

Short Communication

A new 9-point sixth-order accurate compact finite-difference method for the Helmholtz equation

Majid Nabavi, M.H. Kamran Siddiqui*, Javad Dargahi

Department of Mechanical and Industrial Engineering, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8, Canada

Received 28 March 2006; received in revised form 11 July 2006; accepted 27 June 2007
Available online 28 August 2007

Abstract

A new 9-point sixth-order accurate compact finite-difference method for solving the Helmholtz equation in one and two dimensions, is developed and analyzed. This scheme is based on sixth-order approximation to the derivative calculated from the Helmholtz equation. A sixth-order accurate symmetrical representation for the Neumann boundary condition was also developed. The efficiency and accuracy of the scheme is validated by its application to two test problems which have exact solutions. Numerical results show that this sixth-order scheme has the expected accuracy and behaves robustly with respect to the wave number.

© 2007 Elsevier Ltd. All rights reserved.

1. Introduction

The Helmholtz equation, $(\Delta + k^2)u = f$, is an elliptic partial differential equation which is a time-harmonic solution of the wave equation. The Helmholtz equation governs some important physical phenomena. These include the potential in time harmonic acoustic and electromagnetic fields, acoustic wave scattering, noise reduction in silencers, water wave propagation, membrane vibration and radar scattering. Obtaining an efficient and more accurate numerical solution for the Helmholtz equation has been the subject of many studies. The numerical solution of the Helmholtz equation has been developed using different approaches such as the finite-difference method [1], the boundary element method [2], the finite-element method [3] and the spectral-element method [4].

The boundary element method is derived through the discretization of an integral equation that is mathematically equivalent to the original partial differential equation. The disadvantages of boundary element methods are the restriction to linear problems in homogeneous and isotropic media, as well as the large computer storage space required and lengthy processing time needed to solve the inherent problems encountered with characteristic wave numbers.

Finite-element methods are used extensively to solve the Helmholtz equation. In addition to the high-computational cost, another disadvantage of Galerkin finite-element method for solving the Helmholtz

*Corresponding author. Tel.: +1 514 848 2424 x 7940; fax: +1 514 848 3175.
E-mail address: siddiqui@encs.concordia.ca (M.H.K. Siddiqui).

equation is the so-called *pollution effect*, which results in the less accurate solution at higher wave numbers for the given nodes per wavelength. Thus, in order to obtain the same solution accuracy for higher wavenumbers, more nodes per wavelength are needed than that for the lower wavenumbers [5,6]. Although some modification in the standard Galerkin approximation have been developed to minimize the pollution effect [8], finding an optimal method is still a challenge [7].

For spectral element methods, it is shown that it requires fewer grid nodes per wavelength compared to the finite-element methods for the Helmholtz equation [4]. However, due to the less sparse resultant matrix compared to the resulting finite-element matrix, the computational time of both methods are almost the same [4].

For the traditional finite-difference methods, in order to increase the order of accuracy of approximation, the stencil of grid points needs to be enlarged, which is not desirable.

Generally, obtaining a more accurate numerical solution means adding more nodes and using smaller mesh sizes, which requires more computing time and storage space. In order to obtain more accurate results for constant mesh size, we need to increase the order of accuracy of the numerical approximation, which, in turn, means enlarging the stencil of grid points. This, however, leads to some problems including difficult treatments of the boundary conditions and approximation of the points next to the boundaries, and increasing the bandwidth of the stiffness matrix, which makes fast direct solver difficult. Therefore, compact finite difference schemes are desired to solve partial differential equation numerically.

A noticeable work in this field has been done by Turkel and Singer [1]. They developed a fourth-order compact finite-difference method using two schemes. One scheme was based on the generalization of the Padé approximation and the other used the Helmholtz equation to calculate higher-order correction terms. They implemented these schemes for Dirichlet and/or Neumann boundary conditions. In the present study we extended the previous work and developed a new scheme to increase the accuracy to the order of six without enlarging the stencil of grid points. We have developed and implemented a new 9-point stencil, sixth-order accurate compact finite-difference method for solving the Helmholtz equation in the one-dimensional and two-dimensional domain with Dirichlet and/or Neumann boundary conditions. Recently, Sutmann [9] has reported a sixth-order compact finite-difference scheme for Helmholtz equation. However, he implemented his scheme only with Dirichlet boundary conditions. As far as the authors know, the present study is the first that developed a sixth-order compact finite-difference scheme with the Neumann boundary conditions.

2. Nine-point sixth-order accurate compact finite-difference scheme

This method is based on a sixth-order accurate approximation to the derivative calculated from the Helmholtz equation. We developed the scheme for both one-dimensional and two-dimensional uniform Cartesian grids with grid spacing $\Delta x = \Delta y = h$. The notation δ_{x0} is used to denote the first-order central difference with respect to x , which is defined as

$$\delta_{x0}u_i = \frac{u_{i+1} - u_{i-1}}{2h}, \tag{1}$$

where $u_i = u(x_i)$. The standard second-order central difference is denoted by δ_x^2 and is defined as

$$\delta_x^2u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}. \tag{2}$$

Difference operators δ_{y0} and δ_y^2 are defined similarly.

