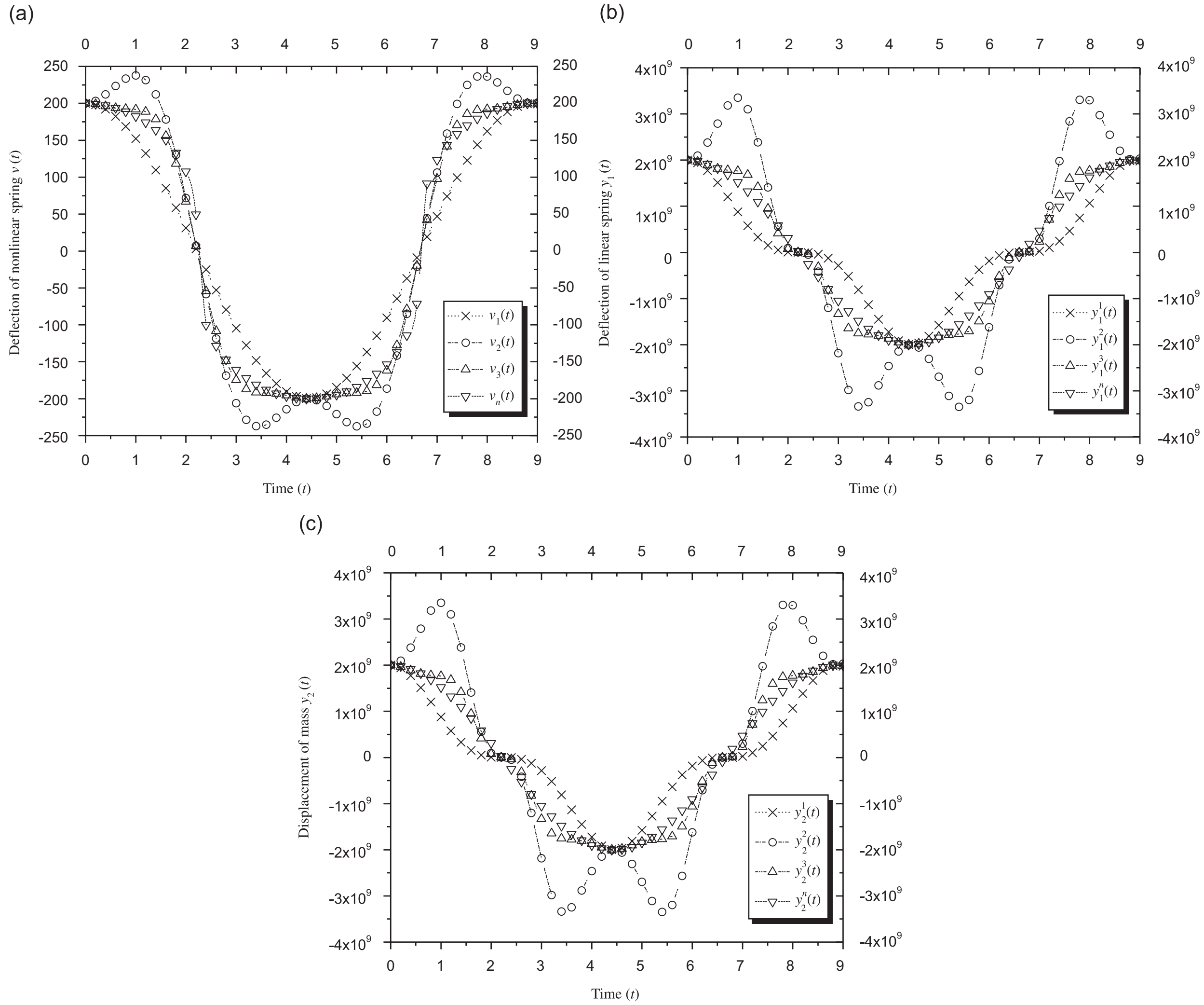

Fig. 4. (a) Comparison of the deflection of nonlinear spring $v(t)$ for various analytical approximations and the numerical integration solutions for $m = 10$, $A = 200$, $\varepsilon = 5$ and $\xi = 50$ ($k_1 = 5$, $k_2 = 250$). (b) Comparison of the deflection of linear spring $y_1(t)$ for various analytical approximations and the numerical integration solutions for $m = 10$, $\varepsilon = 5$ and $\xi = 50$ ($k_1 = 5$, $k_2 = 250$). (c) Comparison of the displacement of mass $y_2(t)$ for various analytical approximations and the numerical integration solutions for $m = 10$, $\varepsilon = 5$ and $\xi = 50$ ($k_1 = 5$, $k_2 = 250$).

