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The exterior matrix method for sequentially coupled fourth-order equations

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Abstract

This paper introduces the exterior matrix method as a means to compute asymptotically the eigenfrequencies for the vibrations of serially connected elements which are each governed by fourth-order equations. The Timoshenko beams and a tapered Euler–Bernoulli beams are just two examples that are solved by this method. The technique involves not just lifting the transfer matrices into the higher dimensional exterior algebra space, but also lifting the original fourth-order equation to generally produce a sixth-order exterior equation, but occasionally the exterior equation is only fifth order. The original fourth-order equation does not have to be solvable for this technique to work, since only the approximate solutions for the exterior equation are needed to compute the eigenfrequencies asymptotically.

1. Introduction

Sequentially coupled partial differential equations frequently arise in many different problems, such as coupled Euler–Bernoulli beams or multi-span cable problems, to name a few. Solving these coupled PDEs usually, but not always, amounts to finding the eigenfrequencies and eigenfunctions for the system. Even in the cases where eigenfrequencies exist, yet the eigenfunctions do not span the solution set, (as in the case $k = 1$ in Ref. [1]), the eigenfrequencies can be used to determine the stability of the system, i.e., if the eigenfrequencies are a bounded distance from the imaginary axis. The most straightforward method used is the transfer matrix method [2,3], where for each component in the structure, the vibrations on one end are related to the vibrations of the other via a matrix. The matrices are then multiplied together to show how the vibrations on the beginning of the structure propagate to the other side. But the product of these matrices can become unwieldy even for a structure with just a handful of components.

Another popular method for studying these complex structures is the Rayleigh–Ritz method [4–6], in which trial functions are produced for each piece of the structure, and then the pieces fit together to produce the vibrations of the whole structure. However, if the joints between the pieces are rigid, choosing the trial functions so that the continuity condition is satisfied becomes very difficult [7]. One solution is to add artificial

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springs between the rigid joints, and then the frequencies of the rigid structure is found by taking the limit as the stiffness of the springs goes to infinity [4,5,7].

The wave propagation method (WPM) is also sometimes used [8], but it still needs to be made mathematically rigorous. Without the mathematical rigor, WPM is still an *ad hoc* method. On the other hand, the finite element method can be used to find the eigenfrequencies, but the asymptotic analysis is often very difficult to obtain without high speed computers.

This paper introduces the exterior matrix method (EMM) as it is applied to the general fourth-order equation, with improved proofs. The exterior matrix method has been used several times before, as in the Euler–Bernoulli beam equation [9,10]. The special properties of the exterior matrices were thought to be a bi-product of the symmetry of the equation. When the same symmetries appeared in studying the dynamics of the nonlinear semi-taught inclined cable equation [11], it became clear that the methods can in fact be applied to *any* fourth-order differential equation. This mathematically rigorous method reproduces the results of the WPM method as a special case, and can even apply to equations for which no exact solution exists. Although the proofs involve rather complicated computations using *Mathematica* [12], it should be noted that the application of this method is not that complicated, as seen in the example of tapered Euler–Bernoulli beams.

Although the main purpose of this paper is to present the exterior matrix method, we will focus on a particular problem as shown in Fig. 1. A beam is composed of 3 segments, the middle of which is a standard Euler–Bernoulli beam, but the other two are tapered. The eigenfunctions of a tapered Euler–Bernoulli beam involve Bessel functions [13], yet with the exterior matrix method, we can find the eigenfrequencies to any degree of accuracy without using Bessel functions. In fact, the governing equation for the vibrations does not even have to have a closed form solution. It should be possible to expand this method to determine whether the spectrum-determined growth condition holds [14], which is a difficult problem in PDEs. The exterior matrix method can be used in tandem with any method that uses transfer matrices, so for example the artificial spring method would benefit from this new methodology [2].

A fourth-order sequentially coupled PDE is a system of partial differential equations for

$$u_i(x,t) \quad 0 \leqslant x \leqslant L_i, \quad t > 0, \quad i = 1, 2, \ldots, m \tag{1}$$

for which the boundary conditions of one function is tied to the boundary conditions of the next

$$\begin{pmatrix} u_i(0,t) \\ u_i'(0,t) \\ u_i''(0,t) \\ u_i'''(0,t) \end{pmatrix} = C_i \begin{pmatrix} u_{i-1}(L_{i-1},t) \\ u_{i-1}'(L_{i-1},t) \\ u_{i-1}''(L_{i-1},t) \\ u_{i-1}'''(L_{i-1},t) \end{pmatrix} \tag{2}$$

with C_i being a 4×4 matrix, which could represent, among other things, a damper, a bend, or a point mass. There would also be two boundary conditions at each end. It should be pointed out that each of the $u_i(x,t)$ can satisfy a *different* PDE. For example, we could have an Euler–Bernoulli beam suspended by two inclined cables. The only restriction is that each of the PDEs be a fourth-order equation in the x variable.

The classical method for solving such a system is to assume the solution is of the form

$$u_i(x,t) = \sum_{\lambda} y_{i,\lambda}(x) \mathrm{e}^{\lambda t}, \tag{3}$$

where the sum is taken over all of the eigenfrequencies λ of the system. Technically, though, such a solution is only guaranteed for Riesz spectral systems, or systems with self-adjoint or skew self-adjoint system operators.

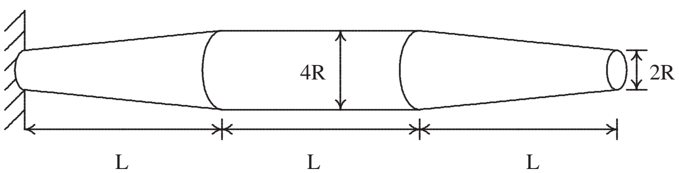

Fig. 1.

In these cases, each $y_{i,\lambda}$ will solve an ODE

$$y_i^{\text{iv}} + p(x,\lambda)y_i''' + q(x,\lambda)y_i'' + r(x,\lambda)y_i' + s(x,\lambda)y_i = 0 \tag{4}$$

formed by plugging Eq. (3) into the corresponding PDE. The main problem is finding the eigenfrequencies λ. Even if we can find the general solution to each of the ODEs,

$$y_{i,\lambda} = c_{i,1}y_{i,1}(x,\lambda) + c_{i,2}y_{i,2}(x,\lambda) + c_{i,3}y_{i,3}(x,\lambda) + c_{i,4}y_{i,4}(x,\lambda) \tag{5}$$

we would still have $4m$ equations with $4m$ unknowns, involving transcendental functions of λ. The eigenfrequencies would be the values for which there is a non-trivial solution, which would amount to finding the determinant of a $4m \times 4m$ matrix [15].

It should be mentioned that there is another issue here: once we have the eigenfrequencies and their corresponding eigenfunctions, do these eigenfunctions form an unconditional basis for the state Hilbert space? If so, the eigenfunctions are said to have the Riesz basis property [16]. Unfortunately, showing that a given linear system has the Riesz basis property is extremely challenging. For example, the second-order string vibration system

$$u_{tt}(x,t) = u_{xx}(x,t), \quad u(0,t) = 0, \quad u_x(1,t) = -u_t(1,t) \tag{6a–c}$$

fails to have the Riesz basis property, because there are no eigenfrequencies. Luckily, the system in Fig. 1 can be considered as a special case of the systems studied in Ref. [16], where the Riesz basis property was proven. In any case, finding the eigenfrequencies goes a long way toward studying the coupled system.

The transfer matrix method can eliminate the need to use large matrices. The main idea behind this method is to choose the four linearly independent solutions in Eq. (5) such that

$$\begin{aligned}
&y_{i,1}(0,\lambda) = 1, \quad y_{i,1}'(0,\lambda) = 0, \quad y_{i,1}''(0,\lambda) = 0, \quad y_{i,1}'''(0,\lambda) = 0,\\
&y_{i,2}(0,\lambda) = 0, \quad y_{i,2}'(0,\lambda) = 1, \quad y_{i,2}''(0,\lambda) = 0, \quad y_{i,2}'''(0,\lambda) = 0,\\
&y_{i,3}(0,\lambda) = 0, \quad y_{i,3}'(0,\lambda) = 0, \quad y_{i,3}''(0,\lambda) = 1, \quad y_{i,3}'''(0,\lambda) = 0,\\
&y_{i,4}(0,\lambda) = 0, \quad y_{i,4}'(0,\lambda) = 0, \quad y_{i,4}''(0,\lambda) = 0, \quad y_{i,4}'''(0,\lambda) = 1.
\end{aligned} \tag{7a–p}$$

We then form the matrix

$$W_i = \begin{pmatrix} y_{i,1}(L_i,\lambda) & y_{i,2}(L_i,\lambda) & y_{i,3}(L_i,\lambda) & y_{i,4}(L_i,\lambda) \\ y_{i,1}'(L_i,\lambda) & y_{i,2}'(L_i,\lambda) & y_{i,3}'(L_i,\lambda) & y_{i,4}'(L_i,\lambda) \\ y_{i,1}''(L_i,\lambda) & y_{i,2}''(L_i,\lambda) & y_{i,3}''(L_i,\lambda) & y_{i,4}''(L_i,\lambda) \\ y_{i,1}'''(L_i,\lambda) & y_{i,2}'''(L_i,\lambda) & y_{i,3}'''(L_i,\lambda) & y_{i,4}'''(L_i,\lambda) \end{pmatrix}, \tag{8}$$

so that

$$\begin{pmatrix} y_{i,\lambda}(L_i) \\ y_{i,\lambda}'(L_i) \\ y_{i,\lambda}''(L_i) \\ y_{i,\lambda}'''(L_i) \end{pmatrix} = W_i \begin{pmatrix} y_{i,\lambda}(0) \\ y_{i,\lambda}'(0) \\ y_{i,\lambda}''(0) \\ y_{i,\lambda}'''(0) \end{pmatrix}. \tag{9}$$

Using Eqs. (2) and (9), we see that

$$\begin{pmatrix} y_{m,\lambda}(L_m) \\ y_{m,\lambda}'(L_m) \\ y_{m,\lambda}''(L_m) \\ y_{m,\lambda}'''(L_m) \end{pmatrix} = W_m C_m W_{m-1} C_{m-1} \cdots C_2 W_1 \begin{pmatrix} y_{1,\lambda}(0) \\ y_{1,\lambda}'(0) \\ y_{1,\lambda}''(0) \\ y_{1,\lambda}'''(0) \end{pmatrix}. \tag{10}$$

Now, we can form a 4×2 matrix B from the two beginning boundary conditions, and a 2×4 matrix F using the boundary conditions at the final end, so that the eigenfrequencies are the solutions to the equation

$$\det(F W_m C_m W_{m-1} C_{m-1} \cdots C_2 W_1 B) = 0. \tag{11}$$

For example, if both ends are clamped, we have $y_{1,\lambda}(0) = y'_{1,\lambda}(0) = 0$, and $y_{m,\lambda}(L_m) = y'_{m,\lambda}(L_m) = 0$. This can be expressed by

$$B^{\text{clamped}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F^{\text{clamped}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{12a,b}$$

Likewise, if there is a free end on either side, the second and third derivatives would be zero, so we would use either

$$B^{\text{free}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad F^{\text{free}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{13a,b}$$

Even though we have a way of finding the eigenfrequencies, Eq. (11) poses some serious problems numerically. The final determinant typically causes the large order terms to cancel, leaving the smaller order terms behind. This means that calculating Eq. (11) via a decimal approximations would be unreliable. Calculating Eq. (11) symbolically eliminates this problem, but the number of terms that cancel increases with m, so even programs such as *Mathematica* would get bogged down if m is large. What we need is a way to take the determinant *before* the matrices are multiplied together, so that the major cancellation occurs first. The exterior matrices provide us with this ability.

2. Defining the exterior matrices

Given a 4×4 matrix M, we can define a linear map M^* sending 4×4 matrices to 4×4 matrices, given by

$$M^*(A) = MAM^{\mathrm{T}}. \tag{14}$$

This forms a homomorphism, since

$$M_1^*(M_2^*(A)) = M_1^*(M_2AM_2^{\mathrm{T}}) = M_1M_2AM_2^{\mathrm{T}}M_1^{\mathrm{T}} = (M_1M_2)A(M_1M_2)^{\mathrm{T}} = (M_1M_2)^*(A). \tag{15}$$

Furthermore, M^* sends anti-symmetric matrices to anti-symmetric matrices, so we can restrict M^* to this subspace. Since M^* is a linear transformation from anti-symmetric matrices to themselves, M^* can in turn be represented by a matrix. A basis for the 4×4 anti-symmetric matrices is

$$e_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{16a–c}$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{16d–f}$$

Note that e_5 breaks the pattern, but the reason for this will be clear later. Using this basis, we find that M^*, when restricted to anti-symmetric matrices, can be expressed by the 6×6 matrix given by

$$\begin{pmatrix}
\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{13} \\ m_{21} & m_{23} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{14} \\ m_{21} & m_{24} \end{vmatrix} & \begin{vmatrix} m_{13} & m_{14} \\ m_{23} & m_{24} \end{vmatrix} & -\begin{vmatrix} m_{12} & m_{14} \\ m_{22} & m_{24} \end{vmatrix} & \begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix} \\
\begin{vmatrix} m_{11} & m_{12} \\ m_{31} & m_{32} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{14} \\ m_{31} & m_{34} \end{vmatrix} & \begin{vmatrix} m_{13} & m_{14} \\ m_{33} & m_{34} \end{vmatrix} & -\begin{vmatrix} m_{12} & m_{14} \\ m_{32} & m_{34} \end{vmatrix} & \begin{vmatrix} m_{12} & m_{13} \\ m_{32} & m_{33} \end{vmatrix} \\
\begin{vmatrix} m_{11} & m_{12} \\ m_{41} & m_{42} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{13} \\ m_{41} & m_{43} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{14} \\ m_{41} & m_{44} \end{vmatrix} & \begin{vmatrix} m_{13} & m_{14} \\ m_{43} & m_{44} \end{vmatrix} & -\begin{vmatrix} m_{12} & m_{14} \\ m_{42} & m_{44} \end{vmatrix} & \begin{vmatrix} m_{12} & m_{13} \\ m_{42} & m_{43} \end{vmatrix} \\
\begin{vmatrix} m_{31} & m_{32} \\ m_{41} & m_{42} \end{vmatrix} & \begin{vmatrix} m_{31} & m_{33} \\ m_{41} & m_{43} \end{vmatrix} & \begin{vmatrix} m_{31} & m_{34} \\ m_{41} & m_{44} \end{vmatrix} & \begin{vmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{vmatrix} & -\begin{vmatrix} m_{32} & m_{34} \\ m_{42} & m_{44} \end{vmatrix} & \begin{vmatrix} m_{32} & m_{33} \\ m_{42} & m_{43} \end{vmatrix} \\
-\begin{vmatrix} m_{21} & m_{22} \\ m_{41} & m_{42} \end{vmatrix} & -\begin{vmatrix} m_{21} & m_{23} \\ m_{41} & m_{43} \end{vmatrix} & -\begin{vmatrix} m_{21} & m_{24} \\ m_{41} & m_{44} \end{vmatrix} & -\begin{vmatrix} m_{23} & m_{24} \\ m_{43} & m_{44} \end{vmatrix} & \begin{vmatrix} m_{22} & m_{24} \\ m_{42} & m_{44} \end{vmatrix} & -\begin{vmatrix} m_{22} & m_{23} \\ m_{42} & m_{43} \end{vmatrix} \\
\begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix} & \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} & \begin{vmatrix} m_{21} & m_{24} \\ m_{31} & m_{34} \end{vmatrix} & \begin{vmatrix} m_{23} & m_{24} \\ m_{33} & m_{34} \end{vmatrix} & -\begin{vmatrix} m_{22} & m_{24} \\ m_{32} & m_{34} \end{vmatrix} & \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix}
\end{pmatrix}. \tag{17}$$

Note that the break in the pattern for e_5 caused some minus signs to appear in this matrix. Otherwise, we are finding all possible determinants of 2×2 submatrices of M. The idea behind forming this matrix is that, by taking all possible 2×2 determinants *before* we multiply the matrices together, we would not have to take a 2×2 determinant at the end.

This idea actually works for larger square matrices of even order, but we have to use alternating covariant tensors, or *exterior forms*, instead of the anti-symmetric matrices [17, p. 202]. For example, if M is a 6×6 matrix, we can form the 20×20 matrix, made up of all possible determinants of 3×3 submatrices. These 400 determinants are arranged according to an appropriate basis for the three dimensional exterior forms on $\mathbb{R}^6$.

Because the matrix in Eq. (17) is derived using a basis of the exterior forms, it is natural to call this matrix the *exterior matrix* of M, and denote it by ext(M).

Because we want to make the exterior matrix method mathematically rigorous, we will need some introductory proofs. The homomorphism $M \to M^*$ gives us the first important property of exterior matrices.

Lemma 1. *If M and N are 4×4 matrices, then*

$$\text{ext}(MN) = \text{ext}(M) \cdot \text{ext}(N). \tag{18}$$

We can now show how the exterior matrices can help us in solving Eq. (11). We begin by relabeling the matrices in Eq. (11). Let $n-1$ be the number of 4×4 matrices, regardless or whether they are coupling matrices or wave matrices. We can then rewrite Eq. (11) as

$$\det(FM_{n-1}M_{n-2}\cdots M_2M_1B) = 0, \tag{19}$$

where the M_j is the jth matrix from the right in Eq. (11). To avoid confusion, we will index the M_j matrices with j instead of i. Thus, if there is a coupling matrix C_i between each W_i as in Eq. (11), then $n = 2m$, $W_i = M_{2i-1}$, and $C_i = M_{2i-2}$.

