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Short Communication

Dynamic characteristics of a structure with multiple attachments: A receptance approach

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Abstract

An application of the analytical approach of receptance for the analysis of vibration and stability of a structure with multiple linear and torsional springs of different stiffness attached to an axially loaded beam is described. The beam–spring combination is a typical analytical model that is often used as the benchmark to verify frequency and buckling data obtained by numerical algorithms and full structural models. Solutions of the analytical model also provide the necessary understanding of system properties in the process of establishing numerical models. Although vibration and buckling characteristics of a structural model with multiple linear and torsional components can be obtained by directly solving governing differential equations, the process can become quite complicated if there are a large number of attachments with a wide range of stiffness values. A receptance procedure proposed in the present study provides a simple and accurate alternative. The approach is an approximation method similar to sub-structuring techniques. By retaining only the essential interactions between the major structure and attached components, the analysis is much simplified. The present study illustrates the simplicity and effectiveness of this approach. A parametric study is also given and compared with the finite element result. Good agreement indicates the accuracy of the approximation.

1. Introduction

An application of the receptance approach for the analysis of vibration and stability of a structure with multiple linear and torsional springs of different stiffness attached to a beam subjected to an axial load is described. The example problem is selected for its simple geometry and solutions available in the limiting cases, although the procedure is intended for more general structural forms with complicated attachments.

The beam–spring combination is a standard analytical model that is often used as the benchmark to verify frequency and buckling data obtained by numerical methods and full structural models. Solutions of the analytical model also provide information on the system properties that is needed in the process of establishing numerical models. Therefore, even with the availability of finite element solutions, analytical studies of simplified models are still useful and necessary.

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An advantage of using the receptance method is that the receptances of the subsystems may be determined by any analytical, experimental, or numerical method. For most applications, it simplifies the analytical procedures and yields accurate solutions. Duncan [1] who originated the method used the term admittance rather than the term receptance that was coined in the text of Bishop and Johnson [2]. Wissenberger [3] reported the effect of local modifications on vibration of linear systems. Jacquot and Soedel [4] studied vibration of elastic surface systems carrying dynamic elements. The topic of beams with a variety of attached elements was treated in the work of and Jacquot and Gibson [5]. Pomazal and Snyder [6], Dowell [7] and Hallquist and Snyder [8] also did fundamental research in the area of local modifications to linear systems. Even though the receptance method is widely used [9–14] to obtain global vibration characteristics of complex structures, a literature survey indicates that it has not been used in the analysis of buckling, although by nature buckling can also be treated as a similar eigenvlaue problem. In the present paper, a structure with multiple attachments is considered. The structure is subjected to axial forces and oscillations. Behaviors of this combination are complicated. There are limiting cases: As the excitation frequency is zero, it is a pure buckling problem; as the axial load becomes zero, it is a free vibration problem. When neither is zero, it is then a general eigenvalue problem.

Following the development of receptance analysis, a parametric study was conducted to show the effectiveness and accuracy of the approximate analysis. Parameters considered include the type, the quantity, the location and the stiffness of springs and their effects on vibration and buckling characteristics. The results obtained agree well with those obtained from the finite element method.

2. The receptance approach

When a sub-structure is added to the main structure, the receptance method is applied by adding the point receptances of the sub-structure to those of the major structure at the interface. For instance, a simply supported beam with n lateral supports, each consists of a linear spring and a torsional spring, as illustrated in Fig. 1, can be viewed as the case where the supports (the sub-structures) are added to the beam (the main structure). At the interface of the beam and the lateral supports, in general, there are the axial and the transverse displacements and the axial slope; therefore, appropriate point force or point moment connections should be considered. However, the receptance due to axial displacements of the beam may be neglected because the magnitude is much smaller, and only the transverse displacements and the axial slope changes need to be considered. The relevant loading connection at the interface of the beam and each support is simplified to a point force connection (representing the linear spring) coupled with a point moment connection (representing the torsional spring), as is shown in Fig. 2.

The point force and point moment at the interface result in linear displacement and angular displacement, respectively. The displacement (or slope) amplitudes, X^u_{Ai} ($u = 1{-}n$, $i = 1$–2) can be expressed as functions of point force (or moment) amplitudes, F^v_{Aj} ($v = 1{-}n, j = 1{-}2$)

$$\{X^u_{Ai}\} = [\alpha^{uv}_{ij}]^{\mathrm{T}}\{F^v_{Aj}\}, \tag{1}$$

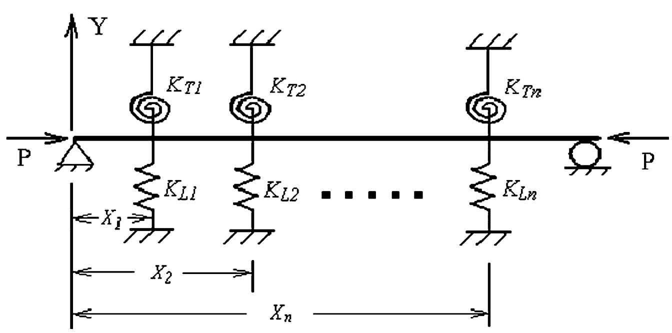

Fig. 1. A simply supported, axially loaded beam is stiffened by n lateral supports; each consists of a linear spring and a torsional spring with different stiffness.