$y_2(t)$ is presented in Figs. 2–6. Figs. 2–4 consider the nonlinear hard-spring cases while Figs. 5 and 6 are the nonlinear soft-spring cases. The curves for $v_n(t)$, $v_{LP}(t)$, $v_{HB}(t)$, $v_1(t)$, $v_2(t)$ and $v_3(t)$ are generated via, respectively, the numerical solution using MATHEMATICA, Eq. (47), Eq. (41e) from Telli and Kopmaz [19], Eqs. (24), (30) and (34). The curves for $y_1(t)$ and $y_2(t)$ can be obtained through simple relationships in Eqs. (7) and (8) accordingly. In Figs. 2a–c, the analytical approximations and numerical integration solutions are highly consistent due to smallness of m, A, ε, k_1 and k_2. In Fig. 3a, the solutions of the LP method is totally invalid. The curves for the HB method, the first- and second-order LHB method apparently deviate from the numerical integration solution. However, the third-order analytical approximation maintains excellent agreement. As accuracy of the linear spring deflection and mass displacement relies heavily on the accuracy of nonlinear spring deflection, the LP solutions in Figs. 3b and 3c also significantly deviate from the other solutions.

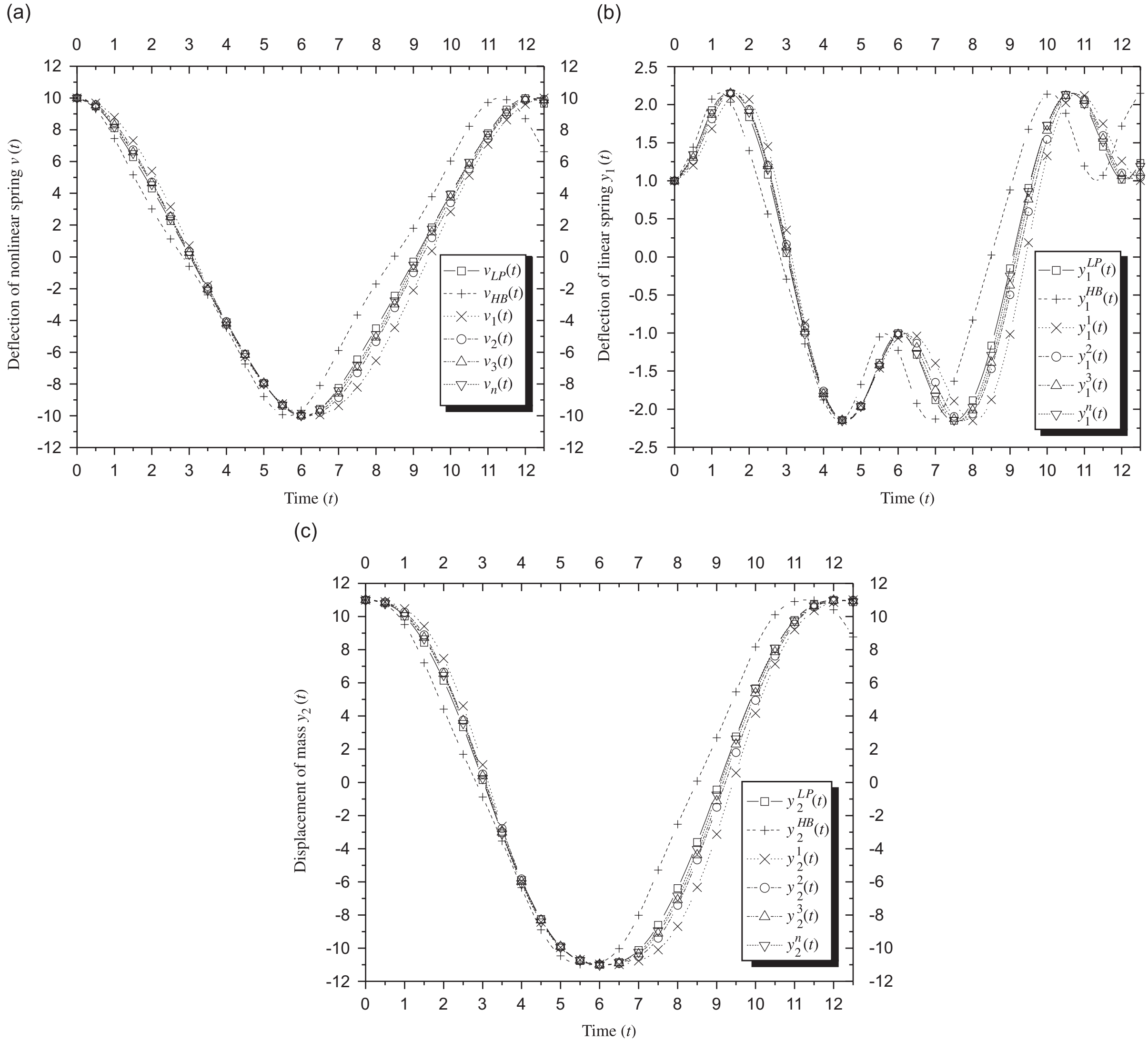

Fig. 5. (a) Comparison of the deflection of nonlinear spring $v(t)$ for various analytical approximations and the numerical integration solutions for $m = 4$, $A = 10$, $\varepsilon = -0.008$ and $\xi = 0.5$ ($k_1 = 6$, $k_2 = 3$). (b) Comparison of the deflection of linear spring $y_1(t)$ for various analytical