2.1. One-dimensional case

In one-dimensional case, the Helmholtz equation becomes the following ordinary differential equation

$$u'' + k^2u = f(x) \quad \text{for } x \in [a, b]. \tag{3}$$

A sixth-order accurate finite-difference estimate of Eq. (2) is

$$\delta_x^2u_i = u''_i + \frac{h^2}{12}u_i^{(4)} + \frac{h^4}{360}u_i^{(6)} + O(h^6). \tag{4}$$

Both $O(h^2)$ and $O(h^4)$ terms are included in Eq. (4), because we want to approximate all of them in order to construct an $O(h^6)$ scheme. Applying δ_x^2 to $u_i^{(4)}$, we get

$$u_i^{(6)} = \delta_x^2 u_i^{(4)} + O(h^2). \tag{5}$$

Substituting Eq. (5) into Eq. (4) yields

$$\delta_x^2 u_i = u_i'' + \frac{h^2}{12} u_i^{(4)} + \frac{h^4}{360} (\delta_x^2 u_i^{(4)} + O(h^2)) + O(h^6). \tag{6}$$

To get a compact $O(h^6)$ approximation, we simply take the appropriate derivative of Eq. (3), that is

$$u_i^{(4)} = -k^2 u_i'' + f_i'', \tag{7}$$

where $f_i = f(x_i)$ and $f_i'' = f''(x_i)$. Inserting Eq. (7) into Eq. (6) will result in

$$\delta_x^2 u_i = u_i'' + \left(\frac{h^2}{12} + \frac{h^4}{360} \delta_x^2 \right) (-k^2 u_i'' + f_i'') + O(h^6). \tag{8}$$

Then, a compact (implicit) approximation for u_i'' with a sixth-order accuracy will be given as

$$u_i'' = \frac{\delta_x^2 u_i - \frac{h^2}{12} \left(1 + \frac{h^2}{30} \delta_x^2 \right) f_i''}{\left(1 - \frac{k^2 h^2}{12} \left(1 + \frac{h^2}{30} \delta_x^2 \right) \right)} + O(h^6). \tag{9}$$

Using this estimate and considering the discrete solution of Eq. (3) which satisfies the approximation, we get

$$\delta_x^2 U_i + k^2 \left(1 - \frac{k^2 h^2}{12} \left(1 + \frac{h^2}{30} \delta_x^2 \right) \right) U_i = \left(1 - \frac{k^2 h^2}{12} \left(1 + \frac{h^2}{30} \delta_x^2 \right) \right) f_i + \frac{h^2}{12} \left(1 + \frac{h^2}{30} \delta_x^2 \right) f_i'', \tag{10}$$

$$k^2 \left(1 - \frac{k^2 h^2}{12} \right) U_i + \left(1 - \frac{k^4 h^4}{360} \right) \delta_x^2 U_i = \left(1 - \frac{k^2 h^2}{12} \right) f_i - \frac{k^2 h^4}{360} \delta_x^2 f_i + \frac{h^2}{12} f_i'' + \frac{h^4}{360} \delta_x^2 f_i'', \tag{11}$$

where U_i is the discrete approximation to u_i satisfying the discrete formulation of Eq. (3) which implies, $u_i = U_i + O(h^6)$. Using Eq. (2) and $\delta_x^2 f_i'' = (f_{i+1}'' - 2f_i'' + f_{i-1}'')/h^2$, we can express the scheme in the following form:

$$d_{10} U_i + d_{11} (U_{i+1} + U_{i-1}) = b_{10} f_i + b_{11} (f_{i+1} + f_{i-1}) + b_{12} f_i'' + b_{13} (f_{i+1}'' + f_{i-1}''), \tag{12}$$

where

$$\begin{aligned} d_{10} &= -2 + k^2 h^2 \left(1 - \frac{28k^2 h^2}{360} \right), & d_{11} &= 1 - \frac{k^4 h^4}{360}, \\ b_{10} &= \left(1 - \frac{28k^2 h^2}{360} \right) h^2, & b_{11} &= \frac{-k^2 h^4}{360}, & b_{12} &= \frac{28h^4}{360}, & b_{13} &= \frac{h^4}{360}. \end{aligned} \tag{13}$$

Since f and f'' are known at every grid point, the right-hand side of Eq. (12) is known for all nodes. The system equations given by Eq. (12) can be written for each node and a resultant linear system of equations is obtained. In the cases where f is not known analytically, only a fourth-order accurate approximation of f'' is required, which can be obtained using $f_i'' = (-f_{i+1} + 16f_{i+1/2} - 30f_i + 16f_{i-1/2} - f_{i-1})/12h^2$.

2.2. Two-dimensional case

Consider the two-dimensional Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u(x, y) = f(x, y) \quad \text{for } x \in [a, b] \text{ and } y \in [c, d]. \tag{14}$$

We want to have a 9-point stencil which is symmetrical in both x and y directions. The central difference scheme for Eq. (14) in two dimensions can be written as

$$\delta_x^2 u_{ij} + \delta_y^2 u_{ij} + k^2 u_{ij} + T_{ij} = f_{ij}, \tag{15}$$

where $u_{ij} = u(x_i, y_j), f_{ij} = f(x_i, y_j)$ and

$$T_{ij} = -\frac{h^2}{12} \left[\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} \right]_{ij} - \frac{h^4}{360} \left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} \right]_{ij} + O(h^6). \tag{16}$$