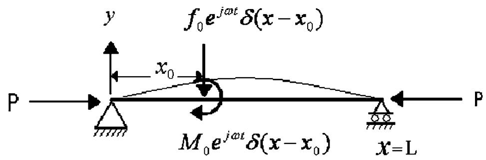

Fig. 2. The axially loaded beam is subjected to a point force and a point moment, due to the effect of the linear spring and the torsional spring, respectively. As the interactive loadings are harmonic, the beam–spring combination will oscillate. If the excitation frequency is zero, the interactive loadings become static.

where the α_{ij}^{uv} are the receptances of system A (the beam); the superscript u indicates the order number of the interface between the spring and the beam; the superscript v indicates the order number of the interactive load. The subscript i represents the type of displacement, namely, 1 for the linear displacement and 2 for the slope. The subscript j represents the type of load, namely, 1 for the force and 2 for the moment. Similarly,

$$\{X_{Bi}^{u}\} = [\beta_{ij}^{uv}]^{\mathrm{T}}\{F_{Bj}^{v}\}, \tag{2}$$

where β_{ij}^{uv} are the receptances of system B (the spring support). For the springs, the receptances β_{ij}^{uv} exist only for $u=v$ and $i=j$, due to zero coupling of the linear spring motion to the torsional spring motion. When the receptance is applied to add a sub-system B to a major system A, with no external forces (or moments) applied to the two systems, the force (or moment) equilibrium and displacement (or slope) compatibility must be satisfied.

Combining Eqs. (1) and (2), and applying the above equalities gives

$$\begin{bmatrix} \alpha_{11}^{11}+\beta_{11}^{11} & \alpha_{11}^{12} & \bullet & \alpha_{11}^{1n} & \alpha_{12}^{11} & \alpha_{12}^{12} & \bullet & \alpha_{12}^{1n} \\ \alpha_{11}^{21} & \alpha_{11}^{22}+\beta_{11}^{22} & \bullet & \alpha_{11}^{2n} & \alpha_{12}^{21} & \alpha_{12}^{22} & \bullet & \alpha_{12}^{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \alpha_{11}^{n1} & \alpha_{11}^{n2} & \bullet & \alpha_{11}^{nn}+\beta_{11}^{nn} & \alpha_{12}^{n1} & \alpha_{12}^{n2} & \bullet & \alpha_{12}^{nn} \\ \alpha_{21}^{11} & \alpha_{21}^{12} & \bullet & \alpha_{21}^{1n} & \alpha_{22}^{11}+\beta_{22}^{11} & \alpha_{22}^{12} & \bullet & \alpha_{22}^{1n} \\ \alpha_{21}^{21} & \alpha_{21}^{22} & \bullet & \alpha_{21}^{2n} & \alpha_{22}^{21} & \alpha_{22}^{22}+\beta_{22}^{22} & \bullet & \alpha_{22}^{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \alpha_{21}^{n1} & \alpha_{21}^{n2} & \bullet & \alpha_{21}^{nn} & \alpha_{22}^{n1} & \alpha_{22}^{n2} & \bullet & \alpha_{22}^{nn}+\beta_{22}^{nn} \end{bmatrix} \begin{Bmatrix} F_{A1}^{1} \\ F_{A1}^{2} \\ \bullet \\ F_{A1}^{n} \\ F_{A2}^{1} \\ F_{A2}^{2} \\ \bullet \\ F_{A2}^{n} \end{Bmatrix} = 0, \tag{3}$$

where F_{A1}^{1}, $F_{A1}^{2}\ldots F_{A1}^{n}$ are the interactive forces, and F_{A2}^{1}, $F_{A2}^{2}\ldots{}_{A2}^{n}$ are the interactive moments. In order to get a non-trivial solution for the force vector, the $2n\times 2n$ determinant of the receptance matrix given in Eq. (3) must be equal to zero. For the general case of a beam with n lateral supports, as shown in Fig. 1, the system frequency equation and the buckling load equation can be expressed as

$$\begin{vmatrix} \alpha_{11}^{11}+\beta_{11}^{11} & \alpha_{11}^{12} & \bullet & \alpha_{11}^{1n} & \alpha_{12}^{11} & \alpha_{12}^{12} & \bullet & \alpha_{12}^{1n} \\ \alpha_{11}^{21} & \alpha_{11}^{22}+\beta_{11}^{22} & \bullet & \alpha_{11}^{2n} & \alpha_{12}^{21} & \alpha_{12}^{22} & \bullet & \alpha_{12}^{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \alpha_{11}^{n1} & \alpha_{11}^{n2} & \bullet & \alpha_{11}^{nn}+\beta_{11}^{nn} & \alpha_{12}^{n1} & \alpha_{12}^{n2} & \bullet & \alpha_{12}^{nn} \\ \alpha_{21}^{11} & \alpha_{21}^{12} & \bullet & \alpha_{21}^{1n} & \alpha_{22}^{11}+\beta_{22}^{11} & \alpha_{22}^{12} & \bullet & \alpha_{22}^{1n} \\ \alpha_{21}^{21} & \alpha_{21}^{22} & \bullet & \alpha_{21}^{2n} & \alpha_{22}^{21} & \alpha_{22}^{22}+\beta_{22}^{22} & \bullet & \alpha_{22}^{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \alpha_{21}^{n1} & \alpha_{21}^{n2} & \bullet & \alpha_{21}^{nn} & \alpha_{22}^{n1} & \alpha_{22}^{n2} & \bullet & \alpha_{22}^{nn}+\beta_{22}^{nn} \end{vmatrix} = 0. \tag{4}$$

The stiffness and receptance of the springs (the linear and the torsional springs) can be different, namely $\beta_{11}^{11}\neq\beta_{11}^{22}\neq\cdots\neq\beta_{11}^{nn}$ in Eq. (4).