approximations and the numerical integration solutions for $m = 4$, $\varepsilon = -0.008$ and $\xi = 0.5$ ($k_1 = 6$, $k_2 = 3$). (c) Comparison of the displacement of mass $y_2(t)$ for various analytical approximations and the numerical integration solutions for $m = 4$, $\varepsilon = -0.008$ and $\xi = 0.5$ ($k_1 = 6$, $k_2 = 3$).

To extend applicability and to show flexibility and accuracy of this LHB method for extremely large parameters, an example for $m = 10$, $A = 200$, $\varepsilon = 5$, $k_1 = 5$ and $k_2 = 250$ is presented in Figs. 4. In Figs. 4a–c, no solution using the LP and HB methods is presented because the angular frequencies obtained are complex and thus invalid. Hence, these methods are inapplicable for such large parameters. It is clearly observed that the third-order analytical approximation is in excellent consistency with the numerical integration solution even for such extremely large parameters. The lower-order approximations are inaccurate at $t = 0$. The second-order analytical approximation shows significant deviation in the crest and trough of the curve.

Figs. 5–6 show that the LP and HB methods are able to provide useful results for soft-spring cases; the HB solutions are very inaccurate. Having said so, the higher-order analytical approximations of the LHB method guarantee sufficient accuracy for both hard- or soft-spring represented by ε and also large parameters of m, A, k_1 and k_1 for which the classical LP and HB methods fail.