Using the following appropriate derivatives of Eq. (14):

$$\frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 f}{\partial x^2} - k^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^2 \partial y^2}, \quad \frac{\partial^4 u}{\partial y^4} = \frac{\partial^2 f}{\partial y^2} - k^2 \frac{\partial^2 u}{\partial y^2} - \frac{\partial^4 u}{\partial y^2 \partial x^2} \tag{17}$$

in Eq. (16), we get

$$T_{ij} = -\frac{h^2}{12} \left(\nabla^2 f_{ij} - 2 \left[\frac{\partial^4 u}{\partial x^2 \partial y^2} \right]_{ij} - k^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]_{ij} \right) - \frac{h^4}{360} \left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} \right]_{ij} + O(h^6). \tag{18}$$

In our derivation of an $O(h^6)$ scheme, we need a fourth-order approximation of $\partial^4 u / \partial x^2 \partial y^2$ in Eq. (18) which can be written as

$$\left[\frac{\partial^4 u}{\partial x^2 \partial y^2} \right]_{ij} = \delta_x^2 \delta_y^2 u_{ij} - \frac{h^2}{12} \left[\frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4} \right]_{ij} + O(h^4). \tag{19}$$

Substituting Eq. (19) into Eq. (18), we get

$$T_{ij} = \frac{h^2}{12} \left(-\nabla^2 f_{ij} + 2\delta_x^2 \delta_y^2 u_{ij} + k^2 f_{ij} - k^4 u_{ij} \right) - \frac{h^4}{360} \left[\frac{\partial^6 u}{\partial x^6} + 5 \frac{\partial^6 u}{\partial x^4 \partial y^2} + 5 \frac{\partial^6 u}{\partial x^2 \partial y^4} + \frac{\partial^6 u}{\partial y^6} \right]_{ij} + O(h^6). \tag{20}$$

Clearly, getting a compact sixth-order approximation requires compact expressions of the four derivatives of order six in Eq. (20), which can be done by further differentiating Eq. (14), that is

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4} + k^2 \frac{\partial^4 u}{\partial x^2 \partial y^2}, \tag{21}$$

$$\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} = \nabla^4 f - k^2 \left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} \right) - \left(\frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4} \right). \tag{22}$$

Substituting Eqs. (17), (21) into Eq. (22) gives us

$$\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} = \nabla^4 f - \frac{\partial^4 f}{\partial x^2 \partial y^2} - k^2 \nabla^2 f + k^4 (-k^2 u + f) + 3k^2 \frac{\partial^4 u}{\partial x^2 \partial y^2}. \tag{23}$$

Using Eqs. (21), (23), we can eliminate all the derivatives of u in Eq. (20), that is

$$T_{ij} = \frac{h^2}{12} \left(-\nabla^2 f_{ij} + 2\delta_x^2 \delta_y^2 u_{ij} + k^2 f_{ij} - k^4 u_{ij} \right) - \frac{h^4}{360} \left(\nabla^4 f_{ij} + 4 \left[\frac{\partial^4 f}{\partial x^2 \partial y^2} \right]_{ij} - k^2 \nabla^2 f_{ij} + k^4 f_{ij} - k^6 u_{ij} - 2k^2 \delta_x^2 \delta_y^2 u_{ij} \right) + O(h^6). \tag{24}$$

The compact sixth-order approximation of the two-dimensional Helmholtz equation can thus be obtained as:

$$\begin{aligned} & \frac{h^2}{6} \left(1 + \frac{k^2 h^2}{30} \right) \delta_x^2 \delta_y^2 U_{ij} + (\delta_x^2 + \delta_y^2) U_{ij} + k^2 \left(1 - \frac{k^2 h^2}{12} + \frac{k^4 h^4}{360} \right) U_{ij} \\ & = \left(1 - \frac{k^2 h^2}{12} + \frac{k^4 h^4}{360} \right) f_{ij} + \left(\frac{h^2}{12} \left(1 - \frac{k^2 h^2}{30} \right) \right) \nabla^2 f_{ij} + \frac{h^4}{360} \nabla^4 f_{ij} + \frac{h^4}{90} \left[\frac{\partial^4 f}{\partial x^2 \partial y^2} \right]_{ij}, \end{aligned} \tag{25}$$

where $U_{i,j}$ is the discrete approximation to $u_{i,j}$ satisfying the discrete formulation of Eq. (14) which means $u_{i,j} = U_{i,j} + O(h^6)$. As it is seen, we can express the equation in the form of

$$d_{20}U_{i,j} + d_{21}D_1 + d_{22}D_2 = b_{20}f_{i,j} + b_{21}\nabla^2 f_{i,j} + b_{22}\nabla^4 f_{i,j} + b_{23}\left[\frac{\partial^4 f}{\partial x^2 \partial y^2}\right]_{i,j}, \tag{26}$$

where

$$\begin{aligned} D_1 &= U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}, \\ D_2 &= U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1} + U_{i-1,j-1}, \\ B_1 &= f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1}, \\ B_2 &= f_{i+1,j+1} + f_{i-1,j+1} + f_{i+1,j-1} + f_{i-1,j-1}, \end{aligned} \tag{27}$$

then we get

$$\begin{aligned} d_{20} &= -\frac{10}{3} + k^2 h^2 \left(\frac{46}{45} - \frac{k^2 h^2}{12} + \frac{k^4 h^4}{360} \right), & d_{21} &= \frac{2}{3} - \frac{k^2 h^2}{90}, & d_{22} &= \frac{1}{6} + \frac{k^2 h^2}{180}, \\ b_{20} &= h^2 \left(1 - \frac{k^2 h^2}{12} + \frac{k^4 h^4}{360} \right), & b_{21} &= \frac{h^4}{12} \left(1 - \frac{k^2 h^2}{30} \right), & b_{22} &= \frac{h^6}{360}, & b_{23} &= \frac{h^6}{90}. \end{aligned} \tag{28}$$