3. Formulation of receptance

3.1. Beam receptances due to displacement and slope caused by harmonically excited interactive force at the interface

For a simply supported, axially loaded beam with one lateral support consisting of a linear spring and a torsional spring, the interactive loadings at the interface are represented by a point force and a point moment, as illustrated in Fig. 2. With a compressive axial load P at each end and a transverse point force $f_0\delta(x-x_0)$ at x_0, the moment M_x on the cross-section of the beam at arbitrary location x is

$$M_x = \begin{cases} -Py + \dfrac{f_0 x}{L}(L-x_0), & 0<x<x_0, \\ -Py + \dfrac{f_0 x}{L}(L-x_0) - f_0(x-x_0), & x_0<x<L, \end{cases} \tag{5}$$

where y is the transverse deflection, f_0 the amplitude of the lateral force, L the length of the beam, $\delta(x-x_0)$ a Dirac-delta function. With the moment on the arbitrary cross-section of the beam denoted in the form of Eq. (5), the nonhomogeneous buckling equation of the beam–spring combination can be expressed as

$$EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + Py = -f_0(x-x_0)u(x-x_0) + \frac{f_0 x}{L}(L-x_0), \tag{6}$$

where $u(x-x_0)$ is a unit step function. In a vibrating beam, the interactive force is harmonic as $f_0 \mathrm{e}^{\mathrm{j}\omega t}\delta(x-x_0)$ and the transverse deflection $y(x,t)$ is $Y(x)\mathrm{e}^{\mathrm{j}\omega t}$. Upon considering the inertia, the differential equation for the transverse vibration of the beam is

$$EI\frac{\mathrm{d}^4 Y(x)}{\mathrm{d}x^4} + P\frac{\mathrm{d}^2 Y(x)}{\mathrm{d}x^2} - \rho A\omega^2 Y(x) = -f_0\delta(x-x_0), \tag{7}$$

in which ρ is the mass density of the material, A the cross-sectional area of the beam, EI its flexural rigidity, and ω the excitation frequency. The complementary solution of Eq. (7) is

$$Y(x) = C_1\cosh(\lambda_1 x) + C_2\sinh(\lambda_1 x) + C_3\cos(\lambda_2 x) + C_4\sin(\lambda_2 x), \tag{8}$$

where

$$\lambda_1 = \left(\sqrt{(P/2EI)^2 + \rho A\omega^2/EI} - P/2EI\right)^{1/2}$$

and

$$\lambda_2 = \left(\sqrt{(P/2EI)^2 + \rho A\omega^2/EI} + P/2EI\right)^{1/2}$$

Substituting the boundary conditions into Eq. (8) and solving its coefficients, we get the time-dependent transverse deflection of the axially loaded beam due to a harmonic lateral force $f_0\mathrm{e}^{\mathrm{j}\omega t}$ at axial location x_0 to be:

$$y_{f_0}(x,t) = \sum_{m=1}^{\infty}\frac{f_0\sin(m\pi x_0/L)\sin(m\pi x/L)\mathrm{e}^{\mathrm{j}\omega t}}{(L/2)[\rho A\omega^2 + P(m\pi/L)^2 - (m\pi/L)^4 EI]}. \tag{9}$$

The receptance of the beam due to its displacement response $y_{f_0}(x,t)$ to the harmonic lateral force $f_0\mathrm{e}^{\mathrm{j}\omega t}$ at the interface is defined as $\alpha_{11}^{uv} = y_{f_0}(x,t)/f_0\mathrm{e}^{\mathrm{j}\omega t}$, which is

$$\alpha_{11}^{uv} = \sum_{m=1}^{\infty}\frac{\sin(m\pi x_u/L)\sin(m\pi x_v/L)}{(L/2)[\rho A\omega^2 + P(m\pi/L)^2 - (m\pi/L)^4 EI]}, \tag{10}$$

where the subscripts $u = 1-n$ and $v = 1-n$.

The receptance due to the slope response of the beam to the harmonic lateral force $f_0 e^{j\omega t}$ at the interface is defined as $\alpha_{21}^{uv} = (\partial y_{f_0}/\partial x)/f_0 e^{j\omega t}$, which is expressed as

$$\alpha_{21}^{uv} = \sum_{m=1}^{\infty} \frac{(m\pi/L)\cos(m\pi x_u/L)\sin(m\pi x_v/L)}{(L/2)[\rho A\omega^2 + P(m\pi/L)^2 - (m\pi/L)^4 EI]}. \tag{11}$$

3.2. Beam receptances due to displacement and slope caused by harmonically excited interactive moment at the interface

A concentrated moment M_0 can be defined by two forces that are separated by an infinitesimally small distance ε and are of the same magnitude f_0 but opposite direction, namely, $\lim_{\varepsilon\to 0} f_0\varepsilon = M_0$. Applying this definition to Eq. (9) and multiplying its nominator and denominator by ε, we have the displacement of a transversely vibrating beam subjected to a harmonic point moment defined as

$$y_{M_0}(x,t) = \lim_{\varepsilon\to 0}(f_0\varepsilon)\sum_{m=1}^{\infty} \frac{e^{j\omega t}[\sin(m\pi x_0/L) - \sin(m\pi(x_0-\varepsilon)/L)]\sin(m\pi x/L)}{\varepsilon(L/2)[\rho A\omega^2 + P(m\pi/L)^2 - (m\pi/L)^4 EI]}. \tag{12}$$

Rearranging Eq. (12) and multiplying its nominator and denominator by $m\pi/L$, we obtain

$$y_{M_0} = \lim_{\varepsilon\to 0}(f_0\varepsilon)\sum_{m=1}^{\infty} \frac{e^{j\omega t}\left\{\sin\left(m\pi x_0/L\right)[1-\cos(m\pi\varepsilon/L)] + \cos(m\pi x_0/L)\sin(m\pi\varepsilon/L)\right\}\sin(m\pi x/L)(m\pi/L)}{(L/2)[\rho A\omega^2 + P(m\pi/L)^2 - (m\pi/L)^4 EI](m\pi\varepsilon/L)}. \tag{13}$$