Proposition 1. *If F is a 2×4 matrix, and B is a 4×2 matrix, and M_j are all 4×4 matrices for $1 \leqslant j \leqslant n-1$, then*

$$\det(FM_{n-1}M_{n-2}\cdots M_2M_1B) = P_nP_{n-1}P_{n-2}\cdots P_1P_0, \tag{20}$$

where

$$P_n = \left(\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \quad \begin{vmatrix} f_{11} & f_{13} \\ f_{21} & f_{23} \end{vmatrix} \quad \begin{vmatrix} f_{11} & f_{14} \\ f_{21} & f_{24} \end{vmatrix} \quad \begin{vmatrix} f_{13} & f_{14} \\ f_{23} & f_{24} \end{vmatrix} \quad -\begin{vmatrix} f_{12} & f_{14} \\ f_{22} & f_{24} \end{vmatrix} \quad \begin{vmatrix} f_{12} & f_{13} \\ f_{22} & f_{23} \end{vmatrix} \right), \tag{21a}$$

$$P_0 = \left(\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \quad \begin{vmatrix} b_{11} & b_{12} \\ b_{31} & b_{32} \end{vmatrix} \quad \begin{vmatrix} b_{11} & b_{12} \\ b_{41} & b_{42} \end{vmatrix} \quad \begin{vmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{vmatrix} \quad -\begin{vmatrix} b_{21} & b_{22} \\ b_{41} & b_{42} \end{vmatrix} \quad \begin{vmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{vmatrix} \right)^{\mathrm{T}} \tag{21b}$$

and $P_j = \mathrm{ext}(M_j)$ *for* $1 \leqslant j \leqslant n-1$.

For example, if both ends are clamped, we would use

$$P_n^{\text{clamped}} = (1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0) \quad \text{and} \quad P_0^{\text{clamped}} = (0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0)^{\mathrm{T}}. \tag{22a,b}$$

If one end was free, we would instead use

$$P_n^{\text{free}} = (0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0) \quad \text{or} \quad P_0^{\text{free}} = (1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)^{\mathrm{T}}. \tag{23a,b}$$

It should be noted that the proposition would still be true if all of the negative signs in P_0, P_n, and P_j were removed, but the presence of the negative signs adds an additional property to the P_j matrix which we will see later. Before we can prove this proposition, we first need to prove the following lemma.

Lemma 2. *If F is a $2 \times n$ matrix, and B is an $n \times 2$ matrix, and we let A be the $n \times n$ matrix defined by $a_{ij} = b_{i1}b_{j2} - b_{i2}b_{j1}$, then*

$$FAF^{\mathrm{T}} = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}, \tag{24}$$

where $d = \det(FB)$.

Proof. Since, by the definition, A is anti-symmetric, it is clear that FAF^{T} will be anti-symmetric. Thus, we only need to proof that the element in the first row and second column of FAF^{T} is d. But this element can be expressed as

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{n} f_{1i}a_{ij}f_{2j} &= \sum_{i=1}^{n}\sum_{j=1}^{n} f_{1i}(b_{i1}b_{j2} - b_{i2}b_{j1})f_{2j} \\ &= \left(\sum_{i=1}^{n} f_{1i}b_{i1}\right)\left(\sum_{j=1}^{n} f_{2j}b_{j2}\right) - \left(\sum_{i=1}^{n} f_{1i}b_{i2}\right)\left(\sum_{j=1}^{n} f_{2j}b_{j1}\right) = d. \qquad \square \end{aligned} \tag{25}$$

This lemma can be extended to allow F to be an $m \times n$ matrix, and B will be an $n \times m$ matrix, except we will have to use m dimensional covariant tensors instead of matrices.

Proof of Proposition 1. We already saw that for each 4×4 matrix M induces a linear map M^* sending 4×4 matrices to 4×4 matrices, given by

$$M^*(A) = MAM^{\mathrm{T}}. \tag{26}$$

Furthermore, we can define the linear map F^* sending 4×4 matrices to 2×2 matrices

$$F^*(A) = FAF^{\mathrm{T}}. \tag{27}$$

As before, F^* sends anti-symmetric 4×4 matrices to 2×2 anti-symmetric matrices. If we pick $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as the single basis element of 2×2 anti-symmetric matrices, the action of F^* can be expressed by

$$P_n = \left(\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \quad \begin{vmatrix} f_{11} & f_{13} \\ f_{21} & f_{23} \end{vmatrix} \quad \begin{vmatrix} f_{11} & f_{14} \\ f_{21} & f_{24} \end{vmatrix} \quad \begin{vmatrix} f_{13} & f_{14} \\ f_{23} & f_{24} \end{vmatrix} \quad -\begin{vmatrix} f_{12} & f_{14} \\ f_{22} & f_{24} \end{vmatrix} \quad \begin{vmatrix} f_{12} & f_{13} \\ f_{22} & f_{23} \end{vmatrix} \right). \tag{28}$$

Now, the anti-symmetric matrix A in Lemma 2 can be expressed as a column vector

$$P_0 = \left(\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \quad \begin{vmatrix} b_{11} & b_{12} \\ b_{31} & b_{32} \end{vmatrix} \quad \begin{vmatrix} b_{11} & b_{12} \\ b_{41} & b_{42} \end{vmatrix} \quad \begin{vmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{vmatrix} \quad -\begin{vmatrix} b_{21} & b_{22} \\ b_{41} & b_{42} \end{vmatrix} \quad \begin{vmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{vmatrix} \right)^{\mathrm{T}} \tag{29}$$

using the same basis for the 4×4 anti-symmetric matrices. Thus, Lemma 2 states that

$$F^* M^*_{n-1} M^*_{n-2} \cdots M^*_2 M^*_1(A) = (FM_{n-1}M_{n-2} \cdots M_2 \cdot M_1)^*(A) = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}, \tag{30}$$

where $d = \det(FM_{n-1}M_{n-2} \cdots M_2 \cdot M_1 B)$. Converting the linear transformations to the corresponding matrices, we have

$$P_n P_{n-1} P_{n-2} \cdots P_1 P_0 = d. \qquad \square \tag{31}$$

To use Proposition 1, for each joint matrix C_i in Eq. (11) we form the matrix P_j using Eq. (17). This will be straightforward, since the joint matrices will not involve complicated functions. Then for each transfer matrix $M_j = W_i$, we create the matrix

$$P_j = \begin{pmatrix}
\begin{vmatrix} y_{j,1} & y_{j,2} \\ y'_{j,1} & y'_{j,2} \end{vmatrix} & \begin{vmatrix} y_{j,1} & y_{j,3} \\ y'_{j,1} & y'_{j,3} \end{vmatrix} & \begin{vmatrix} y_{j,1} & y_{j,4} \\ y'_{j,1} & y'_{j,4} \end{vmatrix} & \begin{vmatrix} y_{j,3} & y_{j,4} \\ y'_{j,3} & y'_{j,4} \end{vmatrix} & -\begin{vmatrix} y_{j,2} & y_{j,4} \\ y'_{j,2} & y'_{j,4} \end{vmatrix} & \begin{vmatrix} y_{j,2} & y_{j,3} \\ y'_{j,2} & y'_{j,3} \end{vmatrix} \\
\begin{vmatrix} y_{j,1} & y_{j,2} \\ y''_{j,1} & y''_{j,2} \end{vmatrix} & \begin{vmatrix} y_{j,1} & y_{j,3} \\ y''_{j,1} & y''_{j,3} \end{vmatrix} & \begin{vmatrix} y_{j,1} & y_{j,4} \\ y''_{j,1} & y''_{j,4} \end{vmatrix} & \begin{vmatrix} y_{j,3} & y_{j,4} \\ y''_{j,3} & y''_{j,4} \end{vmatrix} & -\begin{vmatrix} y_{j,2} & y_{j,4} \\ y''_{j,2} & y''_{j,4} \end{vmatrix} & \begin{vmatrix} y_{j,2} & y_{j,3} \\ y''_{j,2} & y''_{j,3} \end{vmatrix} \\
\begin{vmatrix} y_{j,1} & y_{j,2} \\ y'''_{j,1} & y'''_{j,2} \end{vmatrix} & \begin{vmatrix} y_{j,1} & y_{j,3} \\ y'''_{j,1} & y'''_{j,3} \end{vmatrix} & \begin{vmatrix} y_{j,1} & y_{j,4} \\ y'''_{j,1} & y'''_{j,4} \end{vmatrix} & \begin{vmatrix} y_{j,3} & y_{j,4} \\ y'''_{j,3} & y'''_{j,4} \end{vmatrix} & -\begin{vmatrix} y_{j,2} & y_{j,4} \\ y'''_{j,2} & y'''_{j,4} \end{vmatrix} & \begin{vmatrix} y_{j,2} & y_{j,3} \\ y'''_{j,2} & y'''_{j,3} \end{vmatrix} \\
\begin{vmatrix} y''_{j,1} & y''_{j,2} \\ y'''_{j,1} & y'''_{j,2} \end{vmatrix} & \begin{vmatrix} y''_{j,1} & y''_{j,3} \\ y'''_{j,1} & y'''_{j,3} \end{vmatrix} & \begin{vmatrix} y''_{j,1} & y''_{j,4} \\ y'''_{j,1} & y'''_{j,4} \end{vmatrix} & \begin{vmatrix} y''_{j,3} & y''_{j,4} \\ y'''_{j,3} & y'''_{j,4} \end{vmatrix} & -\begin{vmatrix} y''_{j,2} & y''_{j,4} \\ y'''_{j,2} & y'''_{j,4} \end{vmatrix} & \begin{vmatrix} y''_{j,2} & y''_{j,3} \\ y'''_{j,2} & y'''_{j,3} \end{vmatrix} \\
-\begin{vmatrix} y'_{j,1} & y'_{j,2} \\ y'''_{j,1} & y'''_{j,2} \end{vmatrix} & -\begin{vmatrix} y'_{j,1} & y'_{j,3} \\ y'''_{j,1} & y'''_{j,3} \end{vmatrix} & -\begin{vmatrix} y'_{j,1} & y'_{j,4} \\ y'''_{j,1} & y'''_{j,4} \end{vmatrix} & -\begin{vmatrix} y'_{j,3} & y'_{j,4} \\ y'''_{j,3} & y'''_{j,4} \end{vmatrix} & \begin{vmatrix} y'_{j,2} & y'_{j,4} \\ y'''_{j,2} & y'''_{j,4} \end{vmatrix} & -\begin{vmatrix} y'_{j,2} & y'_{j,3} \\ y'''_{j,2} & y'''_{j,3} \end{vmatrix} \\
\begin{vmatrix} y'_{j,1} & y'_{j,2} \\ y''_{j,1} & y''_{j,2} \end{vmatrix} & \begin{vmatrix} y'_{j,1} & y'_{j,3} \\ y''_{j,1} & y''_{j,3} \end{vmatrix} & \begin{vmatrix} y'_{j,1} & y'_{j,4} \\ y''_{j,1} & y''_{j,4} \end{vmatrix} & \begin{vmatrix} y'_{j,3} & y'_{j,4} \\ y''_{j,3} & y''_{j,4} \end{vmatrix} & -\begin{vmatrix} y'_{j,2} & y'_{j,4} \\ y''_{j,2} & y''_{j,4} \end{vmatrix} & \begin{vmatrix} y'_{j,2} & y'_{j,3} \\ y''_{j,2} & y''_{j,3} \end{vmatrix}
\end{pmatrix}, \tag{32}$$

where $y_{j,1}$, $y_{j,2}$, $y_{j,3}$, and $y_{j,4}$ satisfy Eqs. (7a–p). This seems like more work on the front end, but the reward will be the lack of a determinant after multiplying the matrices together. Hence, once the exterior matrix P_j is found for each type of structural element, that result can be used for a host of different problems.

Proposition 1 can be extended as Lemma 2 was, using alternating covariant tensors, or exterior forms, instead of the anti-symmetric matrices [17, p. 202]. For example, if B is 6×3, F is 3×6, and the M_j are 6×6 matrices, then we can find

$$\det(FM_{n-1}M_{n-2} \cdots M_2 M_1 B) = P_n P_{n-1} P_{n-2} \cdots P_1 P_0, \tag{33}$$

where P_0 is a 1×20 matrix, P_n is a 20×1 matrix, and the P_j are 20×20 matrices, found using an appropriate basis for the three-dimensional exterior forms on $\mathbb{R}^6$. This was in fact done in Ref. [9], where luckily the 20×20 matrices were sparse.

The reason for the basis element e_5 being negative is so that the exterior matrices will have the following property.

Proposition 2. *Let*

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{34}$$

be the 6×6 *matrix with a* 3×3 *identity matrix on the upper right and lower left corners, and zeros elsewhere. If* $P = \mathrm{ext}(M)$, *then* $P^{\mathrm{T}}JPJ = \det(M)I$.

Proof. Let $D = P^{\mathrm{T}}JPJ$. Since $J^{\mathrm{T}} = J$ and $J^2 = I$, it is easy to check that $JDJ = D^{\mathrm{T}}$. Note that JDJ essentially exchanges the upper right and lower left corners of D, as well as the upper left and lower right corners. Hence, we will only have to prove the proposition for the top half of the D, and use this symmetry for the lower half.

We can compute

$$\begin{aligned} d_{11} = {} & \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} \cdot \begin{vmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{vmatrix} - \begin{vmatrix} m_{11} & m_{12} \\ m_{31} & m_{32} \end{vmatrix} \cdot \begin{vmatrix} m_{23} & m_{24} \\ m_{43} & m_{44} \end{vmatrix} + \begin{vmatrix} m_{11} & m_{12} \\ m_{41} & m_{42} \end{vmatrix} \cdot \begin{vmatrix} m_{23} & m_{24} \\ m_{33} & m_{34} \end{vmatrix} \\ & + \begin{vmatrix} m_{31} & m_{32} \\ m_{41} & m_{42} \end{vmatrix} \cdot \begin{vmatrix} m_{13} & m_{14} \\ m_{23} & m_{24} \end{vmatrix} - \begin{vmatrix} m_{21} & m_{22} \\ m_{41} & m_{42} \end{vmatrix} \cdot \begin{vmatrix} m_{13} & m_{14} \\ m_{33} & m_{34} \end{vmatrix} + \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix} \cdot \begin{vmatrix} m_{13} & m_{14} \\ m_{43} & m_{44} \end{vmatrix}, \end{aligned} \tag{35a}$$

$$\begin{aligned} d_{22} = {} & - \begin{vmatrix} m_{11} & m_{13} \\ m_{21} & m_{23} \end{vmatrix} \cdot \begin{vmatrix} m_{32} & m_{34} \\ m_{42} & m_{44} \end{vmatrix} + \begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix} \cdot \begin{vmatrix} m_{22} & m_{24} \\ m_{42} & m_{44} \end{vmatrix} - \begin{vmatrix} m_{11} & m_{13} \\ m_{41} & m_{43} \end{vmatrix} \cdot \begin{vmatrix} m_{22} & m_{24} \\ m_{32} & m_{34} \end{vmatrix} \\ & - \begin{vmatrix} m_{31} & m_{33} \\ m_{41} & m_{43} \end{vmatrix} \cdot \begin{vmatrix} m_{12} & m_{14} \\ m_{22} & m_{24} \end{vmatrix} + \begin{vmatrix} m_{21} & m_{23} \\ m_{41} & m_{43} \end{vmatrix} \cdot \begin{vmatrix} m_{12} & m_{14} \\ m_{32} & m_{34} \end{vmatrix} - \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} \cdot \begin{vmatrix} m_{12} & m_{14} \\ m_{42} & m_{44} \end{vmatrix}, \end{aligned} \tag{35b}$$

$$\begin{aligned} d_{33} = {} & \begin{vmatrix} m_{11} & m_{14} \\ m_{21} & m_{24} \end{vmatrix} \cdot \begin{vmatrix} m_{32} & m_{33} \\ m_{42} & m_{43} \end{vmatrix} - \begin{vmatrix} m_{11} & m_{14} \\ m_{31} & m_{34} \end{vmatrix} \cdot \begin{vmatrix} m_{22} & m_{23} \\ m_{42} & m_{43} \end{vmatrix} + \begin{vmatrix} m_{11} & m_{14} \\ m_{41} & m_{44} \end{vmatrix} \cdot \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} \\ & + \begin{vmatrix} m_{31} & m_{34} \\ m_{41} & m_{44} \end{vmatrix} \cdot \begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix} - \begin{vmatrix} m_{21} & m_{24} \\ m_{41} & m_{44} \end{vmatrix} \cdot \begin{vmatrix} m_{12} & m_{13} \\ m_{32} & m_{33} \end{vmatrix} + \begin{vmatrix} m_{21} & m_{24} \\ m_{31} & m_{34} \end{vmatrix} \cdot \begin{vmatrix} m_{12} & m_{13} \\ m_{42} & m_{43} \end{vmatrix}. \end{aligned} \tag{35c}$$

Now, if we multiply out the terms of d_{11}, we find that we get $\det(M)$. Note that d_{22} looks like the negative of d_{11}, except that the middle two columns of M are exchanged. Thus, $d_{22} = d_{11} = \det(M)$. Likewise, d_{33} is d_{11} with the last three columns of M permuted, so $d_{33} = d_{11} = \det M$. The fact that $JDJ = D^{\mathrm{T}}$ tells us that all of the diagonal elements of D are $\det M$.