If we write the system equations given by Eq. (26) for each node, we can obtain the final linear system of equations. In the cases where f is not known analytically, only a fourth-order accurate approximation of $\nabla^2 f$ and a second-order accurate approximation of $\nabla^4 f$ and $\partial^4 f / \partial x^2 \partial y^2$ are required.

2.3. Sixth-order accurate approximation of the boundary

We like to approximate the boundary conditions with the same accuracy as the interior nodes. For the Dirichlet boundary condition, we can simply put the boundary for every node on the boundary. For the Neumann boundary condition, the sixth-order approximation is conducted for both one-dimensional and two-dimensional cases.

2.3.1. One-dimensional case

For a Neumann boundary condition in one dimension, we assume

$$u'(x_0) = \beta \quad (\beta \text{ a constant}). \tag{29}$$

The sixth-order approximation of Eq. (29) is

$$\delta_{x0}u_i = u'_i + \frac{h^2}{6}u'''_i + \frac{h^4}{120}u^{(5)}_i + O(h^6), \tag{30}$$

$$\delta_{x0}u_i = u'_i + \frac{h^2}{6}u'''_i + \frac{h^4}{120}(\delta_x^2 u'''_i + O(h^2)) + O(h^6). \tag{31}$$

Differentiating Eq. (3), we get

$$u'''_i = -k^2 u'_i + f'. \tag{32}$$

Using Eq. (32) in Eq. (31), we get

$$\delta_{x0}u_i = \left(1 - \frac{k^2 h^2}{6}\right)u'_i - \frac{k^2 h^4}{120}\delta_x^2 u'_i + \frac{h^2}{6}\left(1 + \frac{h^2}{20}\delta_x^2\right)f'_i + O(h^6). \tag{33}$$

Using $\delta_x^2 u'_i = u'''_i + O(h^2)$ and Eq. (32) in Eq. (33) gives us

$$\delta_{x0}u_i = \left(1 - \frac{k^2 h^2}{6} + \frac{k^4 h^4}{120}\right)u'_i + \frac{h^2}{6}\left(1 - \frac{h^2}{20}\right)f'_i + \frac{h^4}{120}\delta_x^2 f'_i + O(h^6). \tag{34}$$

Using $\delta_x^2 f'_i = (f'_{i+1} - 2f'_i + f'_{i-1})/h^2 + O(h^2)$ and Eq. (1) will result in

$$\frac{u_{i+1} - u_{i-1}}{2h} = \left(1 - \frac{k^2 h^2}{6} + \frac{k^4 h^4}{120}\right) u' + \frac{h^2}{6} \left(\frac{9}{10} - \frac{k^2 h^2}{20}\right) f'_i + \frac{h^2}{120} (f'_{i+1} + f'_{i-1}) + O(h^6). \tag{35}$$

Considering discrete formulation and using Eq. (29) for $i = 0$, we get

$$d_{11}(U_1 - U_{-1}) = 2h\beta d_{11} \left(1 - \frac{k^2 h^2}{6} + \frac{k^4 h^4}{120}\right) + \frac{h^3}{3} d_{11} \left(\frac{9}{10} - \frac{k^2 h^2}{20}\right) f'_0 + \frac{h^3}{60} d_{11} (f'_1 + f'_{-1}). \tag{36}$$

We wish to eliminate U_{-1} because we do not have equations at point x_{-1} . Using Eq. (12) for $i = 0$, we get:

$$d_{11}(U_1 + U_{-1}) + d_{10}U_0 = b_{10}f_0 + b_{11}(f_1 + f_{-1}) + b_{12}f''_0 + b_{13}(f''_1 + f''_{-1}). \tag{37}$$

We use Eqs. (36), (37) to eliminate U_{-1} and get the desired approximation for the boundary point x_0 , that is

$$2d_{11}U_1 + d_{10}U_0 = 2h\beta d_{11} \left(1 - \frac{k^2 h^2}{6} + \frac{k^4 h^4}{120}\right) + b_{10}f_0 + b_{11}(f_1 + f_{-1}) + \frac{h^3}{3} d_{11} \left(\frac{9}{10} - \frac{k^2 h^2}{20}\right) f'_0 + \frac{h^3}{60} d_{11} (f'_1 + f'_{-1}) + b_{12}f''_0 + b_{13}(f''_1 + f''_{-1}). \tag{38}$$

All parameters on the right-hand side of Eq. (38) are known. In the cases where f is not known analytically, only a fourth-order accurate approximation of f' is required.