Applying L' Hospital's law, we have

$$\lim_{\varepsilon\to 0}\frac{[1-\cos(m\pi\varepsilon/L)]}{(m\pi\varepsilon/L)} = \lim_{\varepsilon\to 0}\frac{\sin(m\pi\varepsilon/L)}{1} = 0 \tag{14}$$

and

$$\lim_{\varepsilon\to 0}\frac{\sin(m\pi\varepsilon/L)}{(m\pi\varepsilon/L)} = \lim_{\varepsilon\to 0}\frac{\cos(m\pi\varepsilon/L)}{1} = 1. \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (13), the transverse deflection of the beam subjected to a harmonic interactive moment is expressed as

$$y_{M_0}(x,t) = \sum_{m=1}^{\infty} \frac{M_0[(m\pi/L)\sin(m\pi x/L)\cos(m\pi x_0/L)]e^{j\omega t}}{(L/2)[\rho A\omega^2 + P(m\pi/L)^2 - (m\pi/L)^4 EI]}. \tag{16}$$

The receptance of the beam due to its displacement response $y_{M_0}(x,t)$ to the harmonic interactive moment $M_0 e^{j\omega t}$ at the interface is defined as $\alpha_{12}^{uv} = y_{M_0}(x,t)/M_0 e^{j\omega t}$, which is

$$\alpha_{12}^{uv} = \sum_{m=1}^{\infty} \frac{(m\pi/L)\sin(m\pi x_u/L)\cos(m\pi x_v/L)}{(L/2)[\rho A\omega^2 + P(m\pi/L)^2 - (m\pi/L)^4 EI]}. \tag{17}$$

The receptance of the beam due to the slope response to the harmonic interactive moment $M_0 e^{j\omega t}$ at the interface is defined as $\alpha_{22}^{uv} = (\partial y_{M_0}(x,t)/\partial x)/M_0 e^{j\omega t}$, which is expressed as

$$\alpha_{22}^{uv} = \sum_{m=1}^{\infty} \frac{(m\pi/L)^2\cos(m\pi x_u L)\cos(m\pi x_v/L)}{(L/2)[\rho A\omega^2 + P(m\pi/L)^2 - (m\pi/L)^4 EI]}. \tag{18}$$

3.3. Spring receptances

The receptances of the linear springs due to the displacement responses to the interactive forces are defined as $\beta_{11}^{uv}=1/k_t^v$, where k_t^v $(v=1-n)$ are the stiffness of the linear springs, and u must be equal to v. Similarly, the receptances of the torsional springs due to the angular displacement responses to the interactive moments are defined as $\beta_{22}^{uv}=1/k_r^v$, where k_t^v $(v=1-n)$ are the stiffness of the torsional springs.

4. Interpretation of receptance

If the natural frequency of a simply supported beam without axial load is denoted as ω_m $(\omega_m=(m\pi/L)^2\sqrt{EI/\rho A})$, Eqs. (10), (11), (17) and (18) can be expressed in the following forms:

$$\alpha_{11}^{uv}=\sum_{m=1}^{\infty}\frac{-\sin(m\pi x_u/L)\sin(m\pi x_v/L)}{\rho A(L/2)[\omega_m^2-(P/\rho A)(m\pi/L)^2-\omega^2]}, \tag{19}$$

$$\alpha_{21}^{uv}=\sum_{m=1}^{\infty}\frac{-(m\pi/L)\cos(m\pi x_u/L)\sin(m\pi x_v/L)}{\rho A(L/2)[\omega_m^2-(P/\rho A)(m\pi/L)^2-\omega^2]}, \tag{20}$$

$$\alpha_{12}^{uv}=\sum_{m=1}^{\infty}\frac{-(m\pi/L)\sin(m\pi x_u/L)\cos(m\pi x_v/L)}{\rho A(L/2)[\omega_m^2-(P/\rho A)(m\pi/L)^2-\omega^2]}, \tag{21}$$

$$\alpha_{22}^{uv}=\sum_{m=1}^{\infty}\frac{-(m\pi/L)^2\cos(m\pi x_u/L)\cos(m\pi x_v/L)}{\rho A(L/2)[\omega_m^2-(P/\rho A)(m\pi/L)^2-\omega^2]}. \tag{22}$$

In Eqs. (19)–(22), if ω^2 is equal to $\omega_m^2-(P/\rho A)(m\pi/L)^2$, each receptance becomes infinity. It implies that a simply supported, axially loaded beam without intermediate support is resonant. Due to the stiffening effect of the springs, the natural frequencies of the composite structure, which are obtained by solving the system frequency equation (Eq. (4)) in terms of the receptances defined in Eqs. (19)–(22), are higher than those of an un-stiffened beam. A compressive axial load decreases the natural frequencies of a structure, while a tensile axial load (P is negative) increases them.

In a limiting case, as the compressive axial load increases, the frequency of the lowest mode of vibration approaches zero. It implies that transverse buckling of the structure occurs. This mode of zero natural frequency is the first buckling mode. In other words, if the excitation frequency in Eqs. (10), (11), (17) and (18) is zero ($\omega=0$), these equations become

$$\alpha_{11}^{uv}=\sum_{m=1}^{\infty}\frac{(L/m\pi)^2\sin(m\pi x_u/L)\sin(m\pi x_v/L)}{(L/2)(P-(m\pi/L)^2EI)}, \tag{23}$$

$$\alpha_{21}^{uv}=\sum_{m=1}^{\infty}\frac{(L/m\pi)\cos(m\pi x_u/L)\sin(m\pi x_v/L)}{(L/2)(P-(m\pi/L)^2EI)}, \tag{24}$$

$$\alpha_{12}^{uv}=\sum_{m=1}^{\infty}\frac{(L/m\pi)\sin(m\pi x_u/L)\cos(m\pi x_v/L)}{(L/2)(P-(m\pi/L)^2EI)}, \tag{25}$$

$$\alpha_{22}^{uv}=\sum_{m=1}^{\infty}\frac{\cos(m\pi x_u/L)\cos(m\pi x_v/L)}{(L/2)(P-(m\pi/L)^2EI)}. \tag{26}$$

When the compressive axial load P is equal to one of the buckling loads of a simply supported beam (i.e., $P=(m\pi/L)^2EI$), the receptances defined in Eqs. (23)–(26) become infinite. It implies that the subsystem, the simply supported beam, buckles.