To show that the off-diagonal elements of D are all 0, let us look at one particular case.

$$\begin{aligned} d_{12} = {} & - \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} \cdot \begin{vmatrix} m_{32} & m_{34} \\ m_{42} & m_{44} \end{vmatrix} + \begin{vmatrix} m_{11} & m_{12} \\ m_{31} & m_{32} \end{vmatrix} \cdot \begin{vmatrix} m_{22} & m_{24} \\ m_{42} & m_{44} \end{vmatrix} - \begin{vmatrix} m_{11} & m_{12} \\ m_{41} & m_{42} \end{vmatrix} \cdot \begin{vmatrix} m_{22} & m_{24} \\ m_{32} & m_{34} \end{vmatrix} \\ & - \begin{vmatrix} m_{31} & m_{32} \\ m_{41} & m_{42} \end{vmatrix} \cdot \begin{vmatrix} m_{12} & m_{14} \\ m_{22} & m_{24} \end{vmatrix} + \begin{vmatrix} m_{21} & m_{22} \\ m_{41} & m_{42} \end{vmatrix} \cdot \begin{vmatrix} m_{12} & m_{14} \\ m_{32} & m_{34} \end{vmatrix} - \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix} \cdot \begin{vmatrix} m_{12} & m_{14} \\ m_{42} & m_{44} \end{vmatrix}. \end{aligned} \tag{36}$$

Note that this expression does not involve any of the elements from the third column of M. In fact, d_{12} is the same as d_{22}, except that the third column of M is replaced by the second column. So we are essentially taking the determinant of the matrix M' formed by replacing the third column of M with the second column. But M' has two columns the same, so the determinant is 0. Likewise, all of the off diagonal elements of D will be 0. □

Note that the matrix $P/\sqrt{\det(M)}$ almost, but not quite, a symplectic matrix. An $2n \times 2n$ matrix A is *symplectic* if

$$A^{\mathrm{T}} \cdot \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right) \cdot A = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right), \tag{37}$$

where I is an $n \times n$ identity matrix, and $\mathbf{0}$ is an $n \times n$ zero matrix. Proposition 2 shows that if $\det(M) \neq 0$, then $A = P/\sqrt{\det(M)}$ satisfies

$$A^{\mathrm{T}} J A = J. \tag{38}$$

We will call such a matrix *pseudo-orthogonal*. As close as this is to the definition of symplectic matrices, it is distinct, since $\det(J) = -1$, whereas the determinant of $\left(\begin{smallmatrix} \mathbf{0} & I \\ -I & \mathbf{0} \end{smallmatrix}\right)$ is always 1. Hence, no change of basis will convert one definition to the other. (In fact, the pseudo-orthogonal matrices can be described as $GO_6(\mathbb{R}, F)$, where F is the bilinear form $x_1x_4 + x_2x_5 + x_3x_6$.)

Although we have found a relatively easy way of computing the exterior matrix corresponding to the wave matrices W_i, we had to assume that the general solution Eq. (5) was chosen to satisfy Eqs. (7a–p). Although this is always possible in theory, in practice this involves taking the inverse of a 4×4 matrix with functions, which can be very hard to do. However, it is possible to compute the exterior matrix for a wave matrix even if Eqs. (7a–p) are not satisfied, without inverses.

Proposition 3. *If R and S are two 4×4 matrices, with $\det(S) \neq 0$, then*

$$\operatorname{ext}(RS^{-1}) = 1/\det(S) \cdot \begin{pmatrix} \langle r_1, r_2, s_3, s_4 \rangle & \langle r_1, r_2, s_4, s_2 \rangle & \langle r_1, r_2, s_2, s_3 \rangle & \langle r_1, r_2, s_1, s_2 \rangle & \langle r_1, r_2, s_1, s_3 \rangle & \langle r_1, r_2, s_1, s_4 \rangle \\ \langle r_1, r_3, s_3, s_4 \rangle & \langle r_1, r_3, s_4, s_2 \rangle & \langle r_1, r_3, s_2, s_3 \rangle & \langle r_1, r_3, s_1, s_2 \rangle & \langle r_1, r_3, s_1, s_3 \rangle & \langle r_1, r_3, s_1, s_4 \rangle \\ \langle r_1, r_4, s_3, s_4 \rangle & \langle r_1, r_4, s_4, s_2 \rangle & \langle r_1, r_4, s_2, s_3 \rangle & \langle r_1, r_4, s_1, s_2 \rangle & \langle r_1, r_4, s_1, s_3 \rangle & \langle r_1, r_4, s_1, s_4 \rangle \\ \langle r_3, r_4, s_3, s_4 \rangle & \langle r_3, r_4, s_4, s_2 \rangle & \langle r_3, r_4, s_2, s_3 \rangle & \langle r_3, r_4, s_1, s_2 \rangle & \langle r_3, r_4, s_1, s_3 \rangle & \langle r_3, r_4, s_1, s_4 \rangle \\ \langle r_4, r_2, s_3, s_4 \rangle & \langle r_4, r_2, s_4, s_2 \rangle & \langle r_4, r_2, s_2, s_3 \rangle & \langle r_4, r_2, s_1, s_2 \rangle & \langle r_4, r_2, s_1, s_3 \rangle & \langle r_4, r_2, s_1, s_4 \rangle \\ \langle r_2, r_3, s_3, s_4 \rangle & \langle r_2, r_3, s_4, s_2 \rangle & \langle r_2, r_3, s_2, s_3 \rangle & \langle r_2, r_3, s_1, s_2 \rangle & \langle r_2, r_3, s_1, s_3 \rangle & \langle r_2, r_3, s_1, s_4 \rangle \end{pmatrix}, \tag{39}$$

where $\langle r_i, r_j, s_k, s_l \rangle$ denotes the determinant of the 4×4 matrix formed from the ith and jth rows of R, and the kth and lth rows of S.

Proof. Using Lemma 1, $\operatorname{ext}(RS^{-1}) = \operatorname{ext}(R) \cdot (\operatorname{ext}(S))^{-1}$. But Proposition 2 tells us that $(\operatorname{ext}(S))^{-1} = J \cdot \operatorname{ext}(S)^{\mathrm{T}} \cdot J/\det(S)$. Thus, if we let $P = \operatorname{ext}(R) \cdot J \cdot \operatorname{ext}(S)^{\mathrm{T}} \cdot J$, we need to show that each entry of P is a determinant of the above form. But we already showed in Proposition 2 that, when $R = S = M$, each entry of $J \cdot \operatorname{ext}(M) \cdot J \cdot \operatorname{ext}(M)^{\mathrm{T}}$ can be expressed as a determinant involving the rows of M.

The entry in the ith row and jth column of P uses the i row of ext(R), and the $(j + 3\,(\mathrm{Mod}\,6))$th row of ext($S$). However, each row of ext(R) only uses two of the rows of R. So we find that each entry of P involves only the elements from two rows of R, and two rows of S. Since we know that if $R = S = M$, each entry can be expressed as a determinant involving these rows, we can deduce that each entry will be in the form $\langle r_i, r_j, s_k, s_l \rangle$. By examining the matrix ext(R), we can derive the above formula. □

We can use Proposition 3 to find the exterior matrix for the wave transfer matrix. Suppose we have

$$y_{i,\lambda} = c_{i,1}y_{i,1}(x,\lambda) + c_{i,2}y_{i,2}(x,\lambda) + c_{i,3}y_{i,3}(x,\lambda) + c_{i,4}y_{i,4}(x,\lambda), \tag{40}$$

but Eqs. (7a–p) do not hold. Rather than going to the trouble of inverting a 4×4 matrix to get the solution into the form Eq. (5) and Eqs. (7a–p), we can let

$$R_i = \begin{pmatrix} y_{i,1}(L_i,\lambda) & y_{i,2}(L_i,\lambda) & y_{i,3}(L_i,\lambda) & y_{i,4}(L_i,\lambda) \\ y'_{i,1}(L_i,\lambda) & y'_{i,2}(L_i,\lambda) & y'_{i,3}(L_i,\lambda) & y'_{i,4}(L_i,\lambda) \\ y''_{i,1}(L_i,\lambda) & y''_{i,2}(L_i,\lambda) & y''_{i,3}(L_i,\lambda) & y''_{i,4}(L_i,\lambda) \\ y'''_{i,1}(L_i,\lambda) & y'''_{i,2}(L_i,\lambda) & y'''_{i,3}(L_i,\lambda) & y'''_{i,4}(L_i,\lambda) \end{pmatrix} \tag{41a}$$

and

$$S_i = \begin{pmatrix} y_{i,1}(0,\lambda) & y_{i,2}(0,\lambda) & y_{i,3}(0,\lambda) & y_{i,4}(0,\lambda) \\ y'_{i,1}(0,\lambda) & y'_{i,2}(0,\lambda) & y'_{i,3}(0,\lambda) & y'_{i,4}(0,\lambda) \\ y''_{i,1}(0,\lambda) & y''_{i,2}(0,\lambda) & y''_{i,3}(0,\lambda) & y''_{i,4}(0,\lambda) \\ y'''_{i,1}(0,\lambda) & y'''_{i,2}(0,\lambda) & y'''_{i,3}(0,\lambda) & y'''_{i,4}(0,\lambda) \end{pmatrix}. \tag{41b}$$

Then $W_i = R_i S_i^{-1}$, and Proposition 3 can be used to find the exterior matrix of W_i without computing S_i^{-1}. In fact, this is the way that the exterior matrices were computed in Ref. [11].

3. A simple example

An example of a sequentially coupled fourth-order equation is the problem of finding the eigenfrequencies of a sequence of Euler–Bernoulli beams joined together with various types of joints. Each beam would satisfy the equation

$$m_i \frac{\partial^2 u_i}{\partial t^2} + E_i I_i \frac{\partial^4 u_i}{\partial x^4} = 0, \tag{42}$$

where m_i is the mass per unit length of the ith beam, L_i is the length of this beam, and $E_i I_i$ is the fluxural rigidity of this beam.

Using Eq. (3), the general solution is the sum of solutions of the form

$$u_{i,\lambda}(x,t) = y_{i,\lambda}(x) e^{\lambda t} \tag{43}$$

for various eigenfrequencies λ. Then the $y_{i,\lambda}$ will satisfy

$$E_i I_i y_{i,\lambda}^{\mathrm{iv}} + m_i \lambda^2 y_{i,\lambda} = 0. \tag{44}$$

By letting $k_i = \sqrt[4]{(m_i/(E_i I_i))}$ and $\eta = (1-\mathrm{i})\sqrt{\lambda/2}$ (so that $\mathrm{i}\eta^2 = \lambda$), we can easily find the exact general solution to Eq. (44) to be

$$y_{i,\lambda}(x) = A_i e^{k_i \eta x} + B_i e^{\mathrm{i}k_i \eta x} + C_i e^{-k_i \eta x} + D_i e^{-\mathrm{i}k_i \eta x}. \tag{45}$$

In this case, it is not too hard to write the general solution so that it also satisfies Eqs. (7a–p):

$$\begin{aligned} y_{i,\lambda}(x) = {} & a_i \frac{1}{2}(\cosh(k_i\eta x) + \cos(k_i\eta x)) + b_i \frac{1}{2k_i\eta}(\sinh(k_i\eta x) + \sin(k_i\eta x)) \\ & + c_i \frac{1}{2k_i^2\eta^2}(\cosh(k_i\eta x) - \cos(k_i\eta x)) + d_i \frac{1}{2k_i^3\eta^3}(\sinh(k_i\eta x) - \sin(k_i\eta x)), \end{aligned} \tag{46}$$

so we can form the transfer matrix

$$W_i = \frac{1}{k_i^3\eta^3} \begin{pmatrix} k_i^3\eta^3\,\mathrm{Hya}(k_i\eta L_i) & k_i^2\eta^2\,\mathrm{Hyd}(k_i\eta L_i) & k_i\eta\,\mathrm{Hyc}(k_i\eta L_i) & \mathrm{Hyb}(k_i\eta L_i) \\ k_i^4\eta^4\,\mathrm{Hyb}(k_i\eta L_i) & k_i^3\eta^3\,\mathrm{Hya}(k_i\eta L_i) & k_i^2\eta^2\,\mathrm{Hyd}(k_i\eta L_i) & k_i\eta\,\mathrm{Hyc}(k_i\eta L_i) \\ k_i^5\eta^5\,\mathrm{Hyc}(k_i\eta L_i) & k_i^4\eta^4\,\mathrm{Hyb}(k_i\eta L_i) & k_i^3\eta^3\,\mathrm{Hya}(k_i\eta L_i) & k_i^2\eta^2\,\mathrm{Hyd}(k_i\eta L_i) \\ k_i^6\eta^6\,\mathrm{Hyd}(k_i\eta L_i) & k_i^5\eta^5\,\mathrm{Hyc}(k_i\eta L_i) & k_i^4\eta^4\,\mathrm{Hyb}(k_i\eta L_i) & k_i^3\eta^3\,\mathrm{Hya}(k_i\eta L_i) \end{pmatrix}. \tag{47}$$

Here, we introduced the "hybrid functions" from Ref. [9]:

$$\mathrm{Hya}(x) = \frac{\cosh(x) + \cos(x)}{2} = \frac{\mathrm{e}^x + \mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-x} + \mathrm{e}^{-\mathrm{i}x}}{4}, \tag{48a}$$

$$\mathrm{Hyb}(x) = \frac{\sinh(x) - \sin(x)}{2} = \frac{\mathrm{e}^x + \mathrm{i}\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-x} - \mathrm{i}\mathrm{e}^{-\mathrm{i}x}}{4}, \tag{48b}$$

$$\mathrm{Hyc}(x) = \frac{\cosh(x) - \cos(x)}{2} = \frac{\mathrm{e}^x - \mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-x} - \mathrm{e}^{-\mathrm{i}x}}{4}, \tag{48c}$$

$$\mathrm{Hyd}(x) = \frac{\sinh(x) + \sin(x)}{2} = \frac{\mathrm{e}^x - \mathrm{i}\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-x} + \mathrm{i}\mathrm{e}^{-\mathrm{i}x}}{4}. \tag{48d}$$

Using Eq. (17), we can find the exterior matrix P_j corresponding to this W_i by taking determinants of all 2×2 submatrices of $W_i = M_j$. (Recall that there will also be P_j matrices for the joints as well, so we have $j = 2i - 1$.) It is rather clear that some cancellation will take place in these determinants. Even in this simple example, this can be rather tedious, but Proposition 3 gives us an alternative way of finding P_j. Using the solution Eq. (45), we find that $W_i = R_i \cdot S_i^{-1}$, where

$$R_i = \begin{pmatrix} \mathrm{e}^{k_i\eta L_i} & \mathrm{e}^{\mathrm{i}k_i\eta L_i} & \mathrm{e}^{-k_i\eta L_i} & \mathrm{e}^{-\mathrm{i}k_i\eta L_i} \\ k_i\eta\mathrm{e}^{k_i\eta L_i} & \mathrm{i}k_i\eta\mathrm{e}^{\mathrm{i}k_i\eta L_i} & -k_i\eta\mathrm{e}^{-k_i\eta L_i} & -\mathrm{i}k_i\eta\mathrm{e}^{-\mathrm{i}k_i\eta L_i} \\ k_i^2\eta^2\mathrm{e}^{k_i\eta L_i} & -k_i^2\eta^2\mathrm{e}^{\mathrm{i}k_i\eta L_i} & k_i^2\eta^2\mathrm{e}^{-k_i\eta L_i} & -k_i^2\eta^2\mathrm{e}^{-\mathrm{i}k_i\eta L_i} \\ k_i^3\eta^3\mathrm{e}^{k_i\eta L_i} & -\mathrm{i}k_i^3\eta^3\mathrm{e}^{\mathrm{i}k_i\eta L_i} & -k_i^3\eta^3\mathrm{e}^{-k_i\eta L_i} & \mathrm{i}k_i^3\eta^3\mathrm{e}^{-\mathrm{i}k_i\eta L_i} \end{pmatrix} \tag{49a}$$

and

$$S_i = \begin{pmatrix} 1 & 1 & 1 & 1 \\ k_i\eta & \mathrm{i}k_i\eta & -k_i\eta & -\mathrm{i}k_i\eta \\ k_i^2\eta^2 & -k_i^2\eta^2 & k_i^2\eta^2 & -k_i^2\eta^2 \\ k_i^3\eta^3 & -\mathrm{i}k_i^3\eta^3 & -k_i^3\eta^3 & \mathrm{i}k_i^3\eta^3 \end{pmatrix}. \tag{49b}$$

In this case S_i is easily invertible, and in fact $\det(S_i) = -16\mathrm{i}k_i^6\eta^6$. Finding the exterior matrices for the various types of dampers is straightforward. Converting back to the variable λ, we find that

$$P_j = \begin{pmatrix} p_5 & p_2 & p_3 & p_1 & -p_4 & p_3 \\ k_j^4\lambda^2 p_4 & p_6 & p_2 & p_4 & -2p_3 & p_2 \\ k_j^4\lambda^2 p_3 & k_j^4\lambda^2 p_4 & p_5 & p_3 & -p_2 & k_j^4\lambda^2 p_1 \\ k_j^8\lambda^4 p_1 & k_j^4\lambda^2 p_2 & k_j^4\lambda^2 p_3 & p_5 & -k_j^4\lambda^2 p_4 & k_j^4\lambda^2 p_3 \\ -k_j^4\lambda^2 p_2 & -2k_j^4\lambda^2 p_3 & -k_j^4\lambda^2 p_4 & -p_2 & p_6 & -k_j^4\lambda^2 p_4 \\ k_j^4\lambda^2 p_3 & k_j^4\lambda^2 p_4 & k_j^4\lambda^2 p_1 & p_3 & -p_2 & p_5 \end{pmatrix}, \tag{50}$$

where

$$p_1 = \frac{\mathrm{Hya}(k_jL_j\sqrt{2\lambda}) - 1}{2k_j^4\lambda^2}, \quad p_2 = \frac{\mathrm{Hyd}(k_jL_j\sqrt{2\lambda})}{k_j\sqrt{2\lambda}}, \quad p_3 = \frac{\mathrm{Hyc}(k_jL_j\sqrt{2\lambda})}{2k_j^2\lambda}, \tag{51a–c}$$

$$p_4 = \frac{\mathrm{Hyb}(k_jL_j\sqrt{2\lambda})}{k_j^3\sqrt{2\lambda^3}}, \quad p_5 = \frac{1 + \mathrm{Hya}(k_jL_j\sqrt{2\lambda})}{2}, \quad \text{and} \quad p_6 = \mathrm{Hya}(k_jL_j\sqrt{2\lambda}). \tag{51d–f}$$

Note that k and L are now subscripted by j, so to be consistent with the indexing of the P_j matrices. At first, it seems ridiculous to replace the 4×4 transfer matrices with 6×6 matrices. However, with the transfer matrices, there was a 2×2 determinant that still needs to be calculated after the matrices are multiplied, and the highest order terms cancel in this final determinant. Thus, no approximations can be made in the transfer matrices. However, with the exterior matrices, there is no final determinant so we can make some approximations.