2.3.2. Two-dimensional case

For a Neumann boundary condition in two dimensions, we assume

$$\left. \frac{\partial u}{\partial x} \right|_{x=0} = \beta(y). \tag{39}$$

The sixth-order approximation of Eq. (39) is

$$\delta_{x0} u_i = \left[\frac{\partial u}{\partial x} \right]_{ij} + \frac{h^2}{6} \left[\frac{\partial^3 u}{\partial x^3} \right]_{ij} + \frac{h^4}{120} \left[\frac{\partial^5 u}{\partial x^5} \right]_{ij} + O(h^6). \tag{40}$$

Using the following appropriate derivatives of Eq. (14), we get

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial f}{\partial x} - k^2 \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x \partial y^2}, \quad \frac{\partial^5 u}{\partial x^5} = \frac{\partial^3 f}{\partial x^3} - k^2 \frac{\partial^3 u}{\partial x^3} - \frac{\partial^5 u}{\partial x^3 \partial y^2}, \tag{41}$$

In our derivation of an $O(h^6)$ scheme, we need a fourth-order approximation of $(\partial^3 u / \partial x \partial y^2)$ in Eq. (41) which can be written as

$$\left[\frac{\partial^3 u}{\partial x \partial y^2} \right]_{ij} = \delta_x \delta_y^2 u_{ij} - \frac{h^2}{12} \left[\frac{\partial^5 u}{\partial x \partial y^4} + 2 \frac{\partial^5 u}{\partial x^3 \partial y^2} \right]_{ij} + O(h^4). \tag{42}$$

Substituting of Eq. (42) into Eq. (41) gives us

$$\left[\frac{\partial^3 u}{\partial x^3} \right]_{ij} = \left[\frac{\partial f}{\partial x} \right]_{ij} - k^2 \left[\frac{\partial u}{\partial x} \right]_{ij} - \delta_x \delta_y^2 u_{ij} + \frac{h^2}{12} \left[\frac{\partial^5 u}{\partial x \partial y^4} + 2 \frac{\partial^5 u}{\partial x^3 \partial y^2} \right]_{ij} + O(h^4). \tag{43}$$

The second-order approximation of $(\partial^3 u / \partial x^3)$ can be written as

$$\left[\frac{\partial^3 u}{\partial x^3} \right]_{ij} = \left[\frac{\partial f}{\partial x} \right]_{ij} - k^2 \left[\frac{\partial u}{\partial x} \right]_{ij} - \delta_x \delta_y^2 u_{ij} + O(h^2). \tag{44}$$

Using Eqs. (41), (44), the second-order approximation of $(\partial^5 u / \partial x^5)$ can be written as

$$\left[\frac{\partial^5 u}{\partial x^5} \right]_{ij} = \left[\frac{\partial^3 f}{\partial x^3} \right]_{ij} - k^2 \left[\frac{\partial f}{\partial x} \right]_{ij} + k^4 \left[\frac{\partial u}{\partial x} \right]_{ij} + k^2 \delta_x \delta_y^2 u_{ij} - \left[\frac{\partial^5 u}{\partial x^3 \partial y^2} \right]_{ij} + O(h^2). \tag{45}$$

The appropriate derivatives of Eq. (14) gives us:

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial^5 u}{\partial x^3 \partial y^2} + \frac{\partial^5 u}{\partial x \partial y^4} + k^2 \frac{\partial^3 u}{\partial x \partial y^2} \Rightarrow \frac{\partial^5 u}{\partial x^3 \partial y^2} + \frac{\partial^5 u}{\partial x \partial y^4} = \frac{\partial^3 f}{\partial x \partial y^2} - k^2 \frac{\partial^3 u}{\partial x \partial y^2}. \tag{46}$$

Substituting Eqs. (43), (45), (46) into Eq. (40), and after some modifications we get

$$\begin{aligned} \delta_{x0} u_i - \frac{k^4 h^4}{120} \left[\frac{\partial u}{\partial x} \right]_{i,j} + \frac{h^2}{6} \left(1 + \frac{k^2 h^2}{30} \right) \delta_x \delta_y^2 u_{i,j} \\ = \left(1 - \frac{k^2 h^2}{6} \right) \left[\frac{\partial u}{\partial x} \right]_{i,j} + \frac{h^2}{6} \left(1 - \frac{k^2 h^2}{20} \right) \left[\frac{\partial f}{\partial x} \right]_{i,j} + \frac{h^4}{120} \left[\frac{\partial^3 f}{\partial x^3} \right]_{i,j} + \frac{h^4}{72} \left[\frac{\partial^3 f}{\partial x \partial y^2} \right]_{i,j} + O(h^6). \end{aligned} \tag{47}$$

Now for $(\partial u / \partial x)_{i,j}$ on the right-hand side of Eq. (47), we use this approximation

$$[\partial u / \partial x]_{i,j} = \delta_{x0} u_i + \mu h^2 \delta_x \delta_y^2 u_{i,j}, \tag{48}$$

where μ is an arbitrary number. Using Eq. (48) in Eq. (47), multiplying by $2h$ and setting $i = 0$ will result in

$$\begin{aligned} \hat{d}_{21} (U_{1,j} - U_{-1,j}) + \hat{d}_{22} (U_{1,j+1} + U_{1,j-1} - U_{-1,j+1} - U_{-1,j-1}) \\ = \left(1 - \frac{k^2 h^2}{6} \right) \beta_j + \frac{h^2}{6} \left(1 - \frac{k^2 h^2}{20} \right) \left[\frac{\partial f}{\partial x} \right]_{0,j} + \frac{h^4}{120} \left[\frac{\partial^3 f}{\partial x^3} \right]_{0,j} + \frac{h^4}{72} \left[\frac{\partial^3 f}{\partial x \partial y^2} \right]_{0,j}, \end{aligned} \tag{49}$$

where $\beta_j = \beta(y_j)$ and

$$\hat{d}_{21} = 1 - \frac{k^4 h^4}{120} (1 - 2\mu) - \frac{1}{6h} \left(1 + \frac{k^2 h^2}{30} \right), \quad \hat{d}_{22} = -\frac{k^4 h^4}{120} \mu + \frac{1}{12h} \left(1 + \frac{k^2 h^2}{30} \right). \tag{50}$$