5. Numerical results and discussions

The compressive axial load is assumed to be applied at the centroid of the cross-section of a perfect beam. For convenience, all major parameters are non-dimensionalized. The location of the spring is denoted by C, $C = x/L$. The non-dimensional stiffness of the linear spring, k_L, is denoted by $\bar{K}_L$, $\bar{K}_L = k_L L^3/EI$; whereas the non-dimensional stiffness of the torsional spring, k_T, is denoted by $\bar{K}_T$, $\bar{K}_T = k_T L/EI$. The non-dimensional critical buckling load is denoted by $\bar{P}$, $\bar{P} = P_1^s/[(\pi/L)^2 EI]$, where P_1^s is the system critical buckling load of a beam–spring combination, while $(\pi/L)^2 EI$ is the Euler buckling load. The non-dimensional first natural frequency is denoted by $\bar{\Omega}$, $\bar{\Omega} = \omega_1^s/[(\pi/L)^2\sqrt{EI/\rho}]$, where ω_1^s is the system first natural frequency of a beam–spring combination, while $(\pi/L)^2\sqrt{EI/\rho}$ is that of a simply supported beam.

5.1. Case 1—beam with one lateral support consists of a linear spring and a torsional spring

In Figs. 3(a) and (b), the variation of the first system natural frequency with the location and stiffness of the springs is shown. Fig. 3(a) shows that if the beam is free from compressive axial load (i.e., $P = 0$) and with

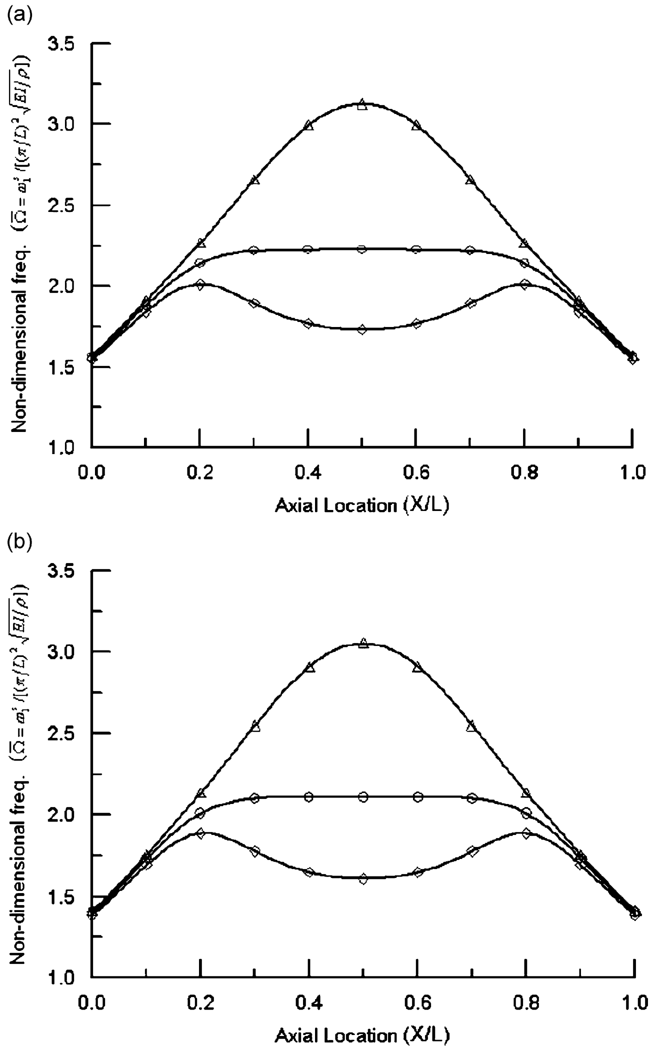

Fig. 3. Without a compressive axial load ($P = 0$, plot (a)) or with a compressive axial load ($P = 0.4P_{cr}$, plot (b)), the first system natural frequency of the composite structure varies with location and stiffness of the spring support. Plot (a) ◇: $\bar{K}_L = \bar{K}_T = 100$, ○: $\bar{K}_L = \bar{K}_T = 200$; and △ $\bar{K}_L = \bar{K}_T = 500$. Plot (b) ◇: $\bar{K}_L = \bar{K}_T = 100$, ○: $\bar{K}_L = \bar{K}_T = 200$; and △: $\bar{K}_L = \bar{K}_T = 500$.

$\bar{K}_L = \bar{K}_T = 100$, the maximum ratio of the first natural frequencies, $\bar{\Omega}$, is 2.012 as the support is located at $C = 0.2$ or 0.8; while $\bar{\Omega}$ is 1.729 when the support is at $C = 0.5$. It indicates that, if both the linear and the torsional springs are soft, the torsional spring, which resists angular displacement, is inactive if $C = 0.5$; therefore, the $C = 0.5$ case effectively only has the linear spring. As the springs become stiffer, the shape of the $\bar{\Omega} - C$ curve changes from concave to convex. As $\bar{K}_L = \bar{K}_T = 200$, the maximum $\bar{\Omega}$ is 2.225, with springs located in the region between $C = 0.42$ and 0.58. As the stiffness of the linear spring increases further, the maximum $\bar{\Omega}$ occurs with the support located at $C = 0.5$. For the cases of $\bar{K}_L = \bar{K}_T = 500$, the maximum $\bar{\Omega}$ is 3.128.