Because the eigenvalues λ will appear in conjugate pairs, we can assume that $\mathrm{Im}(\lambda)\geqslant 0$. Also, because the joint can only dissipate energy, we know that $\mathrm{Re}(\lambda)\leqslant 0$. Thus, we can assume that $\pi/4\leqslant \arg(\sqrt{2\lambda})\leqslant\pi/2$. In this wedge, it is easy to see that for large $|\lambda|$, $|\mathrm{e}^{-k_jL_j\sqrt{2\lambda}}|\leqslant 1$, $|\mathrm{e}^{\mathrm{i}k_jL_j\sqrt{2\lambda}}|<1$, and $|\mathrm{e}^{-\mathrm{i}k_jL_j\sqrt{2\lambda}}|\gg 1$. Thus, we can throw out all occurrences of $\mathrm{e}^{-k_jL_j\sqrt{2\lambda}}$ and $\mathrm{e}^{\mathrm{i}k_jL_j\sqrt{2\lambda}}$, since these are exponentially small compared with $\mathrm{e}^{-\mathrm{i}k_jL_j\sqrt{2\lambda}}$. This produces the approximate exterior matrix

$$\widetilde{P}_j=\begin{pmatrix} k_j^4\lambda^2\tilde{p}_1 & \tilde{p}_2 & \tilde{p}_3 & \tilde{p}_1 & -\tilde{p}_4 & \tilde{p}_3\\ k_j^4\lambda^2\tilde{p}_4 & 2k_j^4\lambda^2\tilde{p}_1 & \tilde{p}_2 & \tilde{p}_4 & -2\tilde{p}_3 & \tilde{p}_2\\ k_j^4\lambda^2\tilde{p}_3 & k_j^4\lambda^2\tilde{p}_4 & k_j^4\lambda^2\tilde{p}_1 & \tilde{p}_3 & -\tilde{p}_2 & k_j^4\lambda^2\tilde{p}_1\\ k_j^8\lambda^4\tilde{p}_1 & k_j^4\lambda^2\tilde{p}_2 & k_j^4\lambda^2\tilde{p}_3 & k_j^4\lambda^2\tilde{p}_1 & -k_j^4\lambda^2\tilde{p}_4 & k_j^4\lambda^2\tilde{p}_3\\ -k_j^4\lambda^2\tilde{p}_2 & -2k_j^4\lambda^2\tilde{p}_3 & -k_j^4\lambda^2\tilde{p}_4 & -\tilde{p}_2 & 2k_j^4\lambda^2\tilde{p}_1 & -k_j^4\lambda^2\tilde{p}_4\\ k_j^4\lambda^2\tilde{p}_3 & k_j^4\lambda^2\tilde{p}_4 & k_j^4\lambda^2\tilde{p}_1 & \tilde{p}_3 & -\tilde{p}_2 & k_j^4\lambda^2\tilde{p}_1, \end{pmatrix} \tag{52}$$

with

$$\tilde{p}_1=\frac{\mathrm{e}^{k_jL_j\sqrt{2\lambda}}+\mathrm{e}^{-\mathrm{i}k_jL_j\sqrt{2\lambda}}}{8k_j^4\lambda^2},\quad \tilde{p}_2=\frac{\mathrm{e}^{k_jL_j\sqrt{2\lambda}}+\mathrm{i}\mathrm{e}^{-\mathrm{i}k_jL_j\sqrt{2\lambda}}}{4k_j\sqrt{2\lambda}}, \tag{53a,b}$$

$$\tilde{p}_3=\frac{\mathrm{e}^{k_jL_j\sqrt{2\lambda}}-\mathrm{e}^{-\mathrm{i}k_jL_j\sqrt{2\lambda}}}{8k_j^2\lambda}\quad\text{and}\quad \tilde{p}_4=\frac{\mathrm{e}^{k_jL_j\sqrt{2\lambda}}-\mathrm{i}\mathrm{e}^{-\mathrm{i}k_jL_j\sqrt{2\lambda}}}{4k_j^3\sqrt{2\lambda^3}}. \tag{53c,d}$$

By removing the exponentially small terms, the exterior matrix goes from rank 6 to rank 2. In fact, we can now express the approximate exterior matrix $\widetilde{P}_j$ as

$$\widetilde{P}_j=\frac{1}{8k_j^4\lambda^2}\begin{pmatrix} 1 & 1\\ k_j\sqrt{2\lambda} & -\mathrm{i}k_j\sqrt{2\lambda}\\ k_j^2\lambda & -k_j^2\lambda\\ k_j^4\lambda^2 & k_j^4\lambda^2\\ -k_j^3\sqrt{2\lambda^3} & -\mathrm{i}k_j^3\sqrt{2\lambda^3}\\ k_j^2\lambda & -k_j^2\lambda \end{pmatrix}\cdot\begin{pmatrix} \mathrm{e}^{k_jL_j\sqrt{2\lambda}} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}k_jL_j\sqrt{2\lambda}} \end{pmatrix}\cdot\begin{pmatrix} k_j^4\lambda^2 & k_j^4\lambda^2\\ k_j^3\sqrt{2\lambda^3} & \mathrm{i}k_j^3\sqrt{2\lambda^3}\\ k_j^2\lambda & -k_j^2\lambda\\ 1 & 1\\ -k_j\sqrt{2\lambda} & \mathrm{i}k_j\sqrt{2\lambda}\\ k_j^2\lambda & -k_j^2\lambda \end{pmatrix}^{\mathrm{T}}. \tag{54}$$

If we now replace every P_j that represents a beam length with

$$\widetilde{N}_j=\begin{pmatrix} \mathrm{e}^{k_jL_j\sqrt{2\lambda}} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}k_jL_j\sqrt{2\lambda}} \end{pmatrix}, \tag{55}$$

and every P_j that represents a joint with

$$\widetilde{N}_j = \frac{1}{8k_{j-1}^2 k_{j+1}^2 \lambda^2} \begin{pmatrix} k_{j+1}^4\lambda^2 & k_{j+1}^4\lambda^2 \\ k_{j+1}^3\sqrt{2\lambda^3} & \mathrm{i}k_{j+1}^3\sqrt{2\lambda^3} \\ k_{j+1}^2\lambda & -k_{j+1}^2\lambda \\ 1 & 1 \\ -k_{j+1}\sqrt{2\lambda} & \mathrm{i}k_{j+1}\sqrt{2\lambda} \\ k_{j+1}^2\lambda & k_{j+1}^2\lambda \end{pmatrix}^{\mathrm{T}} \cdot P_j \cdot \begin{pmatrix} 1 & 1 \\ k_{j-1}\sqrt{2\lambda} & -\mathrm{i}k_{j-1}\sqrt{2\lambda} \\ k_{j-1}^2\lambda & -k_{j-1}^2\lambda \\ k_{j-1}^4\lambda^2 & k_{j-1}^4\lambda^2 \\ -k_{j-1}^3\sqrt{2\lambda^3} & -\mathrm{i}k_{j-1}^3\sqrt{2\lambda^3} \\ k_{j-1}^2\lambda & -k_{j-1}^2\lambda \end{pmatrix} \quad (56)$$

and finally replace

$$\widetilde{N}_n = \frac{1}{2\sqrt{2}k_{n-1}^2\lambda} P_n \cdot \begin{pmatrix} 1 & 1 \\ k_{n-1}\sqrt{2\lambda} & -\mathrm{i}k_{n-1}\sqrt{2\lambda} \\ k_{n-1}^2\lambda & -k_{n-1}^2\lambda \\ k_{n-1}^4\lambda^2 & k_{n-1}^4\lambda^2 \\ -k_{n-1}^3\sqrt{2\lambda^3} & -\mathrm{i}k_{n-1}^3\sqrt{2\lambda^3} \\ k_{n-1}^2\lambda & -k_{n-1}^2\lambda \end{pmatrix}, \quad \widetilde{N}_0 = \frac{1}{2\sqrt{2}k_{n-1}^2\lambda} \begin{pmatrix} k_1^4\lambda^2 & k_1^4\lambda^2 \\ k_1^3\sqrt{2\lambda^3} & \mathrm{i}k_1^3\sqrt{2\lambda^3} \\ k_1^2\lambda & -k_1^2\lambda \\ 1 & 1 \\ -k_1\sqrt{2\lambda} & \mathrm{i}k_1\sqrt{2\lambda} \\ k_1^2\lambda & k_1^2\lambda \end{pmatrix}^{\mathrm{T}} \cdot P_0, \quad (57a,b)$$

we find that we can solve for the eigenvalues approximately by solving

$$\widetilde{N}_n \cdot \widetilde{N}_{n-1} \cdots \widetilde{N}_3 \cdot \widetilde{N}_2 \cdot \widetilde{N}_1 \cdot \widetilde{N}_0 = 0. \quad (58)$$

Hence, we have simplified the problem to the product of only 2×2 matrices, as was done in Ref. [18].

The above method for finding the approximate eigenfrequencies λ is now even simpler than using the wave propagation method [8], and gives the identical results. Yet the exterior matrix method is mathematically rigorous, unlike WPM.

4. Relating infinitesimal generators to the exterior matrices

If the coefficients in Eq. (4) do not depend on x, as in the case of the last example, then it is in fact possible to construct the exterior matrix P_j without first solving Eq. (4). This is because the matrix W_i would have a simple infinitesimal generator:

$$G = L_i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -s(\lambda) & -r(\lambda) & -q(\lambda) & -p(\lambda) \end{pmatrix}. \quad (59)$$

That is,

$$W_i = \mathrm{e}^G = \lim_{h\to 0}(I_4 + hG)^{1/h}, \quad (60)$$

where I_4 is the 4×4 identity matrix. Because of Lemma 1, we can find the infinitesimal generator of $\mathrm{ext}(W_i)$ rather easily.

Proposition 4. *If G is the infinitesimal generator of M, so that* $M = e^G$, *then the limit*

$$Z = \lim_{h\to 0} \frac{\text{ext}(I_4 + hG) - I_6}{h} \tag{61}$$

exists, and $e^Z = \text{ext}(M)$. *That is, Z is the infinitesimal generator of* ext(M).

Proof. Since $\text{ext}(I_4 + xG)$ is clearly a differentiable function of x, and Z is merely the derivative of this with respect to x at $x = 0$, we see that the limit exists. Then

$$e^Z = \lim_{h\to 0}(I_6 + hZ)^{1/h} = \lim_{h\to 0}(\text{ext}(I_4 + hG + O(h^2)))^{1/h} = \lim_{h\to 0} \text{ext}((I_4 + hG + O(h^2))^{1/h}) = \text{ext}(M). \tag{62}$$

Here, $O(h^2)$ means terms of order h^2, which would not affect the final limit. So we see that Z becomes the infinitesimal generator of ext(M). □

We can now apply Proposition 4 to find the exterior matrix for a constant coefficient equation very quickly. Using Eq. (17), we can compute Z to be

$$Z = L_j \cdot \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -r(\lambda) & -q(\lambda) & -p(\lambda) & 0 & -1 & 0 \\ 0 & s(\lambda) & 0 & -p(\lambda) & 0 & r(\lambda) \\ -s(\lambda) & 0 & 0 & -1 & -p(\lambda) & q(\lambda) \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{63}$$

Hence, we can calculate the exterior matrix P_j directly by taking the exponential of a fairly sparse matrix. Interestingly enough, the eigenvalues of Z are precisely the six possible sums of two of the eigenvalues of G.

For example, the Timoshenko [19,20] is a widely used model for studying the transverse vibrations of beams. This model is given by two coupled second-order partial differential equations:

$$(EI\phi_x)_x + kAG(u_x - \phi) - \rho I\phi_{tt} = 0, \tag{64a}$$

$$(kAG(u_x - \phi))_x - \rho A u_{tt} = F(x, t), \tag{64b}$$

where $u(x, t)$ is the lateral displacement at time t of the point at position x, and $\phi(x, t)$ is the cross-sectional rotation due to bending. As in the Euler–Bernoulli beam, E is the Young's modulus, and I is the cross-sectional inertia. This equation introduced the variables G = modulus of elasticity in shear, ρ = mass density per unit length, A = cross-sectional area, k is a constant introduced to account for the geometry-dependent distribution of shearing stress, and $F(x, t)$ is an applied force. If we let $F(x, t) = 0$, we can eliminate $\phi(x, t)$ to produce a single fourth-order equation for each beam in the system

$$m_i \frac{\partial^2 u_i}{\partial t^2} + E_i I_i \frac{\partial^4 u_i}{\partial x^4} + \frac{I_i \rho_i}{k_i G_i} \frac{\partial^4 u_i}{\partial t^4} - \left(\frac{\rho_i E_i I_i}{k_i G_i} + I_i \rho_i\right) \frac{\partial^4 u_i}{\partial x^2 \partial t^2} = 0, \tag{65}$$

where $m_i = \rho_i A_i$ = mass per unit length. This gives rise to the constant coefficient equation for y:

$$y^{\text{iv}}_{i,\lambda}(x) - \left(\frac{\lambda^2 \rho_i}{k_i G_i} + \frac{\lambda^2 \rho_i}{E_i}\right) y''_{i,\lambda}(x) + \left(\frac{\rho_i^2 \lambda^4}{k_i G_i E_i} + \frac{m_i \lambda^2}{E_i I_i}\right) y_{i,\lambda}(x) = 0. \tag{66}$$

Thus, we can compute P_j as

$$\exp\begin{pmatrix} 0 & L_j & 0 & 0 & 0 & 0 \\ 0 & 0 & L_j & 0 & 0 & L_j \\ 0 & g_1 L_j & 0 & 0 & -L_j & 0 \\ 0 & g_2 L_j & 0 & 0 & 0 & 0 \\ -g_2 L_j & 0 & 0 & -L_j & 0 & -g_1 L_j \\ 0 & 0 & 0 & 0 & -L_j & 0 \end{pmatrix}, \tag{67}$$

where

$$g_1 = \frac{\lambda^2 \rho_j}{k_j G_j} + \frac{\lambda^2 \rho_j}{E_j}, \quad g_2 = \frac{\rho_j^2 \lambda^4}{k_j G_j E_j} + \frac{m_j \lambda^2}{E_j I_j} \tag{68a,b}$$

and $j = 2i - 1$. This can be done fairly easily using *Mathematica*. We can express this matrix as

$$P_j = \begin{pmatrix} p_7 & p_8 & p_9 & p_{10} & p_{11} & p_{12} \\ -q_j^2 \lambda^4 p_{11} & p_{13} & p_8 & -p_{11} & p_{14} & p_{15} \\ q_j^2 \lambda^4 p_{12} & p_{16} & p_{17} & p_{12} & -p_{15} & p_{18} \\ q_j^4 \lambda^8 p_{10} & q_j^2 \lambda^4 p_8 & q_j^2 \lambda^4 p_9 & p_7 & q_j^2 \lambda^4 p_{11} & q_j^2 \lambda^4 p_{12} \\ -q_j^2 \lambda^4 p_8 & q_j^2 \lambda^4 p_{14} & q_j^2 \lambda^4 p_{11} & -p_8 & p_{13} & -p_{16} \\ q_j^2 \lambda^4 p_9 & -q_j^2 \lambda^4 p_{11} & q_j^2 \lambda^4 p_{10} & p_9 & -p_8 & p_{17} \end{pmatrix}, \tag{69}$$

where

$$p_7 = \frac{q_j s_j^2 \cosh(L_j \lambda r_j) - q_j r_j^2 \cosh(L_j \lambda s_j) + q_j r_j^2 - q_j s_j^2 + 2 r_j^2 s_j^2}{2 r_j^2 s_j^2}, \quad p_8 = \frac{s_j \sinh(L_j \lambda r_j) + r_j \sinh(L_j \lambda s_j)}{2 \lambda r_j s_j}, \tag{70a,b}$$

$$p_9 = \frac{s_j^2 \cosh(L_j \lambda r_j) + r_j^2 \cosh(L_j \lambda s_j) - r_j^2 - s_j^2}{2 \lambda^2 r_j^2 s_j^2}, \quad p_{10} = \frac{s_j^2 \cosh(L_j \lambda r_j) - r_j^2 \cosh(L_j \lambda s_j) + r_j^2 - s_j^2}{2 \lambda^4 q_j r_j^2 s_j^2}, \tag{70c,d}$$

$$p_{11} = \frac{r_j \sinh(L_j \lambda s_j) - s_j \sinh(L_j \lambda r_j)}{2 \lambda^3 q_j r_j s_j}, \quad p_{12} = \frac{s_j^2 u_j \cosh(L_j \lambda r_j) - r_j^2 v_j \cosh(L_j \lambda s_j) + r_j^4 - s_j^4}{8 \lambda^2 q_j r_j^2 s_j^2}, \tag{70e,f}$$

$$p_{13} = \frac{\cosh(L_j \lambda s_j) + \cosh(L_j \lambda r_j)}{2}, \quad p_{14} = \frac{\cosh(L_j \lambda s_j) - \cosh(L_j \lambda r_j)}{2 \lambda^2 q_j}, \tag{70g,h}$$

$$p_{15} = \frac{s_j u_j \sinh(L_j \lambda r_j) - r_j v_j \sinh(L_j \lambda s_j)}{8 \lambda q_j r_j s_j}, \quad p_{16} = \frac{\lambda s_j u_j \sinh(L_j \lambda r_j) + \lambda r_j v_j \sinh(L_j \lambda s_j)}{8 r_j s_j}, \tag{70i,j}$$

$$p_{17} = \frac{s_j^2 u_j \cosh(L_j \lambda r_j) + r_j^2 v_j \cosh(L_j \lambda s_j) - (r_j^2 - s_j^2)^2}{8 r_j^2 s_j^2},$$
$$p_{18} = \frac{s_j^2 u_j^2 \cosh(L_j \lambda r_j) - r_j^2 v_j^2 \cosh(L_j \lambda s_j) + (r_j^2 - s_j^2)^3}{32 r_j^2 s_j^2 q_j} \tag{70k,l}$$

and we introduced the variables

$$r_j(\lambda) = \sqrt{\frac{\rho_j}{G_j k_j} + \frac{\rho_j}{E_j} + \frac{2\rho_j}{E_j}\sqrt{\frac{m_j E_j}{\lambda^2 \rho_j^2 I_j} + \frac{E_j}{G_j k_j}}}, \quad s_j(\lambda) = \sqrt{\frac{\rho_j}{G_j k_j} + \frac{\rho_j}{E_j} - \frac{2\rho_j}{E_j}\sqrt{\frac{m_j E_j}{\lambda^2 \rho_j^2 I_j} + \frac{E_j}{G_j k_j}}},$$

$$q_j(\lambda) = \frac{\rho_j}{E_j}\sqrt{\frac{m_j E_j}{\lambda^2 \rho_j^2 I_j} + \frac{E_j}{G_j k_j}} = \frac{r_j(\lambda)^2 - s_j(\lambda)^2}{4}, \quad u_j(\lambda) = 3 r_j(\lambda)^2 + s_j(\lambda)^2, \quad v_j(\lambda) = 3 s_j(\lambda)^2 + r_j(\lambda)^2. \tag{71a–e}$$

Thus, we have found the exterior matrix for the Timoshenko beam without having to solve the original fourth-order equation.