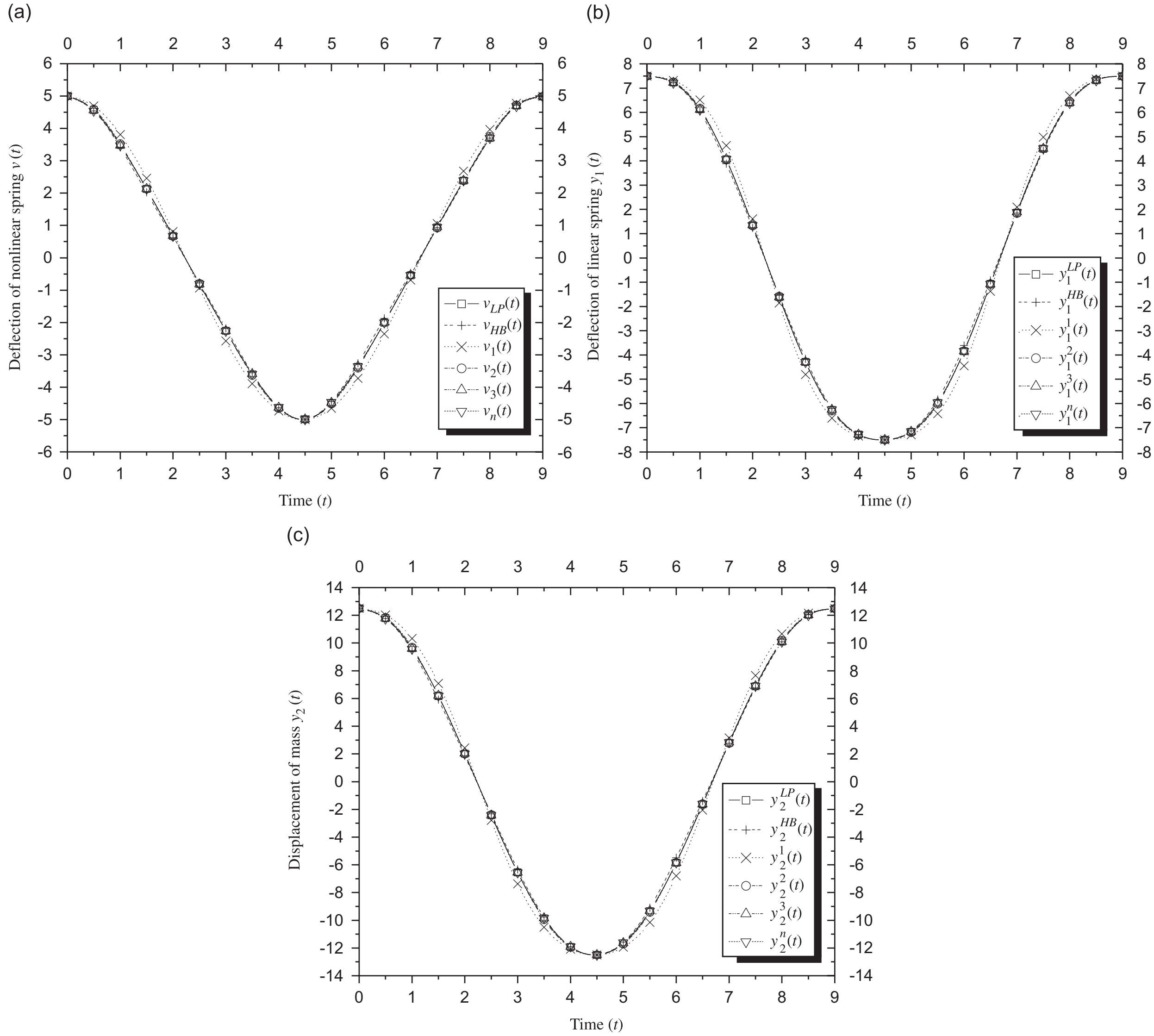

Fig. 6. (a) Comparison of the deflection of nonlinear spring $v(t)$ for various analytical approximations and the numerical integration solutions for $m = 10$, $A = 5$, $\varepsilon = -0.01$ and $\xi = 2$ ($k_1 = 8$, $k_2 = 16$). (b) Comparison of the deflection of linear spring $y_1(t)$ for various analytical approximations and the numerical integration solutions for $m = 10$, $\varepsilon = -0.01$ and $\xi = 2$ ($k_1 = 8$, $k_2 = 16$). (c) Comparison of the displacement of mass $y_2(t)$ for various analytical approximations and the numerical integration solutions for $m = 10$, $\varepsilon = -0.01$ and $\xi = 2$ ($k_1 = 8$, $k_2 = 16$).

For all cases illustrated in the figures, only one period of oscillation are presented. This is because only conservative, nonlinear free oscillation of the mass–spring system is considered. The periodic solution is repetitious and deviations of various analytical approximations with respect to the numerical integration solution are expected to increase as time progresses.