A similar situation happens to the beam–spring combination with a compressive axial load. Fig. 3(b) shows that if the compressive axial load is 40% of the critical buckling load of a simply supported beam, for the case of $\bar{K}_L = \bar{K}_T = 100$, the maximum $\bar{\Omega}$ is 1.888 when the springs are located at $C = 0.2$; while $\bar{\Omega}$ is 1.609 when the springs are located at $C = 0.5$. For the case of $\bar{K}_L = \bar{K}_T = 200$, the maximum $\bar{\Omega}$ is 2.11 with the support located in the region between $C = 0.42$ and 0.58. For the cases of $\bar{K}_L = \bar{K}_T = 500$, the maximum $\bar{\Omega}$ is 3.055. By comparing Fig. 3(b), with Fig. 3(a), we found that the compressive axial load decreases the natural frequencies of the beam–spring combination. If the beam is subjected to a tensile axial load, the system natural frequencies will be raised.

A stiff linear spring functions as a hinged support, which resists transverse displacement of the beam. With a stiff linear spring alone at $C = 0.5$, the simply supported beam of length L becomes a continuous beam with two equal spans of length $L/2$. The critical buckling load and the first natural frequency of the beam–spring combination are four times ($\bar{\Omega} = 4$, $\bar{P} = 4$) those of a beam without intermediate support, as shown in Figs. 4 and 5, respectively. However, a stiff torsional spring alone at $C = 0.5$, which is the anti-node location of the first mode of transverse vibration and buckling, does not raise the critical buckling load or the first natural frequency of the composite structure. A combination of a very stiff linear spring and a very stiff torsional spring function as a full brace support, which constrains the beam from transverse deflection and rotation. In theory, the critical buckling load of a clamped–pinned beam is 2.04 times that of a pinned–pinned beam. In Fig. 4 it is shown that the maximum ratio of the critical buckling loads of the beam–spring combination, with a very stiff support ($\bar{K}_L = \bar{K}_T$) at $C = 0.5$, can be 8.17, which matches the exact solution for the buckling load of a pinned–pinned beam with a length of $L/2$ (i.e., $4 \times 2.04 = 8.16$). As shown in Fig. 5, the first system natural frequency of the beam–spring combination without axial load varies with the stiffness of the springs in a similar manner. However, the maximum ratio of the first frequencies is 6.2, which is smaller than the maximum ratio for the critical buckling loads.

Without the axial load, the ratio of the first system natural frequency ($\bar{\Omega}$) is larger than 1 due to the stiffening effect of the springs. When there is a compressive axial load, the system natural frequency is reduced.

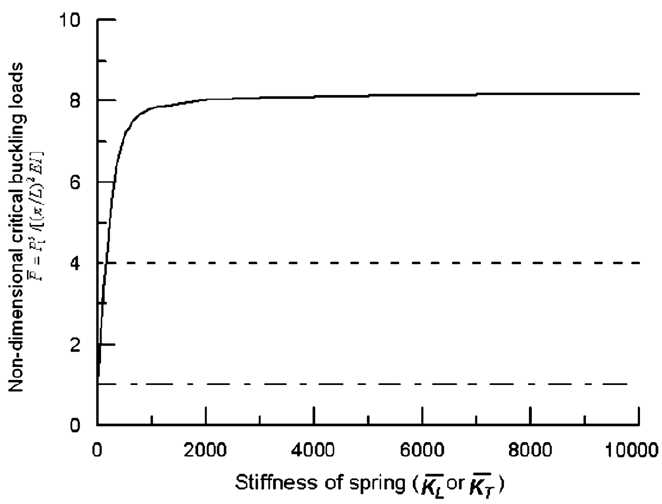

Fig. 4. The critical buckling load of the composite structure varies with stiffness of the springs at $C = 0.5$. $\bar{K}_T$ alone, – · –; $\bar{K}_L$ alone, - - - -; and $\bar{K}_L = \bar{K}_T$, ——.

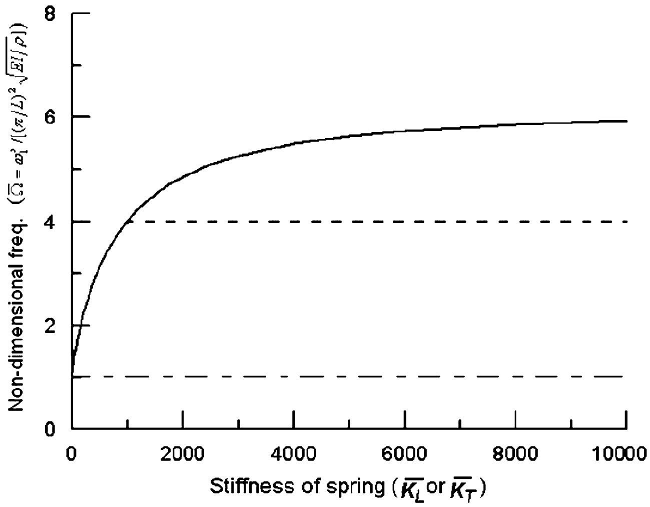

Fig. 5. The first natural frequency of the composite structure without axial load varies with the stiffness of springs at $C = 0.5$. $\bar{K}_T$ alone, – · –; $\bar{K}_L$ alone, - - - -; and $\bar{K}_L = \bar{K}_T$, ——.

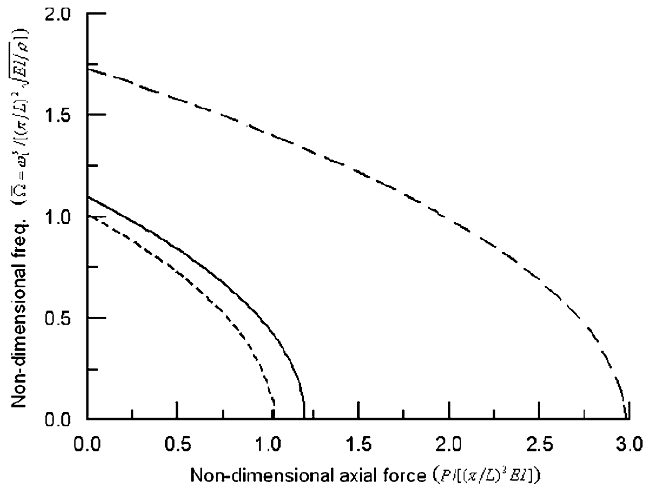

Fig. 6. The ratio of frequencies varies with the compressive axial load and the stiffness of springs, one support at $C = 0.5$. $\bar{K}_L = \bar{K}_T = 1$, - - - -; $\bar{K}_L = \bar{K}_T = 10$, ——; and $\bar{K}_L = \bar{K}_T = 100$, – – –.