There are other applications to using the infinitesimal generators. In Ref. [10], the infinitesimal generator for an Euler–Bernoulli beam was added to the infinitesimal generator of a bend in the beam. By exponentiating this sum, we get a matrix representing a beam that is curved to form an arc of a circle. Using Eq. (45) allows us to find the exterior matrix for a curved beam directly, by exponentiating a 6×6 matrix.

Although infinitesimal generators only work if the differential equations are constant coefficient, for non-constant coefficient equations we can, in principle, use a method akin to the finite element method. By dividing the beam into small pieces, each piece can be approximated with a constant coefficient equation, and so the exterior matrix for that small piece can easily be computed. The problem then amounts to multiplying many 6×6 matrices together, each of which is close to the identity matrix. But since we do not have to take the determinant at the end, this is not hard to do numerically. This method is one way to explore the problem of a tapering Euler–Bernoulli beam, whose radius changes over the distance of the beam.

5. Bypassing the solution to the differential equations

We have seen how using the 6×6 exterior matrices instead of the 4×4 transfer matrices saves us from having to take a determinant at the end of multiplying the matrices together. We pushed the numerically difficult aspect of the problem forward, allowing us to make approximations earlier on, without the worry of cancellation errors. The major cancellation that used to take place in the final determinant now takes place in the construction of the exterior matrix P_j. In a sense, we are lifting the transfer matrices into the exterior algebra space, and working the problem in this higher dimensional space. This works well if we have the *exact* solution to the differential equation (4). Unfortunately, this is not always possible. For example, the equation for an inclined cable with a small amount of sag cannot be solved exactly. Rather, some crude approximations were used to express the solution in terms of Airy functions [21]. But if the solutions to the differential equations are approximations, one wonders how accurate the 4×4 determinants will be in Proposition 3, because of all of the cancellations. Even if we do have the exact solutions to Eq. (4), they may be so ugly that calculating the exterior matrix would be difficult.

Interestingly enough, the Airy functions canceled out in the computation of the exterior matrix for the inclined cable problem [11]. This suggests that it may be possible to push the numerically difficult aspect even further forward, by lifting Eq. (4) into the exterior algebra space. We expect the new equation to be sixth order, but we would need only the *approximate* solutions to this new equation, and construct the approximate exterior matrix $\widetilde{P}_j$ from these solutions. In which case, we will never have to solve Eq. (4) in the first place, and cancellation errors never have a chance to show up.

The key is to use the symmetry in the problem. The matrix W_i expresses the solution to Eq. (4) at $x=L_i$ in terms of the solution at $x=0$. In order to exploit the symmetry, we add a parameter ξ to the problem by determining the solution at $x=L_i$ in terms of the solution at $x=\xi$. The added difficulty often comes in finding W_i with the additional parameter, since we have to replace the 0's in Eqs. (41a,b) with ξ's. For constant coefficient equations, this amounts to replacing every L_i with $L_i-\xi$ in W_i. For equations which are not constant coefficient, we can use Eq. (9) to see that $W_i(L_i,\xi,\lambda)=W_i(L_i)\cdot W_i^{-1}(\xi)$.

Once we have $W_i(L_i,\xi,\lambda)$ with the added parameter, we can take advantage of the symmetry. Note that $W_i(\xi,L_i,\lambda)$ finds the solution to the original equation at $x=\xi$ in terms of the solution at $x=L_i$. Thus, $W_i(\xi,L_i,\lambda)=W_i^{-1}(L_i,\xi,\lambda)$. We can now use Lemma 1 to show that the exterior matrix $P_j(\xi,L_i,\lambda)=P_j^{-1}(L_i,\xi,\lambda)$. But Proposition 2 gives us another way to find the inverse of P_j:

$$P_j^{-1}=\frac{JP_j^{\mathrm{T}}J}{\det(W_i)}. \tag{72}$$

Since we can calculate the determinant of W_i using Abel's formula [22, p. 223] to be

$$\Delta_j(L_j,\xi,\lambda)=\exp\left(-\int_{\xi}^{L_j}p(x,\lambda)\,\mathrm{d}x\right), \tag{73}$$

we have

$$\Delta_j(L_j,\xi,\lambda)P_j(\xi,L_j,\lambda)=J\cdot P_j^{\mathrm{T}}(L_j,\xi,\lambda)\cdot J. \tag{74}$$

The extra variable gives us other interesting properties of the exterior matrices. Since $W_j(L_j,\xi)=W_j(L_j)\cdot W_j^{-1}(\xi)$, by Lemma 1 we have that $P_j(L_j,\xi,\lambda)=P_j(L_j)\cdot P_j^{-1}(\xi)$. Thus,

$$P_j(\xi_1,\xi_2,\lambda)\cdot P_j(\xi_2,\xi_3,\lambda)=P_j(\xi_1,\xi_3,\lambda). \tag{75}$$

But there is yet another interesting relationship between the elements of P_j. It is clear that

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{vmatrix} y_{i,1}(x,\lambda) & y_{i,2}(x,\lambda) \\ y'_{i,1}(x,\lambda) & y'_{i,2}(x,\lambda) \end{vmatrix} = \begin{vmatrix} y_{i,1}(x,\lambda) & y_{i,2}(x,\lambda) \\ y''_{i,1}(x,\lambda) & y''_{i,2}(x,\lambda) \end{vmatrix}. \tag{76}$$

By the same token, if we let $P_{j,uv}$ denote the (u,v) entry in the matrix $P_j(L_i,\xi,\lambda)$, then by looking at the entries of Eq. (32), we find that

$$\begin{aligned}
&\frac{\partial}{\partial L_j}P_{j,1v} = P_{j,2v}, \quad \frac{\partial}{\partial L_j}P_{j,6v} = -P_{j,5v}, \quad \frac{\partial}{\partial L_j}P_{j,2v} = P_{j,3v} + P_{j,6v},\\
&\frac{\partial}{\partial L_j}P_{j,3v} = -P_{j,5v} - p(L_j,\lambda)P_{j,3v} - q(L_j,\lambda)P_{j,2v} - r(L_j,\lambda)P_{j,1v},\\
&\frac{\partial}{\partial L_j}P_{j,4v} = -p(L_j,\lambda)P_{j,4v} + r(L_j,\lambda)P_{j,6v} + s(L_j,\lambda)P_{j,2v},\\
&\frac{\partial}{\partial L_j}P_{j,5v} = -P_{j,4v} - p(L_j,\lambda)P_{j,5v} + q(L_j,\lambda)P_{j,6v} - s(L_j,\lambda)P_{j,1v}.
\end{aligned} \tag{77a−f}$$

Eqs. (77a–d) can be used to find $P_{j,3v}$ in terms of $P_{j,1v}$:

$$2\frac{\partial}{\partial L_j}P_{j,3v} + p(L_j,\lambda)P_{j,3v} = \frac{\partial^3}{\partial L_j^3}P_{j,1v} - q(L_j,\lambda)\frac{\partial}{\partial L_j}P_{j,1v} - r(L_j,\lambda)P_{j,1v}. \tag{78}$$

This can be solved, using $\Delta_j^{-1/2}$ as the integrating factor, and integrating by parts:

$$\begin{aligned}
P_{j,3v} = {}& \frac{1}{2}\frac{\partial^2}{\partial L_j^2}P_{j,1v} - \frac{1}{4}p(L_j)\frac{\partial}{\partial L_j}P_{j,1v} + \frac{1}{4}p'(L_j)P_{j,1v} + \frac{1}{8}p(L_j)^2P_{j,1v} - \frac{1}{2}q(L_j)P_{j,1v}\\
&+ \frac{1}{16}\sqrt{\Delta_j}\int \Delta_j^{-1/2}Q(L_j)P_{j,1v}\,\mathrm{d}L_j + C_v(\xi)\sqrt{\Delta_j},
\end{aligned} \tag{79}$$

where

$$Q(x) = -p(x)^3 + 4p(x)\,q(x) - 8r(x) - 6p(x)\,p'(x) + 8q'(x) - 4p''(x). \tag{80}$$

If we are fortunate enough so that $Q(x)\equiv 0$, then the integral in Eq. (79) cancels out. We can then continue to use Eqs. (77e,f) to find a differential equation for which $P_{i,1v}$ satisfies for all six elements. Surprisingly enough, this turns out to be only a fifth-order equation, since one integral was used in Eq. (79). We then can reconstruct the entire matrix P_j with just one solution to the new equation.

Theorem 1. *If the coefficients of Eq.* (4) *for some i satisfy $Q(x)\equiv 0$, where $Q(x)$ is defined by Eq.* (80), *then we can let $f(L_j,\xi)$ be the unique solution to the fifth-order ordinary differential equation*:

$$\begin{aligned}
&16\frac{\partial^5}{\partial L_j^5}f + (32q(L_j) - 12p(L_j)^2 - 48p'(L_j))\frac{\partial^3}{\partial L_i^3}f + (48q'(L_j) - 36p(L_j)p'(L_j) - 72p''(L_j))\frac{\partial^2}{\partial L_j^2}f\\
&+ (p(L_j)^4 - 8p(L_j)^2q(L_j) + 16q(L_j)^2 - 64s(L_j) - 32q(L_j)p'(L_j) - 12p'(L_j)^2 + 16p(L_j)q'(L_j)\\
&+ 48q''(L_j) - 44p(L_j)p''(L_j) - 56p'''(L_j))\frac{\partial}{\partial L_i}f\\
&+ (16q(L_j)q'(L_j) - 16r(L_j)p'(L_j) - 12p(L_j)p'(L_j)^2 - 4p(L_j)^2q'(L_j) + 8p'(L_j)q'(L_j) - 32s'(L_j)\\
&+ 8p(L_j)q''(L_j) - 16q(L_j)p''(L_j) - 24p'(L_j)p''(L_j) + 16q'''(L_j) - 16p(L_j)p'''(L_j) - 16p^{\mathrm{iv}}(L_j))f = 0
\end{aligned} \tag{81}$$

using the initial conditions

$$f\big|_{L_j=\xi} = \frac{\partial}{\partial L_j}f\bigg|_{L_j=\xi} = \frac{\partial^2}{\partial L_j^2}f\bigg|_{L_j=\xi} = \frac{\partial^3}{\partial L_j^3}f\bigg|_{L_j=\xi} = 0, \quad \frac{\partial^4}{\partial L_j^4}f\bigg|_{L_j=\xi} = 2. \tag{82a,b}$$

Then $f(L_j, \xi)$ will be a symmetrical function in L_j and ξ, that is, $f(L_j, \xi) = f(\xi, L_j)$, and we can construct the exterior matrix P_j corresponding to $W_i = M_j$ using the equation

$$P_j = \sqrt{\Delta_j}(D_{L_j}((J \cdot D_\xi(f(L_j, \xi)))^{\mathrm{T}}) - G(L_j)^{\mathrm{T}} \cdot G(\xi) \cdot J/2). \tag{83}$$

Here, $G(x)$ is the row matrix

$$G(x) = \tfrac{1}{4}(0 \;\; 0 \;\; -4 \;\; [4q(x) - p(x)^2 - 2p'(x)] \;\; 2p(x) \;\; 4) \tag{84}$$

and D_x is the operator matrix

$$D_x = \begin{pmatrix} 1 \\ \dfrac{\partial}{\partial x} - \dfrac{p(x)}{2} \\ \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2} - \dfrac{3}{4}p(x)\dfrac{\partial}{\partial x} + \dfrac{3p(x)^2 - 4q(x)}{8} \\ d_1 \\ d_2 \\ \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2} - \dfrac{1}{4}p(x)\dfrac{\partial}{\partial x} + \dfrac{4q(x) - 4p'(x) - p(x)^2}{8} \end{pmatrix}, \tag{85}$$

where

$$\begin{aligned} d_1 = {} & \frac{1}{2}\frac{\partial^4}{\partial x^4} - \frac{1}{4}p(x)\frac{\partial^3}{\partial x^3} + \frac{4q(x) - 5p'(x) - p(x)^2}{4}\frac{\partial^2}{\partial x^2} \\ & + \frac{p(x)^3 - 4p(x)q(x) - 6p(x)p'(x) + 16q'(x) - 20p''(x)}{16}\frac{\partial}{\partial x} \\ & + \frac{2q(x)^2 - 4s(x) - 3q(x)p'(x) - p(x)p''(x) + 2q''(x) - 2p'''(x)}{4} \\ & + \frac{p(x)^4 - 8p(x)^2q(x) + 6p(x)^2p'(x)}{32}, \end{aligned} \tag{86a}$$

and

$$d_2 = -\frac{1}{2}\frac{\partial^3}{\partial x^3} + \frac{1}{2}p(x)\frac{\partial^2}{\partial x^2} + \frac{3p'(x) - 2q(x)}{4}\frac{\partial}{\partial x} + \frac{4p(x)q(x) - p(x)^3 - 8q'(x) + 8p''(x)}{16}. \tag{86b}$$

Here, the partial derivatives are applied to whatever follows. Thus, D_x can be applied to a single function, producing a column matrix, or can be applied to a row matrix, producing a matrix with 6 rows. Also, the variable x can be replaced by any variable, so that D_ξ would have all partial derivatives taken with respect to ξ.

At first it would seem that the case where $Q(x) = 0$ would be rather rare. However, in both the Euler–Bernoulli beam and Timoshenko beam equations, the $Q(x)$ in Eq. (80) is exactly zero. Before continuing with the proof, let us demonstrate how Theorem 1 is used. We find that for the Timoshenko equation (65), $p(x) = r(x) = 0$,

$$q(x) = -\lambda^2\rho\left(\frac{1}{kG} + \frac{1}{E}\right) \quad \text{and} \quad s(x) = \left(\frac{\rho^2\lambda^4}{kGE} + \frac{m\lambda^2}{EI}\right). \tag{87a,b}$$

Eq. (81) gives us

$$16f^{v}(L_j) - 32\lambda^2\rho\left(\frac{1}{kG} + \frac{1}{E}\right)f'''(L_j) + \left(16\lambda^4\rho^2\left(\frac{1}{kG} + \frac{1}{E}\right)^2 - 64\left(\frac{\rho^2\lambda^4}{kGE} + \frac{m\lambda^2}{EI}\right)\right)f'(L_j) = 0. \tag{88}$$

Solving this equation with the initial conditions $f(\xi)=f'(\xi)=f''(\xi)=f'''(\xi)=0$ and $f^{\text{iv}}(\xi)=2$ yields the unique solution

$$f=\left(\frac{\rho}{Gk\psi}+\frac{\rho}{E\psi}-\frac{2\rho}{E\psi}\sqrt{\frac{mE}{\lambda^2\rho^2 I}+\frac{E}{Gk}}\right)\cosh\left(\lambda(L_j-\xi)\sqrt{\frac{\rho}{Gk}+\frac{\rho}{E}+\frac{2\rho}{E}\sqrt{\frac{mE}{\lambda^2\rho^2 I}+\frac{E_j}{Gk}}}\right)$$
$$-\left(\frac{\rho}{Gk\psi}+\frac{\rho}{E\psi}+\frac{2\rho}{E\psi}\sqrt{\frac{mE}{\lambda^2\rho^2 I}+\frac{E_j}{Gk}}\right)\cosh\left(\lambda(L_j-\xi)\sqrt{\frac{\rho}{Gk}+\frac{\rho}{E}-\frac{2\rho}{E}\sqrt{\frac{mE}{\lambda^2\rho^2 I}+\frac{E}{Gk}}}\right)$$
$$+\frac{4\rho}{E\psi}\sqrt{\frac{mE}{\lambda^2\rho^2 I}+\frac{E}{Gk}}, \tag{89}$$

where

$$\psi=\lambda^4\frac{2\rho}{E}\sqrt{\frac{mE}{\lambda^2\rho^2 I}+\frac{E}{Gk}}\left(\left(\frac{\rho}{Gk}+\frac{\rho}{E}\right)^2-\frac{4\rho^2}{E^2}\left(\frac{mE}{\lambda^2\rho^2 I}+\frac{E}{Gk}\right)\right). \tag{90}$$

Then f is indeed a symmetric function of L_j and ξ. The row matrix G for this example would be given by

$$G=\begin{pmatrix}0 & 0 & -1 & \left(\frac{-\lambda^2\rho}{kG}+\frac{-\lambda^2\rho}{E}\right) & 0 & 1\end{pmatrix}, \tag{91}$$

while the operator D_x would simplify to

$$D_x=\begin{pmatrix} 1 \\ \frac{\partial}{\partial x} \\ \frac{1}{2}\frac{\partial^2}{\partial x^2}+\left(\frac{\lambda^2\rho}{2kG}+\frac{\lambda^2\rho}{2E}\right) \\ \frac{1}{2}\frac{\partial^4}{\partial x^4}+\left(\frac{-\lambda^2\rho}{kG}+\frac{-\lambda^2\rho}{E}\right)\frac{\partial^2}{\partial x^2}+\left(\frac{\lambda^4\rho^2}{2k^2G^2}+\frac{\lambda^4\rho^2}{2E^2}-\frac{m\lambda^2}{EI}\right) \\ -\frac{1}{2}\frac{\partial^3}{\partial x^3}+\left(\frac{\lambda^2\rho}{2kG}+\frac{\lambda^2\rho}{2E}\right)\frac{\partial}{\partial x} \\ \frac{1}{2}\frac{\partial^2}{\partial x^2}+\left(\frac{-\lambda^2\rho}{2kG}+\frac{-\lambda^2\rho}{2E}\right)\end{pmatrix}. \tag{92}$$

Then Eq. (83) will find $P_j(L_j,\xi)$ for the Timoshenko beam. Finally, setting $\xi=0$ reproduces Eq. (69). Note that Eq. (89) contains all of the information to analyze a Timoshenko beam structure, yet is much more concise than Eq. (69).