4. Conclusions

In summary, a new method of linearized HB has been developed to construct an analytical approximation for nonlinear free vibration of a system with linear and nonlinear stiffness. As exact solutions for many nonlinear oscillation systems are frequently unavailable, the new approach is advantageous because approximate analytical solutions can be obtained. An avenue to analytically investigate the steady-state response of the system is thus possible. This analytical approach does not require numerical integration as it yields a set of simple, algebraic equations depending on initial conditions. Moreover, these approximate analytical frequencies are valid for small as well as large amplitudes of oscillation. Unlike the perturbation and classical HB methods, the proposed method is simple and it also avoids complicated numerical integration. Furthermore, it does not require a known initial condition at the outset, which is a required condition for all other numerical methods. Because the initial conditions in many practical cases may not be known a priori, this method could be more preferable in analyzing certain nonlinear systems. The results concluded that the third-order analytical approximation provides very accurate solutions with respect to the numerical integration solutions.

Acknowledgment

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Appendix A

The third-order analytical approximation is obtained from the quartic equation of in Eq. (35) which is simplified and presented as Eq. (A.1). Here, $\omega = \omega_3$ for simplicity. It can be solved directly by substituting the corresponding coefficients of Fourier series in any symbolic softwares such as MATHEMATICA. The Fourier-series coefficients a_{2i+1}, b_{2i+1}, c_{2i}, $d_{2(i+1)}$, e_{2i} and f_{2i} for $i = 0,1,2,\ldots$ in Eqs. (A.1)–(A.3) can be determined from Eqs. (15) to (20) where $v_1(\tau) = \mathrm{A}\cos\tau$ is replaced by $v_2(\tau)$ given in Eq. (30). The respective relations are presented in Eqs. (A.4)–(A.28):

$$\begin{aligned}
&-2\big[\big(9\omega^2c_0-\omega^2c_2-\omega^2c_4+9\omega^2c_6+\omega^2d_2-\omega^2d_4+3\omega^2d_6-\omega^2e_0+\omega^2e_2\\
&\quad+\omega^2e_4-\omega^2e_6-f_0+f_2+f_4-f_6\big)\big(9\omega^2c_2-16\omega^2c_4-25\omega^2c_6+3\omega^2d_2-2\omega^2d_4\\
&-5\omega^2d_6-\omega^2e_2+\omega^2e_6-f_2+f_6\big)+\big(-\omega^2c_0+8\omega^2c_2+9\omega^2c_4+2\omega^2d_2+3\omega^2d_4\\
&+\omega^2e_0-\omega^2e_4+f_0-f_4\big)\big(-9\omega^2c_0+25\omega^2c_2-9\omega^2c_6+25\omega^2c_8+5\omega^2d_2-3\omega^2d_6\\
&+5\omega^2d_8+\omega^2e_0-\omega^2e_2+\omega^2e_6-\omega^2e_8+f_0-f_2+f_6-f_8\big)\big]\big[\big(\omega^2a_5+b_5\big)\big(9\omega^2c_2\\
&-16\omega^2c_4-25\omega^2c_6+3\omega^2d_2-2\omega^2d_4-5\omega^2d_6-\omega^2e_2+\omega^2e_6-f_2+f_6\big)\\
&+\omega^2a_1\big(25\omega^2c_0-9\omega^2c_2-9\omega^2c_8+25\omega^2c_{10}+3\omega^2d_2-3\omega^2d_8+5\omega^2d_{10}-\omega^2e_0\\
&+\omega^2e_2+\omega^2e_8-\omega^2e_{10}-f_0+f_2+f_8-f_{10}\big)+b_1\big(25\omega^2c_0-9\omega^2c_2-9\omega^2c_8+25\omega^2c_{10}\\
&+3\omega^2d_2-3\omega^2d_8+5\omega^2d_{10}-\omega^2e_0+\omega^2e_2+\omega^2e_8-\omega^2e_{10}-f_0+f_2+f_8-f_{10}\big)\big]\\
&+2\big[\big(\omega^2a_3+b_3\big)\big(9\omega^2c_2-16\omega^2c_4-25\omega^2c_6+3\omega^2d_2-2\omega^2d_4-5\omega^2d_6-\omega^2e_2+\omega^2e_6\\
&-f_2+f_6\big)+\omega^2a_1\big(-9\omega^2c_0+25\omega^2c_2-9\omega^2c_6+25\omega^2c_8+5\omega^2d_2-3\omega^2d_6+5\omega^2d_8\\
&+\omega^2e_0-\omega^2e_2+\omega^2e_6-\omega^2e_8+f_0-f_2+f_6-f_8\big)+b_1(-9\omega^2c_0+25\omega^2c_2-9\omega^2c_6\\
&+25\omega^2c_8+5\omega^2d_2-3\omega^2d_6+5\omega^2d_8+\omega^2e_0-\omega^2e_2+\omega^2e_6-\omega^2e_8+f_0-f_2+f_6\\
&-f_8)\big][(9\omega^2c_2-16\omega^2c_4-25\omega^2c_6+3\omega^2d_2-2\omega^2d_4-5\omega^2d_6-\omega^2e_2+\omega^2e_6-f_2+f_6\big)\\
&\times(9\omega^2c_2-\omega^2c_4-\omega^2c_6+9\omega^2c_8-3\omega^2d_2+\omega^2d_4-\omega^2d_6+3\omega^2d_8-\omega^2e_2+\omega^2e_4\\
&+\omega^2e_6-\omega^2e_8-f_2+f_4+f_6-f_8)+(-\omega^2c_0+8\omega^2c_2+9\omega^2c_4+2\omega^2d_2+3\omega^2d_4\\
&+\omega^2e_0-\omega^2e_4+f_0-f_4)(25\omega^2c_0-9\omega^2c_2-9\omega^2c_8+25\omega^2c_{10}+3\omega^2d_2-3\omega^2d_8\\
&+5\omega^2d_{10}-\omega^2e_0+\omega^2e_2+\omega^2e_8-\omega^2e_{10}-f_0+f_2+f_8-f_{10})\big]=0.
\end{aligned}\tag{A.1}$$