Setting $i = 0$ in Eq. (26), we get

$$\begin{aligned} d_{20} U_{0,j} + d_{21} (U_{1,j} + U_{-1,j} + U_{0,j+1} + U_{0,j-1}) + d_{22} (U_{1,j+1} + U_{-1,j+1} + U_{1,j-1} + U_{-1,j-1}) \\ = b_{20} f_{0,j} + b_{21} \nabla^2 f_{0,j} + b_{22} \nabla^4 f_{0,j} + b_{23} \left[\frac{\partial^4 f}{\partial x^2 \partial y^2} \right]_{0,j}. \end{aligned} \tag{51}$$

In order to eliminate all elements with $i = -1$, we define a constant η such that $\eta = d_{21} / \hat{d}_{21} = d_{22} / \hat{d}_{22}$, which can be obtained by

$$\eta = \frac{1}{1 - \frac{k^4 h^4}{120}}. \tag{52}$$

If we multiply Eq. (49) with η and add to Eq. (51) we will get the final formula for boundary nodes,

$$\begin{aligned} d_{20} U_{0,j} + d_{21} (2U_{1,j} + U_{0,j+1} + U_{0,j-1}) + d_{22} (U_{1,j+1} + U_{1,j-1}) \\ = b_{20} f_{0,j} + b_{21} \nabla^2 f_{0,j} + b_{22} \nabla^4 f_{0,j} + b_{23} \left[\frac{\partial^4 f}{\partial x^2 \partial y^2} \right]_{0,j} \\ + \eta \left(\left(1 - \frac{k^2 h^2}{6} \right) \beta_j + \frac{h^2}{6} \left(1 - \frac{k^2 h^2}{20} \right) \left[\frac{\partial f}{\partial x} \right]_{0,j} + \frac{h^4}{120} \left[\frac{\partial^3 f}{\partial x^3} \right]_{0,j} + \frac{h^4}{72} \left[\frac{\partial^3 f}{\partial x \partial y^2} \right]_{0,j} \right). \end{aligned} \tag{53}$$

All parameters on the right-hand side of Eq. (53) are known. In the cases where f is not known analytically, only a fourth-order accurate approximation of $\partial f / \partial x$ and a second-order approximation of $\partial^3 f / \partial x^3$ and $\partial^3 f / \partial x \partial y^2$ are required.

3. Numerical results

In order to validate our sixth-order accurate scheme and examine its behavior, we developed the scheme on two-model problems in two dimensions. In *problem A*, we solved

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u(x, y) = (k^2 - 2\pi^2) \sin(\pi x) \sin(\pi y) \quad 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1, \tag{54}$$

with the pure Dirichlet boundary conditions on all sides of a unit square, that is

$$u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0, \tag{55}$$

and in problem B, we solved

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u(x, y) = (k^2 - 2\pi^2) \cos(\pi x) \sin(\pi y) \quad 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1, \tag{56}$$

with the Neumann boundary condition on the left side of a unit square and Dirichlet boundary conditions on the remaining three sides, that is

$$u_x(0, y) = 0 \quad u(1, y) = -\sin(\pi y) \quad u(x, 0) = u(x, 1) = 0. \tag{57}$$

The exact solutions for problems A and B are $u(x, y) = \sin(\pi x) \sin(\pi y)$ and $u(x, y) = \cos(\pi x) \sin(\pi y)$, respectively.

The next step is to solve the resultant linear set of equations. We used LU-decomposition by Gaussian elimination with pivoting for each set of equations to find values of u at $N \times N$ nodes. The code was written in MATLAB environment using version 7.1. The code was executed on a Pentium 4, 3 Ghz PC.

In order to compare the numerical solution $U_{i,j}$ to the exact solution $u_{i,j}$, we use two performance metrics namely l_2 -norm and order. The metric l_2 -norm of the error vector e , is defined a

$$\|e\|_2 = \frac{1}{N} \sqrt{\sum_{i,j=0}^N e_{i,j}^2}, \tag{58}$$

where $e_{i,j} = u_{i,j} - U_{i,j}$ and N is the number of nodes. The metric order is defined as

$$\text{Order}(n, n + 1) = \log_2 \frac{\|e\|_\infty(n)}{\|e\|_\infty(n + 1)}, \tag{59}$$

and measures the order of accuracy of numerical solutions. $\|e\|_\infty$ in Eq. (59) is called l_∞ -norm of the error vector and is defined as

$$\|e\|_\infty = \max_{0 \leq i,j \leq N} e_{i,j} \tag{60}$$

The l_2 -norm of the error and the order of accuracy, for different values of N and for $k = 10$, are presented in Tables 1 and 2 for problems A and B, respectively. It is clearly seen that the norm of the error behaves like the order of the scheme. As we multiply N by two, the error decreases by $2^6 = 64$. It means that our scheme has really the accuracy of order six. This trend can also be seen in Figs. 1 and 2, where the $\log_2 \|e\|_\infty$ is plotted versus $\log_2 N$ for both problems A and B, respectively. The slope of the line is -6 which means that the order of accuracy is 6.