Proof of Theorem 1. The key to the proof is to define $f(L_i,\xi,\lambda)$ in terms of one of the entries of the matrix P_j, and then use this definition to show that it possesses all of the required properties. If we let $f(L_i,\xi,\lambda)=P_{j,14}(L_i,\xi,\lambda)/\sqrt{\Delta_j(L_i,\xi)}$, then since the element $P_{j,14}$ is fixed by the transformation $P_j\to J\cdot P_j^{\mathrm{T}}\cdot J$, Eq. (74) shows that $f(L_i,\xi,\lambda)$ would indeed be a symmetrical function in L_j and ξ.

To show that $f(L_i,\xi)$ satisfies the initial conditions, we use the fact that P_j becomes the identity matrix when $L_j=\xi$. That is,

$$P_{j,uv}\big|_{L_j=\xi}=\begin{cases}0 & \text{if } u\neq v,\\ 1 & \text{if } u=v.\end{cases} \tag{93}$$

Using Eqs. (77a–f), one can find any number of partial derivatives of any element of P_j in terms of an expression without derivatives. In particular, $P_{j,14}$ is seen to satisfy Eqs. (82a,b), and since $\Delta(\xi,\xi)=1$, $f(L_j,\xi,\lambda)$ will also satisfy Eqs. (82a,b).

Again using Eq. (93), it is clear that the $C_v(\xi)$ in Eq. (79) is zero when $v = 4$ and $Q = 0$. Then, using Eq. (79) with Eqs. (77a–f), we can find a fifth-order equation for which $P_{j,14}$ is a solution:

$$\begin{aligned}
&4\frac{\partial^5}{\partial L_j^5}P_{j,14} + 10p(L_j)\frac{\partial^4}{\partial L_j^4}P_{j,14} + (7p(L_j)^2 + 8q(L_j) + 8p'(L_j))\frac{\partial^3}{\partial L_j^3}P_{j,14}\\
&\quad + (14p(L_j)q(L_j) - 4r(L_j) + 16q'(L_j))\frac{\partial^2}{\partial L_j^2}P_{j,14}\\
&\quad + (5p(L_j)^2q(L_j) - p(L_j)^4 + 4q(L_j)^2 - 2p(L_j)r(L_j) - 16s(L_j) - 9p(L_j)^2p'(L_j) + 4q(L_j)p'(L_j) - 6p'(L_j)^2\\
&\quad + 18p(L_j)q'(L_j) - 10p(L_j)p''(L_j) + 12q''(L_j) - 4p'''(L_j))\frac{\partial}{\partial L_j}P_{j,14}\\
&\quad + (4p(L_j)q(L_j)^2 - p(L_j)^3q(L_j) + p(L_j)^2r(L_j) - 4q(L_j)r(L_j) - 8p(L_j)s(L_j) - 2p(L_j)^3p'(L_j)\\
&\quad - 4p(L_j)q(L_j)p'(L_j) + 2r(L_j)p'(L_j) - 6p(L_j)p'(L_j)^2 + 3p(L_j)^2q'(L_j) + 8q(L_j)q'(L_j) + 2p'(L_j)q'(L_j)\\
&\quad - 8s'(L_j) - 6p(L_j)^2p''(L_j) - 2q(L_j)p''(L_j) - 8p'(L_j)p''(L_j) + 8p(L_j)q''(L_j) - 6p(L_j)p'''(L_j)\\
&\quad + 4q'''(L_j) - 2p^{\text{iv}}(L_j))P_{j,14} = 0. \qquad (94)
\end{aligned}$$

Making the substitution $P_{j,14} = f\sqrt{\Delta_j}$ yields Eq. (81). Since there is a unique solution which also satisfies Eq. (93), we see that the two definitions of $f(L_j, \xi, \lambda)$ are equivalent.

Finally, we must show how to express the entire matrix P_j in terms of $f(L_j, \xi, \lambda)$. It is clear that $P_{j,14}$ can be expressed easily, and in fact Eqs. (77a–f) allows us to express the fourth column of P_j in terms of $f(L_j, \xi, \lambda)$. But to get the rest of the matrix P_j, we apply the symmetry property of Eq. (74). For example, if we apply Eq. (74) to the element $P_{j,15}$, we find that

$$\Delta_j P_{j,15}(\xi, L_j) = P_{j,24}(L_j, \xi) = \frac{\partial}{\partial L_j}P_{j,14} = \frac{\partial}{\partial L_j}(f\sqrt{\Delta_j}) = -\sqrt{\Delta_j}\,\frac{p(L_j)}{2}f + \sqrt{\Delta_j}\frac{\partial}{\partial L_j}f. \qquad (95)$$

Here, we used that $\partial\Delta_j/\partial L_j = -p(L_j)$. Now, exchanging the variables L_j and ξ, and using the fact that $\Delta(\xi, L_j) = \Delta(L_j, \xi)^{-1}$, we get

$$P_{j,15}(L_j, \xi) = -\sqrt{\Delta_j}\frac{p(\xi)}{2}f + \sqrt{\Delta_j}\frac{\partial}{\partial \xi}f, \qquad (96)$$

where we use the symmetry property of $f(L_j, \xi) = f(\xi, L_j)$. In this way, we can determine the first row of P_j in terms of f.

Finding the element $P_{j,36}$ is a bit trickier. When we plug in the initial conditions Eq. (93) into Eq. (79) with $Q = 0$, we find that when $v = 6$, C_v is $-1/2$. Thus,

$$P_{j,36} = \frac{1}{2}\frac{\partial^2}{\partial L_j^2}P_{j,16} - \frac{1}{4}p(L_j)\frac{\partial}{\partial L_j}P_{j,16} + \frac{1}{4}p'(L_j)P_{j,16} + \frac{1}{8}p(L_j)^2P_{j,16} - \frac{1}{2}q(L_j)P_{j,16} - \frac{1}{2}\sqrt{\Delta_j}. \qquad (97)$$

Now we can use Eqs. (77a–f) to find the sixth column of P_j in terms of f, which when "rotated" through Eq. (74) also gives us the third row of P_j. With both the first and the third rows determined in terms of $f(L_j, \xi, \lambda)$, Eqs. (77a–f) determine the whole matrix. *Mathematica* can verify that, in fact, Eq. (83) produces the exact same result, so the theorem is proved. The fact that P_j factors nicely in Eq. (83) is not too surprising given that, first of all, P_j becomes a linear (but not homogeneous) function of $f(L_i, \xi, \lambda)$, and secondly, going up and down a column always uses derivatives with respect to L_i, whereas going across a row uses derivatives with respect to ξ. □

6. The general case $Q \neq 0$

The proof to Theorem 1 introduced the idea that Eq. (74) can be used to "rotate" any equation involving elements in the same column of P_j, and produce an equation involving elements in the same row. We can apply this technique to Eq. (79), and we find that

$$\begin{aligned}\Delta_j P_{j,\bar{v}6}(\xi, L_j) = P_{j,3v}(L_j,\xi) &= \frac{1}{2}\frac{\partial^2}{\partial L_j^2}P_{j,1v}(L_j,\xi) - \frac{1}{4}p(L_j)\frac{\partial}{\partial L_j}P_{j,1v}(L_j,\xi) \\ &+ \left(\frac{1}{4}p'(L_j) + \frac{1}{8}p(L_j)^2 - \frac{1}{2}q(L_j)\right)P_{j,1v}(L_j,\xi) \\ &+ \frac{1}{16}\sqrt{\Delta_j}\int \Delta_j^{-1/2}Q(L_j)P_{j,1v}(L_j,\xi)\,\mathrm{d}L_j + C_v(\xi)\sqrt{\Delta_j}. \end{aligned} \tag{98}$$

Here, $\bar{v} = v + 3$ or $v - 3$, whichever is in the range of 1–6. Exchanging L_j and ξ causes Δ_j to become Δ_j^{-1}, so we have

$$\begin{aligned}\Delta_j^{-1}P_{j,\bar{v}6}(L_j,\xi) = \frac{1}{2}\frac{\partial^2}{\partial \xi^2}P_{j,1v}(\xi, L_j,) - \frac{1}{4}p(\xi)\frac{\partial}{\partial \xi}P_{j,1v}(\xi, L_j) + \left(\frac{1}{4}p'(\xi) + \frac{1}{8}p(\xi)^2 - \frac{1}{2}q(\xi)\right)P_{j,1v}(\xi, L_j) \\ + \frac{1}{16}\Delta_j^{-1/2}\int \Delta_j^{1/2}Q(\xi)P_{j,1v}(\xi, L_j)\,\mathrm{d}\xi + C_v(L_j)\Delta_j^{-1/2}. \end{aligned} \tag{99}$$

But $P_{j,1v}(\xi, L_j) = P_{j,\bar{v}4}(L_j,\xi)/\Delta_j$. Making this substitution, and finally multiplying by Δ_j gives us

$$\begin{aligned}P_{j,\bar{v}6} = \frac{1}{2}\frac{\partial^2}{\partial \xi^2}P_{j,\bar{v}4} - \frac{5}{4}p(\xi)\frac{\partial}{\partial \xi}P_{j,\bar{v}4} - \frac{1}{4}p'(\xi)P_{j,\bar{v}4} + \frac{7}{8}p(\xi)^2P_{j,\bar{v}4} - \frac{1}{2}q(\xi)P_{j,\bar{v}4} \\ + \frac{1}{16}\sqrt{\Delta_j}\int \Delta_j^{-1/2}Q(\xi)P_{j,\bar{v}4}\,\mathrm{d}\xi + C_v(L_j)\sqrt{\Delta_j}. \end{aligned} \tag{100}$$

Eqs. (79) and (100) are the key to extending Theorem 1 to the case where $Q \neq 0$.

Theorem 2. *If the coefficients of Eq.* (4) *are such that* $Q \neq 0$, *where* $Q(x)$ *is defined in Eq.* (80), *then we can let* $f(L_j, \xi, \lambda)$ *be the unique solution to the sixth-order ordinary differential equation*

$$\begin{aligned} &16R(L_j)\frac{\partial^6 f}{\partial L_j^6} + 80R'(L_j)\frac{\partial^5 f}{\partial L_j^5} + ((32q - 48p' - 12p^2)R(L_j) + 160R''(L_j))\frac{\partial^4 f}{\partial L_j^4} \\ &+ ((48q' - 72p'' - 36pp')R(L_j) + (96q - 144p' - 36p^2)R'(L_j) + 160R'''(L_j))\frac{\partial^3 f}{\partial L_j^3} \\ &+ ((3p^4 - 16p^2q + 16q^2 + 16pr - 64s + 12p^2p' - 32qp' - 12(p')^2 - 36pp'' + 48q'' - 56p''')R(L_j) \\ &+ (16pq - 4p^3 - 32r - 96pp' + 128q' - 160p'')R'(L_j) + (96q - 144p' - 36p^2)R''(L_j) + 80R^{\mathrm{iv}}(L_j))\frac{\partial^2 f}{\partial L_j^2} \\ &+ ((3p^3p' - 12pqp' + 8rp' + 6p(p')^2 - 4p^2q' + 16qq' - 16p'q' - 32s' - 16qp'' - 12p'p'' + 8pq'' \\ &- 16pp''' + 16q''' - 16p^{\mathrm{iv}})R(L_j) \\ &+ (2p^4 - 12p^2q + 16q^2 + 8pr - 64s + 6p^2p' - 32qp' - 12(p')^2 + 8pq' - 40pp'' + 48q'' - 56p''')R'(L_j) \\ &+ (8pq - 2p^3 - 16r - 48pp' + 64q' - 80p'')R''(L_j) + (32q - 12p^2 - 48p')R'''(L_j) \\ &+ 16R^{v}(L_j))\frac{\partial f}{\partial L_j} - \frac{Q(L_j)}{4}f = 0 \end{aligned} \tag{101}$$

using the initial conditions $f(\xi, \xi) = -128$,

$$\left.\frac{\partial}{\partial L_j}f\right|_{L_j=\xi} = \left.\frac{\partial^2}{\partial L_j^2}f\right|_{L_j=\xi} = \left.\frac{\partial^3}{\partial L_j^3}f\right|_{L_j\xi} = \left.\frac{\partial^4}{\partial L_j^4}f\right|_{L_j=\xi} = \left.\frac{\partial^5}{\partial L_j^5}f\right|_{L_j=\xi} = 0. \tag{102}$$

Here, $R(L_j) = 1/Q(L_j)$, and the functions p, q, r, and s are all evaluated at L_j. Then $f(L_j, \xi)$ will again be a symmetrical function, $f(L_j, \xi, \lambda) = f(\xi, L_j, \lambda)$, and we can construct the exterior matrix P_j corresponding to

$W_i = M_j$ by the equation

$$P_j = \sqrt{\Delta_j} D_{L_j}((J \cdot D_\xi(f(L_j, \xi)))^{\mathrm{T}}), \tag{103}$$

where we have

$$D_x = \begin{pmatrix} R(x)\dfrac{\partial}{\partial x} \\ R(x)\dfrac{\partial^2}{\partial^2 x} + \dfrac{-p(x)R(x) + 2R'(x)}{2}\dfrac{\partial}{\partial x} \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} \tag{104}$$

and

$$d_3 = \frac{R(x)}{2}\frac{\partial^3}{\partial x^3} + \frac{-3p(x)R(x) + 4R'(x)}{4}\frac{\partial^2}{\partial x^2} + \frac{3p(x)^2R(x) - 4q(x)R(x) - 6p(x)R'(x) + 4R''(x)}{8}\frac{\partial}{\partial x} + \frac{1}{16}, \tag{105a}$$

$$\begin{aligned} d_4 = {} & \frac{R(x)}{2}\frac{\partial^5}{\partial x^5} + \frac{-p(x)R(x) + 8R'(x)}{4}\frac{\partial^4}{\partial x^4} \\ & + \frac{-p(x)^2R(x) + 4q(x)R(x) - 5p'(x)R(x) - 3p(x)R'(x) + 12R''(x)}{4}\frac{\partial^3}{\partial x^3} \\ & + \frac{p(x)^3 - 4p(x)q(x) + 4r(x) + 4q'(x) - 8p''(x)}{8}R(x)\frac{\partial^2}{\partial x^2} \\ & + \frac{8q(x)R'(x) - 2p(x)^2R'(x) - 10p'(x)R'(x) - 3p(x)R''(x) + 8R'''(x)}{4}\frac{\partial^2}{\partial x^2} \\ & + \frac{p(x)^3 - 4p(x)q(x) - 6p(x)p'(x) + 16q'(x) - 20p''(x)}{16}R'(x)\frac{\partial}{\partial x} \\ & + \frac{4q(x)R''(x) - p(x)^2R''(x) - 5p'(x)R''(x) - p(x)R'''(x) + 2R^{\mathrm{iv}}(x)}{4}\frac{\partial}{\partial x} \\ & + \frac{2q(x)^2 - 4s(x) - 3q(x)p'(x) - p(x)p''(x) + 2q''(x) - 2p'''(x)}{4}R(x)\frac{\partial}{\partial x} \\ & + \frac{p(x)^4 - 8p(x)^2q(x) + 6p(x)^2p'(x)}{32}R(x)\frac{\partial}{\partial x} + \frac{p(x)^2 - 4q(x) + 2p'(x)}{64}, \end{aligned} \tag{105b}$$

$$\begin{aligned} d_5 = {} & \frac{-R(x)}{2}\frac{\partial^4}{\partial x^4} + \frac{p(x)R(x) - 3R'(x)}{2}\frac{\partial^3}{\partial x^3} + \frac{-2q(x)R(x) + 3p'(x)R(x) + 4p(x)R'(x) - 6R''(x)}{4}\frac{\partial^2}{\partial x^2} \\ & + \frac{4p(x)q(x) - p(x)^3 - 4r(x) - 3p(x)p'(x) + 2p''(x)}{8}R(x)\frac{\partial}{\partial x} \\ & + \frac{3p'(x)R'(x) - 2q(x)R'(x) + 2p(x)R''(x) - 2R'''(x)}{4}\frac{\partial}{\partial x} - \frac{p(x)}{32}, \end{aligned} \tag{105c}$$

$$d_6 = \frac{R(x)}{2}\frac{\partial^3}{\partial x^3} + \frac{-p(x)R(x) + 4R'(x)}{4}\frac{\partial^2}{\partial x^2} + \frac{(4q(x) - p(x)^2 - 4p'(x))R(x) - 2p(x)R'(x) + 4R''(x)}{8}\frac{\partial}{\partial x} - \frac{1}{16}. \tag{105d}$$

Eq. (101) (or Eq. (81) in the case $Q = 0$) is called the *exterior equation* for the differential equation (4). This equation is essentially the original equation lifted up into the exterior algebra space.