The constants x_2 and x_3 in Eq. (34) are derived in terms of the coefficients of Fourier series as follows:

$$\begin{aligned} x_2(A) = & -[2\omega^2 a_3(9\omega^2 c_2 - 16\omega^2 c_4 - 25\omega^2 c_6 + 3\omega^2 d_2 - 2\omega^2 d_4 - 5\omega^2 d_6 - \omega^2 e_2 \\ & + \omega^2 e_6 - f_2 + f_6) + 2b_3(9\omega^2 c_2 - 16\omega^2 c_4 - 25\omega^2 c_6 + 3\omega^2 d_2 - 2\omega^2 d_4 - 5\omega^2 d_6 \\ & - \omega^2 e_2 + \omega^2 e_6 - f_2 + f_6) + 2\omega^2 a_1(-9\omega^2 c_0 + 25\omega^2 c_2 - 9\omega^2 c_6 + 25\omega^2 c_8 \\ & + 5\omega^2 d_2 - 3\omega^2 d_6 + 5\omega^2 d_8 + \omega^2 e_0 - \omega^2 e_2 + \omega^2 e_6 - \omega^2 e_8 + f_0 - f_2 + f_6 - f_8) \\ & + 2b_1(-9\omega^2 c_0 + 25\omega^2 c_2 - 9\omega^2 c_6 + 25\omega^2 c_8 + 5\omega^2 d_2 - 3\omega^2 d_6 + 5\omega^2 d_8 + \omega^2 e_0 \\ & - \omega^2 e_2 + \omega^2 e_6 - \omega^2 e_8 + f_0 - f_2 + f_6 - f_8)]/[(9\omega^2 c_0 - \omega^2 c_2 - \omega^2 c_4 + 9\omega^2 c_6 \\ & + \omega^2 d_2 - \omega^2 d_4 + 3\omega^2 d_6 - \omega^2 e_0 + \omega^2 e_2 + \omega^2 e_4 - \omega^2 e_6 - f_0 + f_2 + f_4 - f_6)(9\omega^2 c_2 \\ & - 16\omega^2 c_4 - 25\omega^2 c_6 + 3\omega^2 d_2 - 2\omega^2 d_4 - 5\omega^2 d_6 - \omega^2 e_2 + \omega^2 e_6 - f_2 + f_6) + (-\omega^2 c_0 \\ & + 8\omega^2 c_2 + 9\omega^2 c_4 + 2\omega^2 d_2 + 3\omega^2 d_4 + \omega^2 e_0 - \omega^2 e_4 + f_0 - f_4)(-9\omega^2 c_0 + 25\omega^2 c_2 \\ & - 9\omega^2 c_6 + 25\omega^2 c_8 + 5\omega^2 d_2 - 3\omega^2 d_6 + 5\omega^2 d_8 + \omega^2 e_0 - \omega^2 e_2 + \omega^2 e_6 - \omega^2 e_8 + f_0 \\ & - f_2 + f_6 - f_8)]. \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} x_3(A) = & -(2\omega^2 a_1 + 2b_1 - \omega^2 c_0 x_2 + 8\omega^2 c_2 x_2 + 9\omega^2 c_4 x_2 + 2\omega^2 d_2 x_2 + 3\omega^2 d_4 x_2 \\ & + \omega^2 e_0 x_2 - \omega^2 e_4 x_2 + f_0 x_2 - f_4 x_2)/(-9\omega^2 c_2 + 16\omega^2 c_4 + 25\omega^2 c_6 - 3\omega^2 d_2 + 2\omega^2 d_4 \\ & + 5\omega^2 d_6 + \omega^2 e_2 - \omega^2 e_6 + f_2 - f_6), \end{aligned} \tag{A.3}$$