We also examined the behavior of our scheme for different values of k . Figs. 3 and 4 show $\log_2 \|e\|_2$ versus k with three different value of N for both problems A and B, respectively. Fig. 3 shows that for problem A, except for $4 \leq k \leq 5$ in which the scheme is more sensitive to the value of k , the method behaves robustly with respect to the wave number. Fig. 4 shows that compared to problem A, the scheme for problem B is more sensitive to the value of k . However, for any given value of N , the overall error does not increase with the value

Table 1
Numerical results for the problem A, $k = 10$

N	$\ e\ _2$	$\ e\ _\infty$	Order	Execution time of the proposed scheme (s)	Execution time of the 4-order FEM (s)
8	1.8561E - 7	3.7586E - 6		0.021	0.84
16	1.4556E - 9	5.2585E - 8	6.16	0.031	3.66
32	1.1750E - 11	7.9975E - 10	6.04	0.094	15.86
64	9.4278E - 14	1.2433E - 11	6.00	0.50	84.14
128	9.5151E - 016	1.9691E - 013	5.98	4.18	502.84

Table 2
Numerical results for the problem B, $k = 10$

N	$\ e\ _2$	$\ e\ _\infty$	Order	Execution time of the proposed scheme (s)	Execution time of the 4-order FEM (s)
8	$2.611E - 7$	$3.368E - 6$		0.021	0.86
16	$2.048E - 9$	$8.112E - 8$	6.05	0.031	3.71
32	$1.653E - 11$	$1.2337E - 9$	6.04	0.11	17.47
64	$1.3264E - 13$	$1.9248E - 11$	6.00	0.58	95.11
128	$1.0560E - 015$	$3.0507E - 013$	5.98	4.42	541.15

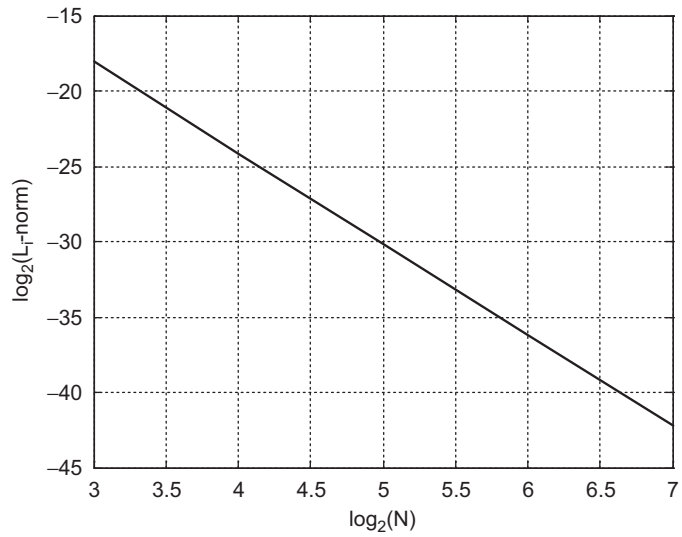


Fig. 1. $\log_2\|e\|_\infty$ versus $\log_2 N$ for problem A.

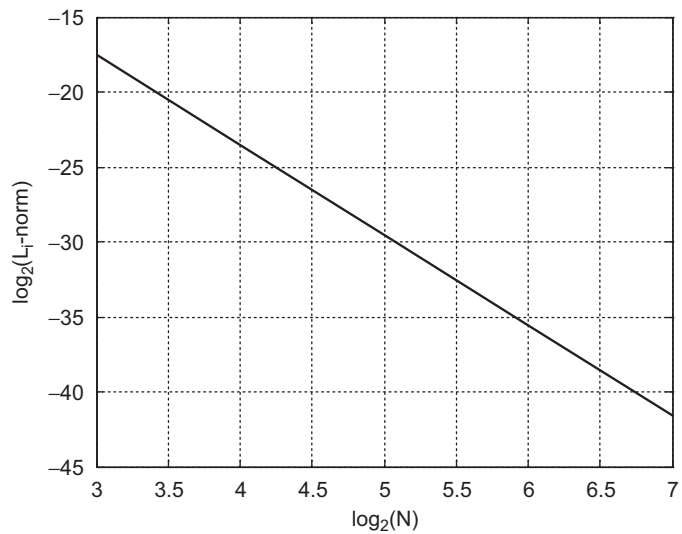


Fig. 2. $\log_2\|e\|_\infty$ versus $\log_2 N$ for problem B.

of k . As the figures show, at some particular values of k in both problems, the accuracy of the method is poor. This fact can be explained through eigenvalue analysis. The approximate eigenvalues for problem A are given as, $\lambda_{Am,n} = k^2 - \pi^2(m^2 + n^2)$ and for problem B, $\lambda_{Bm,n} = k^2 - \pi^2((m + 1/2)^2 + n^2)$ (see Ref. [1]). For example, near $k = 4.44$ where $\lambda_{A1,1} \rightarrow 0$, problem A is unstable and near $k = 5.663$ where $\lambda_{B1,1} \rightarrow 0$, problem B is unstable and the scheme has poor accuracy.