Proof. Once again, we will define $f(L_j,\xi)$ in a totally different way using the matrix P_j, and then show the two definitions are equivalent. In order to be able to evaluate the integrals in Eq. (79) and Eq. (100), we let

$$f(L_j,\xi,\lambda)=\int_0^{\xi}\int_0^{L_j}\Delta_j^{-1/2}(s,t)P_{j,14}(s,t,\lambda)Q(s)Q(t)\,\mathrm{d}s\,\mathrm{d}t+C_1(L_j)+C_2(\xi)+C_3, \tag{106}$$

where the functions $C_1(L_j)$, $C_2(\xi)$, and the constant C_3 will be determined later. Then $P_{j,14}$ can be expressed in terms of this new function

$$P_{j,14}(L_j,\xi,\lambda)=\frac{\sqrt{\Delta_j(L_j,\xi)}}{Q(L_j)Q(\xi)}\frac{\partial^2}{\partial L_j\,\partial\xi}f(L_j,\xi,\lambda), \tag{107}$$

but also the integrals in Eq. (79) and Eq. (100) can also be done. If fact, we can choose $C_2(\xi)$ so that

$$P_{j,34}=\frac{1}{2}\frac{\partial^2}{\partial L_j^2}P_{j,14}-\frac{1}{4}p(L_j)\frac{\partial}{\partial L_j}P_{j,14}+\frac{1}{4}p'(L_j)P_{j,14}+\frac{1}{8}p(L_j)^2P_{j,14}-\frac{1}{2}q(L_j)P_{j,14}+\frac{\sqrt{\Delta_j}}{16Q(\xi)}\frac{\partial}{\partial\xi}f(L_j,\xi). \tag{108}$$

Likewise, we can choose $C_1(L_j)$ so that

$$P_{j,16}=\frac{1}{2}\frac{\partial^2}{\partial\xi^2}P_{j,14}-\frac{5}{4}p(\xi)\frac{\partial}{\partial\xi}P_{j,14}-\frac{1}{4}p'(\xi)P_{j,14}+\frac{7}{8}p(\xi)^2P_{j,14}-\frac{1}{2}q(\xi)P_{j,14}+\frac{\sqrt{\Delta_j}}{16Q(L_j)}\frac{\partial}{\partial L_j}f(L_j,\xi), \tag{109}$$

so we now have determined $f(L_j,\xi)$ up to an arbitrary constant C_3. In order to find $P_{j,36}$, we have to integrate either

$$\int\Delta_j^{-1/2}Q(L_j)P_{j,16}\,\mathrm{d}L_j\quad\text{or}\quad\int\Delta_j^{-1/2}Q(\xi)P_{j,34}\,\mathrm{d}\xi. \tag{110a,b}$$

To do these integrals, we express $P_{j,34}$ and $P_{j,16}$ solely in terms of the function $f(L_j,\xi)$.

$$\begin{aligned}P_{j,34}=\frac{\sqrt{\Delta_j}}{Q(\xi)}\Bigg(&\frac{1}{16}\frac{\partial}{\partial\xi}f(L_j,\xi)+\left(\frac{3p(L_j)^2}{8Q(L_j)}-\frac{q(L_j)}{2Q(L_j)}+\frac{3p(L_j)Q'(L_j)}{4Q(L_j)^2}+\frac{Q'(L_j)^2}{Q(L_j)^3}-\frac{Q''(L_j)}{2Q(L_j)^2}\right)\frac{\partial^2}{\partial L_j\,\partial\xi}f(L_j,\xi)\\&+\left(\frac{-3p(L_j)}{4Q(L_j)}-\frac{Q'(L_j)}{Q(L_j)^2}\right)\frac{\partial^3}{\partial L_j^2\partial\xi}f(L_j,\xi)+\frac{1}{2Q(L_j)}\frac{\partial^4}{\partial L_j^3\partial\xi}f(L_j,\xi)\Bigg).\\P_{j,16}=\frac{\sqrt{\Delta_j}}{Q(L_j)}\Bigg(&\frac{1}{16}\frac{\partial}{\partial L_j}f(L_j,\xi)+\left(\frac{3p(\xi)^2}{8Q(\xi)}-\frac{q(\xi)}{2Q(\xi)}+\frac{3p(\xi)Q'(\xi)}{4Q(\xi)^2}+\frac{Q'(\xi)^2}{Q(\xi)^3}-\frac{Q''(\xi)}{2Q(\xi)^2}\right)\frac{\partial^2}{\partial L_j\,\partial\xi}f(L_j,\xi)\\&+\left(\frac{-3p(\xi)}{4Q(\xi)}-\frac{Q'(\xi)}{Q(\xi)^2}\right)\frac{\partial^3}{\partial L_j\,\partial\xi^2}f(L_j,\xi)+\frac{1}{2Q(\xi)}\frac{\partial^4}{\partial L_j\,\partial\xi^3}f(L_j,\xi)\Bigg).\end{aligned} \tag{111a, b}$$

We then find, expressing the answer in terms of both $f(L_j,\xi)$ and $P_{j,14}$, that

$$\begin{aligned}P_{j,36}=&\frac{7p(L_j)^2p(\xi)^2-28q(L_j)p(\xi)^2-4p(L_j)^2q(\xi)+16q(L_j)q(\xi)}{64}P_{j,14}(L_j,\xi)\\&+\frac{7p'(L_j)p(\xi)^2-4p'(L_j)q(\xi)-p(L_j)^2p'(\xi)+4q(L_j)p'(\xi)-2p'(L_j)p'(\xi)}{32}P_{j,14}(L_j,\xi)\\&+\frac{p(\xi)(20q(L_j)-10p'(L_j)-5p(L_j)^2)}{32}\frac{\partial}{\partial\xi}P_{j,14}(L_j,\xi)+\frac{p(L_j)(4q(\xi)+2p'(\xi)-7p(\xi)^2)}{32}\frac{\partial}{\partial L_j}P_{j,14}(L_j,\xi)\\&+\frac{p(L_j)^2-4q(L_j)+2p'(L_j)}{16}\frac{\partial^2}{\partial\xi^2}P_{j,14}(L_j,\xi)+\frac{7p(\xi)^2-4q(\xi)-2p'(\xi)}{16}\frac{\partial^2}{\partial L_j^2}P_{j,14}(L_j,\xi)\end{aligned}$$

$$+\frac{5p(L_j)p(\xi)}{16}\frac{\partial^2}{\partial L_j\,\partial\xi}P_{j,14}(L_j,\xi)-\frac{p(L_j)}{8}\frac{\partial^3}{\partial L_j\,\partial\xi^2}P_{j,14}(L_j,\xi)-\frac{5p(\xi)}{8}\frac{\partial^3}{\partial L_j^2\,\partial\xi}P_{j,14}(L_j,\xi)$$
$$+\frac{1}{4}\frac{\partial^4}{\partial L_j^2\,\partial\xi^2}P_{j,14}(L_j,\xi)+\frac{\sqrt{\Delta_j}}{32Q(L_j)}\frac{\partial^3}{\partial L_j^3}f(L_j,\xi)+\frac{\sqrt{\Delta_j}}{32Q(\xi)}\frac{\partial^3}{\partial\xi^3}f(L_j,\xi)$$
$$+\sqrt{\Delta_j}\left(\frac{-3p(L_j)}{64Q(L_j)}-\frac{Q'(L_j)}{16Q(L_j)^2}\right)\frac{\partial^2}{\partial L_j^2}f(L_j,\xi)+\sqrt{\Delta_j}\left(\frac{-3p(\xi)}{64Q(\xi)}-\frac{Q'(\xi)}{16Q(\xi)^2}\right)\frac{\partial^2}{\partial\xi^2}f(L_j,\xi)$$
$$+\sqrt{\Delta_j}\left(\frac{3p(L_j)^2}{128Q(L_j)}-\frac{q(L_j)}{32Q(L_j)}+\frac{3p(L_j)Q'(L_j)}{64Q(L_j)^2}+\frac{Q'(L_j)^2}{16Q(L_j)^3}-\frac{Q''(L_j)}{32Q(L_j)^2}\right)\frac{\partial}{\partial L_j}f(L_j,\xi)$$
$$+\sqrt{\Delta_j}\left(\frac{3p(\xi)^2}{128Q(\xi)}-\frac{q(\xi)}{32Q(\xi)}+\frac{3p(\xi)Q'(\xi)}{64Q(\xi)^2}+\frac{Q'(\xi)^2}{16Q(\xi)^3}-\frac{Q''(\xi)}{32Q(\xi)^2}\right)\frac{\partial}{\partial\xi}f(L_j,\xi)+\frac{\sqrt{\Delta_j}}{256}f(L_j,\xi). \tag{112}$$

If we use Eq. (79) to compute this equation, there is an extra $C_6(\xi)\sqrt{\Delta_j}$ term, where $C_6(\xi)$ is a function of only ξ. However, if we instead use Eq. (100) to derive Eq. (112), there is an extra $C_6(L_j)\sqrt{\Delta_j}$ term, where $C_6(L_j)$ depends only on L_j. Thus, the extra term would be of the form $k\sqrt{\Delta_j}$, where k depends neither on L_j nor ξ. But then we still have C_3 left undetermined in Eq. (106), so we can choose C_3 so that this extra term is absorbed into the last term of Eq. (112).

We now can use Eqs. (77a–f) to compute the entire matrix P_j in terms of the function $f(L_j,\xi)$ as we did in Theorem 1. *Mathematica* then can verify that Eq. (103) holds for this f. But we still need to show that this f is the same as the one defined in the theorem. We can do this by determining the new $f(L_j,\xi)$ from the matrix P_j. Using Eqs. (107)–(112), and then eliminating the derivatives via Eqs. (77a–f), we find that

$$\begin{aligned}f(L_j,\xi)=\Delta_j^{-1/2}&((8p(L_j)^2p'(\xi)+8p(\xi)^2p'(L_j))P_{j,14}-(16p(L_j)^2q(\xi)+16p(\xi)^2q(L_j))P_{j,14}\\&+4p(L_j)^2p(\xi)^2P_{j,14}+64q(L_j)q(\xi)P_{j,14}-(32p'(L_j)q(\xi)+32p'(\xi)q(L_j))P_{j,14}+16p'(L_j)p'(\xi)P_{j,14}\\&+(16p(L_j)^2P_{j,13}+16p(\xi)^2P_{j,64})-(64q(L_j)P_{j,13}+64q(\xi)P_{j,64})+(32p'(L_j)P_{j,13}+32p'(\xi)P_{j,64})\\&+(64q(L_j)P_{j,16}+64q(\xi)P_{j,34})-(16p(L_j)^2P_{j,16}+16p(\xi)^2P_{j,34})-(32p'(L_j)P_{j,16}+32p'(\xi)P_{j,34})\\&+(32q(L_j)p(\xi)P_{j,15}+32q(\xi)p(L_j)P_{j,24})-(8p(L_j)^2p(\xi)P_{j,15}+8p(\xi)^2p(L_j)P_{j,24})\\&-(16p'(L_j)p(\xi)P_{j,15}+16p'(\xi)p(L_j)P_{j,24})-(32p(L_j)P_{j,23}+32p(\xi)P_{j,65})\\&+(32p(L_j)P_{j,26}+32p(\xi)P_{j,35})-(64P_{j,33}+64P_{j,66})+16p(L_j)p(\xi)P_{j,25}+64P_{j,63}+64P_{j,36}).\end{aligned} \tag{113}$$

Eq. (113) is grouped so as to emphasize the symmetric properties of $f(L_j,\xi)$. Using Eq. (74), it can be seen that $f(L_j,\xi)=f(\xi,L_j)$.

We can also find the sixth order equation for which $f(L_j,\xi)$ satisfies. By plugging Eqs. (107) and (111a,b) into Eqs. (77a–f), we derive Eq. (101). However, the last step in deriving Eq. (101) from Eq. (107) and Eqs. (111a,b) is to integrate with respect to ξ, so in principle there could be a non-homogeneous term $g(L_j)$ on the right-hand side of Eq. (101). However, by plugging Eq. (113) into Eq. (101), and simplifying via Eqs. (77a–f), we find that the non-homogeneous term is indeed zero, so Eq. (101) is correct. (The trick to avoid a huge mess is to realize that since $g(L_j)$ does not depend on ξ, one can set $\xi=L_j$ after taking the derivatives to simplify the expression.)

Using Eq. (113), we find that indeed $f(\xi,\xi)=-128$. Also, using Eq. (109) along with Eqs. (77a–f), we find that Eq. (102) also holds. Thus, the $f(L_j,\xi)$ we defined in the proof is the same as the one defined in the theorem. □

7. Bypassing the initial conditions

Unfortunately, the exterior Eqs. (81) and (101) can rarely be solved exactly. But since we can get by with an asymptotic approximation to P_j, we only need an approximate solution to $f(L_i,\xi)$, since the derivative operations will preserve the asymptotic estimates. That is, if we have an asymptotic approximation valid for $L_j>\xi$ as either $|\lambda|\to\infty$, or perhaps as some other perturbation parameter $\varepsilon\to 0$, then we can compute the asymptotic approximation for the whole matrix P_j, which in turn will allow us to compute the eigenvalues for the system. However, with only the approximate solutions, the initial conditions will be useless. However, we can use the symmetry property $f(L_j, \xi) = f(\xi, L_j)$ instead of the initial conditions to determine the arbitrary constants. Let us break down the steps in how to do this.

Suppose we can find the linearly independent approximate solutions to Eq. (81) or Eq. (101) as a function of $L_j > \xi$ valid as either $|\lambda| \to \infty$ or as $\varepsilon \to 0$. We can then express an approximation to $f(L_j, \xi, \lambda)$ as

$$f(L_j, \xi, \lambda) \sim C_1(\xi, \lambda) f_1(L_j, \lambda) + C_2(\xi, \lambda) f_2(L_j, \lambda) + \cdots + C_k(\xi, \lambda) f_k(L_j, \lambda). \tag{114}$$

Because the asymptotic approximation is not valid if $\xi = L_j$, the initial conditions are useless in determining C_1 through C_k. However, $f(L_j, \xi, \lambda)$ was designed to be symmetrical in the variables L_j and ξ. Since we also know the linear dependence on ξ, we have that

$$f(L_j, \xi, \lambda) \sim F(\xi, \lambda) \cdot S(\lambda) \cdot F(L_j, \lambda)^{\mathrm{T}}, \tag{115}$$

where $F(x, \lambda)$ is the row vector $(f_1(x), f_2(x), \ldots, f_k(x))$, and $S(\lambda)$ is a symmetric matrix that does not depend on either ξ nor L_j.

At this point, we can use Eq. (83) or Eq. (103) to calculate an asymptotic approximation for P_j. We will call this approximate matrix $\widetilde{P}_j(L_j, \xi, \lambda)$. In fact, because $S(\lambda)$ and $F(L_j)$ do not depend on ξ, we see that $D_\xi(F(\xi) \cdot S(\lambda) \cdot F(L_j))^{\mathrm{T}} = (D_\xi(F(\xi)) \cdot S(\lambda) \cdot F(L_j)^{\mathrm{T}}$. Continuing in this way, we see that Eq. (83) tells us

$$P_j \sim \sqrt{\Delta_j}\, M(L_j) \cdot S(\lambda) \cdot M(\xi)^{\mathrm{T}} \cdot J - \sqrt{\Delta_j}\, G(L_j)^{\mathrm{T}} \cdot G(\xi) \cdot J/2, \tag{116}$$

where $M(x) = D_x(F(x))$.

However, we still have the answer in terms of the undetermined coefficients of the matrix $S(\lambda)$. But we also have the result from Proposition 2:

$$P_j(L_j, \lambda)^{\mathrm{T}} \cdot J \cdot P_j(L_j, \lambda) \cdot J = \Delta_j I. \tag{117}$$

This means that

$$\Delta_j^{-1} \widetilde{P}_j(L_j, \lambda)^{\mathrm{T}} \cdot J \cdot \widetilde{P}_j(L_j, \lambda) = O(1) \quad \text{as } |\lambda| \to \infty \tag{118}$$

meaning that the exponentially increasing functions must cancel on the right-hand side. We also have the result from Eq. (75), indicating that if $\xi_1 > \xi_2 > \xi_3$, then $\widetilde{P}_j$ will be an approximation for P_j as $|\lambda| \to \infty$, so we have

$$\widetilde{P}_j(\xi_1, \xi_2, \lambda) \cdot \widetilde{P}_j(\xi_2, \xi_3, \lambda) \sim \widetilde{P}_j(\xi_1, \xi_3, \lambda) \quad \text{as } |\lambda| \to \infty. \tag{119}$$

We now can take advantage of Eqs. (118) and (119). Plugging Eq. (116) into Eq. (118), and assuming that the elements of G are subdominant to the elements of F as $|\lambda| \to \infty$ (a fairly safe assumption), we obtain

$$M(\xi) \cdot S(\lambda) \cdot M(L_j)^{\mathrm{T}} \cdot J \cdot M(L_j) \cdot S(\lambda) \cdot M(\xi)^{\mathrm{T}} = O(1). \tag{120}$$

Also, Eq. (119) gives us

$$M(\xi_1) \cdot S(\lambda) \cdot M(\xi_2)^{\mathrm{T}} \cdot J \cdot M(\xi_2) \cdot S(\lambda) \cdot M(\xi_3)^{\mathrm{T}} \sim M(\xi_1) \cdot S(\lambda) \cdot M(\xi_3)^{\mathrm{T}}. \tag{121}$$

If the matrix $M(x)$ has full rank (another safe assumption, since $f_1(x), f_2(x), \ldots, f_k(x)$ are linearly independent), this tells us that

$$S(\lambda) \cdot M(\xi_2)^{\mathrm{T}} \cdot J \cdot M(\xi_2) \cdot S(\lambda) \sim S(\lambda). \tag{122}$$

This greatly simplifies Eq. (120) to

$$M(\xi) \cdot S(\lambda) \cdot M(\xi)^{\mathrm{T}} = O(1). \tag{123}$$

This equation will determine most of the undetermined coefficients of $S(\lambda)$. It is not hard to use Eq. (122) to solve for the remaining coefficients.