The larger the compressive axial load, the smaller the first system natural frequency. Fig. 6 depicts how the first natural frequency and the buckling load of the composite structure vary with the compressive axial load and the spring stiffness. For example, when $\bar{K}_L = \bar{K}_T = 10$, $\bar{\Omega}$ is 1.098, the axial load is zero and $\bar{\Omega}$ approaches zero as a compressive axial load approaches 1.2 times of the Euler buckling load. It should be noted that the horizontal intercepts of the curves with various $\bar{K}_L = \bar{K}_T$ define the curve of the ratio of the critical buckling shown in Fig. 4. The vertical intercepts of the curves with various $\bar{K}_L = \bar{K}_T$ become the curve of the ratio of the first natural frequency in Fig. 5. This demonstrates that in vibration and buckling of the beam–spring combination, a vibration mode with zero frequency is in essence the first buckling mode.

5.2. Case 2—beam with two separated lateral supports

The stiffening effect with two supports, one at $C_1 = 1/3$ and the other at $C_2 = 2/3$, is similar but stronger than with one support at $C = 0.5$. Fig. 7 depicts how the first natural frequency and the critical buckling load

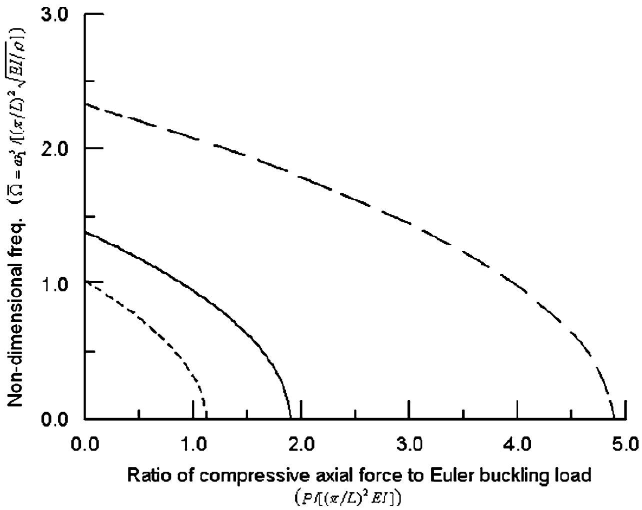

Fig. 7. The ratio of frequencies varies with the compressive axial load and the stiffness of springs, two supports, one support at $C_1 = 1/3$, the other at $C_2 = 2/3$, $\bar{K}_L = \bar{K}_T = 1$, - - - -; $\bar{K}_L = \bar{K}_T = 10$, ——; and $\bar{K}_L = \bar{K}_T = 100$, – – –.

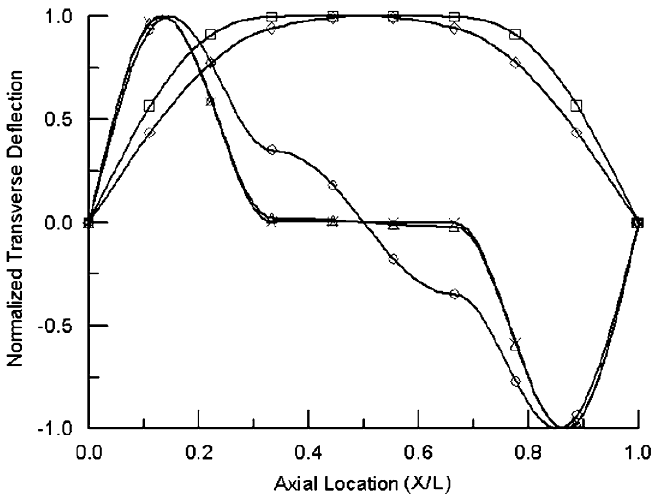

Fig. 8. The first buckling mode shape of the beam–spring structure with one support at $C_1 = 1/3$ and the other at $C_2 = 2/3$. ◇: $\bar{K}_L = \bar{K}_T = 2e+1$; □: $\bar{K}_L = \bar{K}_T = 2e+2$; ○: $\bar{K}_L = \bar{K}_T = 2e+3$; △: $\bar{K}_L = \bar{K}_T = 2e+4$; and ×: $\bar{K}_L = \bar{K}_T = 2e+5$.

of the composite structure vary with the axial load and the stiffness of the two supports. For example, if the stiffness $\bar{K}_L = \bar{K}_T = 10$, $\bar{\Omega}$ is 1.388 when the axial load is zero, compared to 1.098 when there is one support at $C = 0.5$; $\bar{\Omega}$ approaches zero as a compressive axial load approaches 1.89 times of the Euler buckling load, compared to 1.2 with one support at $C = 0.5$.

In Fig. 8, the first buckling mode of the beam–spring structure with two supports, one at $C_1 = 1/3$ and the other at $C_2 = 2/3$, with $\bar{K}_L = \bar{K}_T = 20$, 2×10^2, 2×10^3, 2×10^4, 2×10^5 is shown. When the springs are soft ($\bar{K}_L = \bar{K}_T = 20$ and 200), the shape of the critical buckling mode is a half-wave. When the stiffness increases to $\bar{K}_L = \bar{K}_T = 2000$, the buckling mode resembles a full sine curve with a couple of kinks, which are generated by the torsional springs. When the linear and the torsional springs are both very stiff ($\bar{K}_L = \bar{K}_T = 2 \times 10^5$), the two spring supports function as clamped supports and the buckling mode shape exhibits no displacement and slope in the middle section of the beam.