where

$$a_1 = -\frac{\tilde{A}(4 + 3\tilde{A}^2 z\varepsilon - 3\tilde{A}x_1 z\varepsilon + 6x_1^2 z\varepsilon)}{4}, \tag{A.4}$$

$$a_3 = \frac{9[-\tilde{A}^3 z\varepsilon + 3x_1^3 z\varepsilon + x_1(4 + 6\tilde{A}^2 z\varepsilon)]}{4}, \tag{A.5}$$

$$a_5 = \frac{75\tilde{A}x_1 z\varepsilon(\tilde{A} - x_1)}{4}, \tag{A.6}$$

$$b_1 = \frac{\tilde{A}\omega_e^2(4 + 3\tilde{A}^2\varepsilon - 3\tilde{A}x_1\varepsilon + 6x_1^2\varepsilon)}{4}, \tag{A.7}$$

$$b_3 = \frac{\omega_e^2[\tilde{A}^3\varepsilon - 3x_1^3\varepsilon - 2x_1(2 + 3\tilde{A}^2\varepsilon)]}{4}, \tag{A.8}$$

$$b_5 = -\frac{3\tilde{A}x_1\varepsilon\omega_e^2(\tilde{A} - x_1)}{4}, \tag{A.9}$$

$$c_0 = 2 + 3\tilde{A}^2 z\varepsilon + 3x_1^2 z\varepsilon, \tag{A.10}$$

$$c_2 = \frac{3\tilde{A}z\varepsilon(\tilde{A} - 2x_1)}{2}, \tag{A.11}$$

$$c_4 = -3\tilde{A}x_1 z\varepsilon, \tag{A.12}$$

$$c_6 = \frac{3x_1^2 z\varepsilon}{2}, \tag{A.13}$$

$$c_8 = c_{10} = 0, \tag{A.14}$$

$$d_2 = -6\tilde{A}z\varepsilon(\tilde{A} - 2x_1), \tag{A.15}$$

$$d_4 = 24\tilde{A}x_1 z\varepsilon, \tag{A.16}$$

$$d_6 = -18x_1^2 z\varepsilon, \tag{A.17}$$

$$d_8 = d_{10} = 0, \tag{A.18}$$

$$e_2 = -6\tilde{A}z\varepsilon(\tilde{A} - 2x_1), \tag{A.19}$$

$$e_4 = 48\tilde{A}x_1 z\varepsilon, \tag{A.20}$$

$$e_6 = -54x_1^2 z\varepsilon, \tag{A.21}$$

$$e_0 = e_8 = e_{10} = 0, \tag{A.22}$$

$$f_0 = \left(2 + 3\tilde{A}^2\varepsilon + 3x_1^2\varepsilon\right)\omega_e^2, \tag{A.23}$$

$$f_2 = \frac{3\tilde{A}\varepsilon\omega_e^2(\tilde{A} - 2x_1)}{2}, \tag{A.24}$$

$$f_4 = -3\tilde{A}x_1\varepsilon\omega_e^2, \tag{A.25}$$

$$f_6 = \frac{3x_1^2\varepsilon\omega_e^2}{2}, \tag{A.26}$$

$$f_8 = f_{10} = 0, \tag{A.27}$$

$$\tilde{A} = A + x_1. \tag{A.28}$$

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