One of the important advantages of the proposed scheme is that in comparison with the finite-element, boundary-element or spectral-element methods, this method is very fast. A quantitative comparison in terms of the execution time between the present scheme and fourth-order accurate finite-element method (FEM) is presented in the last two columns of Tables 1 and 2. The results show that for large number of nodes, the present scheme is more that 100 times faster than the fourth-order finite-element method.

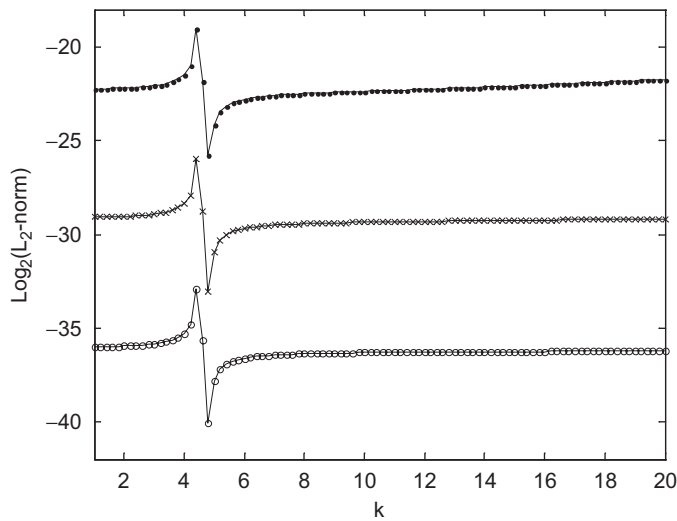


Fig. 3. $\log_2\|e\|_2$ versus k for problem A; ●, $N = 8$; ×, $N = 16$; ○, $N = 32$; k varies in units of 0.2.

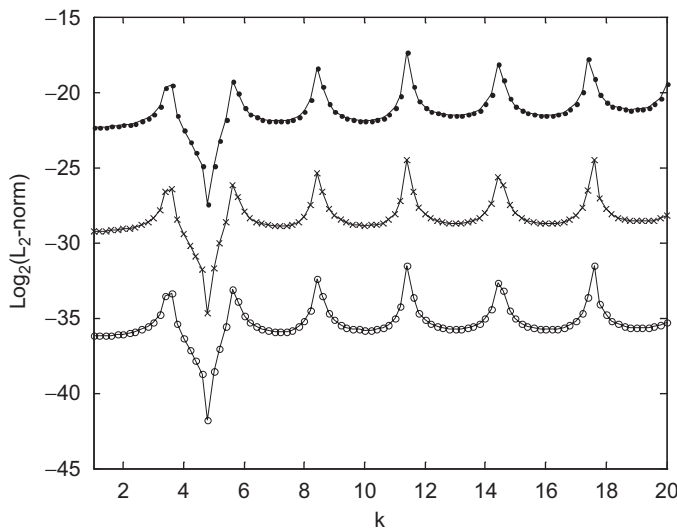


Fig. 4. $\log_2\|e\|_2$ versus k for problem B; ●, $N = 8$; ×, $N = 16$; ○, $N = 32$; k varies in units of 0.2.

4. Conclusions

A new 9-point sixth-order accurate compact finite difference scheme for the Helmholtz equation was developed. A sixth-order accurate symmetrical representation for the Neumann boundary condition was also developed. Numerical results show that our scheme has the expected accuracy and is more than 100 times faster than the fourth-order finite-element method. The results also show that the overall error does not increase with an increase in the wave number.

Acknowledgements

This research is funded by the grants from Natural Science and Engineering Research Council of Canada (NSERC) and Concordia University.

References

- [1] I. Singer, E. Turkel, High-order finite difference method for the helmholtz equation, *Computer Methods in Applied Mechanics and Engineering* 163 (1998) 343–358.
- [2] R.P. Shaw, Integral equation methods in acoustics, *Boundary Elements* 4 (1988) 221–244.
- [3] I. Harari, T.J.R. Hughes, Finite element methods for the helmholtz equation in an exterior domain: model problem, *Computer Methods in Applied Mechanics and Engineering* 87 (1991) 59–96.
- [4] O. Mehdizadeh, M. Paraschivoiu, Investigation of a two-dimensional spectral method for helmholtz's equation, *Journal of Computational Physics* 189 (2003) 111–129.
- [5] F. Ihlenburg, I. Babuska, Finite-element solution to the Helmholtz equation with high wavenumber part I: the h-version of the FEM, *Computers and Mathematics with Applications* 30 (1995) 9–37.
- [6] F. Ihlenburg, I. Babuska, Finite-element solution of the Helmholtz equation with high wave number part II: the h-p-version of the FEM, *Computers and Mathematics with Applications* 34 (1997) 315–358.
- [7] I. Harari, C.L. Nogueira, Reducing dispersion of linear triangular elements for the helmholtz equation, *Journal of Engineering Mechanics* 128 (2002) 351–358.
- [8] A. Oberai, P. Pinsky, A numerical comparison of finite-element methods for the helmholtz equation, *Journal of Computational Acoustics* 8 (2000) 211–221.
- [9] G. Sutmann, Compact finite difference schemes of sixth order for the helmholtz equation, *Journal of Computational and Applied Mathematics* 203 (2007) 15–31.