With one support moved from $C_2 = 2/3$ to 0.8, the critical buckling modes of the beam–spring combination change shapes from the ones shown in Fig. 8 to those shown in Fig. 9. When the linear and the torsional

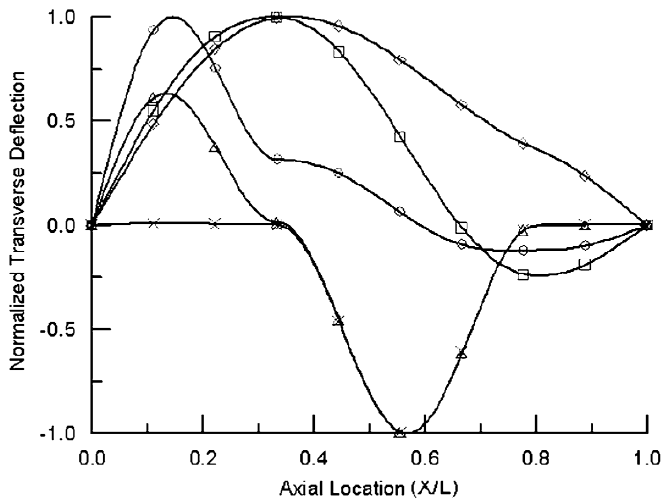

Fig. 9. The first buckling mode shape of the beam–spring structure with one support at $C_1 = 1/3$ and the other at $C_2 = 0.8$. ◇: $\bar{K}_L = \bar{K}_T = 2e+1$; □: $\bar{K}_L = \bar{K}_T = 2e+2$; ○: $\bar{K}_L = \bar{K}_T = 2e+3$; △: $\bar{K}_L = \bar{K}_T = 2e+4$; and ×: $\bar{K}_L = \bar{K}_T = 2e+5$.

springs are both very stiff ($\bar{K}_L = \bar{K}_T = 2 \times 10^5$), the buckling mode shape exhibits a half-wave in the middle section, and there is no displacement and slope in the left and right sections of the beam.

In Figs. 8 and 9, it is shown that vibration and buckling mode shapes vary with the locations and stiffness of the supports. Vibration and buckling modes with unsymmetrical shapes can be generated by either support with equal stiffness but at unsymmetrical locations on the beam, or supports with unequal stiffness but at symmetrical locations on the beam. In theory, the higher-order buckling modes can be obtained, however, in reality, it is only possible for the first buckling mode to occur.

6. Conclusions

The receptance method, a concept for sub-structuring analysis, is applied to analyze the buckling and modal characteristics of a structure with attachments subjected to compressive axial loads. The effects of the type, the quantity, the location and the stiffness of the attachments on the elastic stability and vibration characteristics of the beam are investigated.

Results obtained in the study agree well with those obtained by the finite element method, confirming the receptance method as a useful analytical approach in solving general eigenvalue problems. It should be noted that the intent of this approach is to establish the receptance method for combined vibration and buckling problems. Examples include plates with stiffening ribs, and general shells with stiffening rings.

References

[1] W.J. Duncan, The admittance method for obtaining the natural frequencies of systems, *Philosophical Magazine* 32 (1941) 401–409.
[2] R.E.D. Bishop, D.C. Johnson, *The Mechanics of Vibration*, Cambridge University Press, London, 1960.
[3] J.T. Wissenberger, Effect of local modifications on vibration characteristics of linear systems, *Journal of Applied Mechanics* 35 (1968) 327–332.
[4] R.G. Jacquot, W. Soedel, Vibration of elastic surface systems carrying dynamic elements, *Journal of the Acoustical Society of America* 47 (1970) 1354–1358.
[5] R.G. Jacquot, J.D. Gibson, The effect of discrete masses and elastic supports on continuous beam natural frequencies, *Journal of Sound and Vibration* 23 (1972) 237–244.
[6] R.J. Pomazal, V.W. Snyder, Local modifications of damped linear systems, *American Institute of Aeronautics and Astronautics Journal* 9 (1971) 2216–2221.
[7] E.H. Dowell, Free vibration of an arbitrary structure in terms of component modes, *Journal of Applied Mechanics* 39 (1972) 727–732.
[8] J. Hallquist, V.W. Snyder, Linear damped vibratory structures with arbitrary support conditions, *Journal of Applied Mechanics* 40 (1973) 312–313.

[9] D. Wilken, W. Soedel, Simplified prediction of the modal characteristics of ring-stiffened cylindrical shell, *Journal of Sound and Vibration* 44 (1976) 577–589.
[10] S. Azimi, J.F. Hamilton, W. Soedel, The receptance method applied to the free vibration of continuous rectangular plates, *Journal of Sound and Vibration* 93 (1984) 9–29.
[11] S.C. Huang, W. Soedel, Effects of Coriolis acceleration on the forced vibration of rotating cylindrical shells, *Journal of Applied Mechanics* 55 (1988) 231–233.
[12] S.C. Huang, B.S. Hsu, Resonant phenomena of a rotating cylindrical shell subjected to a harmonic moving load, *Journal of Sound and Vibration* 136 (2) (1990) 215–228.
[13] S.C. Huang, L.H. Chen, Vibration of a spinning cylindrical shell with internal/external ring stiffeners, *Journal of Vibration and Acoustics* 118 (1996) 227–236.
[14] C.K. Su, S.C. Huang, Receptance method to the sensitivity of critical speeds to rotor supports stiffness, *Journal of Engineering Gas Turbines and Power* 119 (1996) 736–739.