8. One final example

The Euler–Bernoulli equation for a beam with variable coefficients [13] is

$$\frac{\partial^2}{\partial x^2}\left(E_i I_i(x)\frac{\partial^2 u_i}{\partial x^2}\right) + m_i(x)\frac{\partial^2 u_i}{\partial t^2} = 0, \tag{124}$$

where now $m_i(x)$ and $I_i(x)$ can vary along the beam. Even in this generalized form, the Riesz bases condition is satisfied [16,23]. We will consider the special case where a beam with constant density ρ_i whose cross section at x is a solid circle of radius $\varepsilon_i x + \beta_i$, then $m_i(x) = \rho_i \pi(\varepsilon_i x + \beta_i)^2$, and the cross-sectional moment of inertial about the y-axis is $I_i(x) = \rho_i \pi(\varepsilon_i x + \beta_i)^4/4$. If we let $k_i = \sqrt[4]{(m_i(0)/(E_i I_i(0)))}$, then we get the fourth-order equation:

$$y_i^{iv} + \frac{8\varepsilon_i}{\varepsilon_i x + \beta_i} y_i''' + \frac{12\varepsilon_i^2}{(\varepsilon_i x + \beta_i)^2} y_i'' + \frac{\lambda^2 k_i^4 \beta_i^2}{(\varepsilon_i x + \beta_i)^2} y_i = 0. \tag{125}$$

This actually does have an exact solution in terms of Bessel functions [13]. If we let $\lambda = i\eta^2$ as in Section 3, then

$$y_i = \frac{1}{\varepsilon_i x + \beta}(c_1 I_2(\eta\theta_i(x)) + c_2 J_2(\eta\theta_i(x)) + c_3 K_2(\eta\theta_i(x)) + c_4 Y_2(\eta\theta_i(x)), \tag{126}$$

where $\theta_i(x) = 2k_i\sqrt{\varepsilon_i\beta_i x + \beta_i^2}/\varepsilon_i$. However, it would be difficult to generate $P_{j,14}$ from this solution. Besides, if the tapered beam were hollow, and the inside and outside radii of the cross sections were not a constant ratio, then the differential equation cannot even be solved via Bessel functions [13]. However, we find from Eq. (125) that $Q(x) = 0$. (This happens for all equations coming from Eq. (124).)

Applying Eq. (81) to Eq. (125), we find that $f(L_j, \xi)$ satisfies

$$16 f^{v}(x) - \frac{64 k_j^4 \beta_j^2 \lambda^2}{(\varepsilon_j x + \beta_j)^2} f'(x) + \frac{64\varepsilon_j k_j^4 \beta_j^2 \lambda^2}{(\varepsilon_j x + \beta_j)^3} f(x) = 0. \tag{127}$$

Letting $z = k_j^2 \beta_j \lambda(\varepsilon_j x + \beta_j)/\varepsilon_j^2$ simplifies this equation to

$$f^{v}(z) - \frac{4}{z^2} f'(z) + \frac{4}{z^3} f(z) = 0 \tag{128}$$

Note that as either $|\lambda| \to \infty$ or as $\varepsilon_j \to 0$, then $z \to \infty$. Thus, we need to find the asymptotic approximations to the solutions of Eq. (128) at the irregular singular point $z = \infty$. We substitute $f = e^{S(z)}$, $f' = e^{S(z)} S'(z)$, $f'' = e^{S(z)}(S''(z) + S'(z)^2)$, etc., and use the method of dominant balance [24, p. 80]. Along with the exact solution $S' = 1/z$, we find the approximate solutions

$$S'(z) \sim i^n\sqrt{\frac{2}{z}} + \frac{1}{z} + \frac{(-i)^n 15}{16\sqrt{2}} z^{-3/2} + \frac{(-1)^n 15}{32 z^2} + \cdots, \quad \text{where} \quad n = 1, 2, 3, \text{ or } 4. \tag{129}$$

Converting back to the original variables, we have the approximate solutions

$$f_n(x) = e^{2i^n k_j\sqrt{2\lambda(\varepsilon_j\beta_j x + \beta_j^2)}/\varepsilon_j}(\varepsilon_j x + \beta_j)\left(1 - \frac{(-i)^n 15\varepsilon_j}{8k_j\sqrt{2\lambda(\varepsilon_j\beta_j x + \beta_j^2)}} + \frac{(-1)^n 105\varepsilon_j^2}{128 k_j^2(2\lambda(\varepsilon_j\beta_j x + \beta_j^2))} + \cdots\right) \tag{130}$$

for $n = 1, 2, 3,$ or 4. The fifth solution, $(\varepsilon_j x + \beta_j)$, is subdominant to another solution regardless of the argument of λ, so it can be left out.

All of the differential operators of D_x will not change the exponentially growing component of the functions. Thus, for Eq. (123) to be satisfied, we must have

$$(e^{i\zeta}\ e^{-\zeta}\ e^{-i\zeta}\ e^{\zeta})\cdot\begin{pmatrix} C_1(\lambda) & C_2(\lambda) & C_3(\lambda) & C_4(\lambda)\\ C_2(\lambda) & C_5(\lambda) & C_6(\lambda) & C_7(\lambda)\\ C_3(\lambda) & C_6(\lambda) & C_8(\lambda) & C_9(\lambda)\\ C_4(\lambda) & C_7(\lambda) & C_9(\lambda) & C_{10}(\lambda)\end{pmatrix}\cdot\begin{pmatrix} e^{i\zeta}\\ e^{-\zeta}\\ e^{-i\zeta}\\ e^{\zeta}\end{pmatrix} = O(1)\quad \text{as } |\lambda|\to\infty, \tag{131}$$

where $\zeta = 2k_j\sqrt{2\lambda(\varepsilon_j\beta_j x + \beta_j^2)}/\varepsilon_j$. This quickly tells us that $C_1 = C_2 = C_4 = C_5 = C_6 = C_8 = C_9 = C_{10} = 0$. With only 2 unknowns left, it is easy to use Eq. (122) to find that C_3 and C_7 can either be 0 or $1/(8\beta_j^2k_j^4\lambda^2)$. But in order for $\widetilde{P}_j$ to end up with a rank of at least two, we find that both C_3 and C_7 must be non-zero. (We can also see this by noting that when $\varepsilon_j = 0$, the classical Euler–Bernoulli solution must be obtained.) Thus, we have a unique reasonable solution for $f(L_j,\xi)$, which can generate the matrix, namely

$$\begin{aligned} f(L_j,\xi) &= \frac{1}{8\beta_j^2k_j^4\lambda^2}(f_1(\xi)\ f_2(\xi)\ f_3(\xi)\ f_4(\xi))\cdot\begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\end{pmatrix}\cdot\begin{pmatrix} f_1(L_j)\\ f_2(L_j)\\ f_3(L_j)\\ f_4(L_j)\end{pmatrix}\\ &= \sum_{n=1}^{4}\frac{e^{2i^n k_j\beta_j\sqrt{2\lambda}(L_j-\xi)/(\sqrt{\varepsilon_j\beta_jL_j+\beta_j^2}+\sqrt{\varepsilon_j\beta_j\xi+\beta_j^2})}}{8\beta_j^2k_j^4\lambda^2(\varepsilon_jL_j+\beta_j)^{-1}(\varepsilon_j\xi+\beta_j)^{-1}}\\ &\quad\times\left(1-\frac{(-i)^n15\varepsilon_j}{8k_j\sqrt{2\lambda(\varepsilon_j\beta_jL_j+\beta_j^2)}}+\frac{(-i)^n15\varepsilon_j}{8k_j\sqrt{2\lambda(\varepsilon_j\beta_j\xi+\beta_j^2)}}+\cdots\right), \end{aligned} \tag{132}$$

where we rationalized the numerators in the exponential functions. If we now make the same assumptions that we did for the uniform Euler–Bernoulli equation, that $\pi/4\leqslant \arg\sqrt{\lambda}\leqslant\pi/2$ and $L_j>\xi$, then the two terms $n=1$ and 2 become exponentially small as $|\lambda|\to\infty$.

Dropping these two terms and setting $\xi = 0$ will create the exterior matrix $\widetilde{P}_j$. As a 6×6 matrix, this is hard to display, but Eq. (83) gives us bonus information. Because only two of the four exponential functions were kept under the assumptions $\pi/4\leqslant\sqrt{\lambda}\leqslant\pi/2$, we know that only two of the four non-zero entries of $S(\lambda)$ were important. This means that, no matter how many terms of the asymptotic series we keep, the matrix for $\widetilde{P}_j$ will have rank two. Dropping the terms of $f(L_j,\xi)$ that are exponentially small gives us

$$\widetilde{f}(L_j,\xi) = \frac{1}{8\beta_j^2k_j^4\lambda^2}(f_3(L_j)f_1(\xi)+f_4(L_j)f_2(\xi)). \tag{133}$$

Plugging this into Eq. (83) tells us that

$$\widetilde{P}_j\sim\frac{\sqrt{\Delta_j}}{8\beta_j^2k_j^4\lambda^2}(D_{L_j}(f_3(L_j))\cdot(J\cdot D_\xi(f_1(\xi)))^{\mathrm{T}}+D_{L_j}(f_4(L_j))\cdot(J\cdot D_\xi(f_2(\xi)))^{\mathrm{T}}). \tag{134}$$

Plugging in $\xi=0$, we can express

$$\widetilde{P}_j = \frac{1}{8\beta_j^2r_j^8\lambda^2k_j^4}U\cdot\begin{pmatrix} e^{z_j} & 0\\ 0 & e^{-iz_j}\end{pmatrix}\cdot V^{\mathrm{T}}, \tag{135}$$

where

$$U = \left(\frac{D_{L_j}(f_4(L_j))}{e^{2\sqrt{2\lambda}k_jr_j\beta_j/\varepsilon_j}}\middle|\frac{D_{L_j}(f_3(L_j))}{e^{-2i\sqrt{2\lambda}k_jr_j\beta_j/\varepsilon_j}}\right) \tag{136a}$$

and

$$V = J \cdot \left(\frac{D_\xi(f_2(\xi))}{\mathrm{e}^{-2\sqrt{2\lambda}k_j\beta_j/\varepsilon_j}} \middle| \frac{D_\xi(f_1(\xi))}{\mathrm{e}^{2\mathrm{i}\sqrt{2\lambda}k_j\beta_j/\varepsilon_j}} \right) \Bigg|_{\xi=0}. \tag{136b}$$

Here, we introduced the variables

$$r_j = \sqrt{\beta_j\varepsilon_j L_j + \beta_j^2}/\beta_j \quad \text{and} \quad z_j = 2k_j\beta_j\sqrt{2\lambda}L_j/(\sqrt{\varepsilon_j\beta_j L_j + \beta_j^2} + \beta_j). \tag{136a,b}$$

Evaluating U and V gives us

$$U = \frac{1}{8k_j r_j^3\sqrt{2\lambda}} \begin{pmatrix} 8\sqrt{2\lambda}\beta_j k_j r_j^5 - 15\varepsilon_j r_j^4 & 8\sqrt{2\lambda}\beta_j k_j r_j^5 - 15\mathrm{i}\varepsilon_j r_j^4 \\ 16\lambda\beta_j k_j^2 r_j^4 - 39\sqrt{2\lambda}\varepsilon_j k_j r_j^3 & -16\mathrm{i}\lambda\beta_j k_j^2 r_j^4 - 39\sqrt{2\lambda}\varepsilon_j k_j r_j^3 \\ 8\sqrt{2\lambda^3}\beta_j k_j^3 r_j^3 - 99\lambda\varepsilon_j k_j^2 r_j^2 & -8\sqrt{2\lambda^3}\beta_j k_j^3 r_j^3 + 99\mathrm{i}\lambda\varepsilon_j k_j^2 r_j^2 \\ 8\sqrt{2\lambda^5}\beta_j k_j^5 r_j - 63\lambda^2\varepsilon_j k_j^4 & 8\sqrt{2\lambda^5}\beta_j k_j^5 r_j - 63\mathrm{i}\lambda^2\varepsilon_j k_j^4 \\ -16\lambda^2\beta_j k_j^4 r_j^2 + 67\sqrt{2\lambda^3}\varepsilon_j k_j^3 r_j & -16\mathrm{i}\lambda^2\beta_j k_j^4 r_j^2 - 67\sqrt{2\lambda^3}\varepsilon_j k_j^3 r_j \\ 8\sqrt{2\lambda^3}\beta_j k_j^3 r_j^3 - 35\lambda\varepsilon_j k_j^2 r_j^2 & -8\sqrt{2\lambda^3}\beta_j k_j^3 r_j^3 + 35\mathrm{i}\lambda\varepsilon_j k_j^2 r_j^2 \end{pmatrix} \tag{138a}$$

and

$$V = \frac{1}{8k_j\sqrt{2\lambda}} \begin{pmatrix} 8\sqrt{2\lambda^5}\beta_j k_j^5 + 63\lambda^2\varepsilon_j k_j^4 & 8\sqrt{2\lambda^5}\beta_j k_j^5 + 63\mathrm{i}\lambda^2\varepsilon_j k_j^4 \\ 16\lambda^2\beta_j k_j^4 + 67\sqrt{2\lambda^3}\varepsilon_j k_j^3 & 16\mathrm{i}\lambda^2\beta_j k_j^4 - 67\sqrt{2\lambda^3}\varepsilon_j k_j^3 \\ 8\sqrt{2\lambda^3}\beta_j k_j^3 + 35\lambda\varepsilon_j k_j^2 & -8\sqrt{2\lambda^3}\beta_j k_j^3 - 35\mathrm{i}\lambda\varepsilon_j k_j^2 \\ 8\sqrt{2\lambda}\beta_j k_j + 15\varepsilon_j & 8\sqrt{2\lambda}\beta_j k_j + 15\mathrm{i}\varepsilon_j \\ -16\lambda\beta_j k_j^2 - 39\sqrt{2\lambda}\varepsilon_j k_j & 16\mathrm{i}\lambda\beta_j k_j^2 - 39\sqrt{2\lambda}\varepsilon_j k_j \\ 8\sqrt{2\lambda^3}\beta_j k_j^3 + 99\lambda\varepsilon_j k_j^2 & -8\sqrt{2\lambda^3}\beta_j k_j^3 - 99\mathrm{i}\lambda\varepsilon_j k_j^2 \end{pmatrix}. \tag{138b}$$

Thus, we can find the eigenfrequences of a sequence of tapered Euler–Bernoulli beams via multiplying two by two matrices together, as we did for the uniform beams in Section 3.

9. Numerical results

Fig. 1 shows beam consisting of 3 segments, each of length L, with the beginning of the first segment fixed, and the end of the last segment free. However, the first and third segments are tapered, ranging from a radius of R to a radius of $2R$. We will let E be the Young's modulus for all three segments.

For the first segment, we can express the radius at the point x to be $r = Rx/L + R$, $(0 \leqslant x \leqslant L)$, so $\varepsilon_1 = R/L$, and $\beta_1 = R$. The radius of the third segment can be expressed as $r = 2R - Rx/L$, $(0 \leqslant x \leqslant L)$, so $\varepsilon_3 = -R/L$, and $\beta_3 = 2R$. Then $r_1 = \sqrt{2}$, while $r_3 = \sqrt{2}/2$. Also, since $k_j = \sqrt[4]{4/E_j\beta_j^2}$, we have $k_1 = E^{-1/4}\sqrt{2}/\sqrt{R}$, whereas $k_3 = E^{-1/4}/\sqrt{R}$. Yet z_1 and z_3 turn out to be equal,

$$z_1 = z_3 = \frac{4\sqrt{\lambda}L}{E^{1/4}\sqrt{R}(1+\sqrt{2})}, \tag{139}$$

a byproduct of the fact that the first and last segments are mirror images of each other. If we let $\eta = (1-\mathrm{i})LE^{-1/4}\sqrt{\lambda/(2R)}$, so that $\mathrm{i}\eta^2 = L^2\lambda/(R\sqrt{E})$, then we can express the U and V matrices as

$$U_1 = \frac{R}{16\eta L^4}\begin{pmatrix} 32L^4\eta + (15\mathrm{i}-15)L^4 & 32L^4\eta - (15+15\mathrm{i})L^4 \\ (32+32\mathrm{i})L^3\eta^2 - 78L^3\eta & (32-32\mathrm{i})L^3\eta^2 - 78L^3\eta \\ 32\mathrm{i}L^2\eta^3 - (99+99\mathrm{i})L^2\eta^2 & -32\mathrm{i}L^2\eta^3 + (99\mathrm{i}-99)L^2\eta^2 \\ -32\eta^5 + (63-63\mathrm{i})\eta^4 & -32\eta^5 + (63+63\mathrm{i})\eta^4 \\ (32-32\mathrm{i})L\eta^4 + 134\mathrm{i}L\eta^3 & (32+32\mathrm{i})L\eta^4 - 134\mathrm{i}L\eta^3 \\ 32\mathrm{i}L^2\eta^3 - (35+35\mathrm{i})L^2\eta^2 & -32\mathrm{i}L^2\eta^3 + (35\mathrm{i}-35)L^2\eta^2 \end{pmatrix}, \tag{140a}$$

$$U_3 = \frac{R}{32\eta L^4}\begin{pmatrix} 32L^4\eta + (15-15\mathrm{i})\sqrt{2}L^4 & 32L^4\eta + (15+15\mathrm{i})\sqrt{2}L^4 \\ (32+32\mathrm{i})\sqrt{2}L^3\eta^2 + 156L^3\eta & (32-32\mathrm{i})\sqrt{2}L^3\eta^2 + 156L^3\eta \\ 64\mathrm{i}L^2\eta^3 + (198+198\mathrm{i})\sqrt{2}L^2\eta^2 & -64\mathrm{i}L^2\eta^3 + (198-198\mathrm{i})\sqrt{2}L^2\eta^2 \\ -128\eta^5 + (252\mathrm{i}-252)\sqrt{2}\eta^4 & -128\eta^5 - (252+252\mathrm{i})\sqrt{2}\eta^4 \\ (64-64\mathrm{i})\sqrt{2}L\eta^4 - 536\mathrm{i}L\eta^3 & (64+64\mathrm{i})\sqrt{2}L\eta^4 + 536\mathrm{i}L\eta^3 \\ 64\mathrm{i}L^2\eta^3 + (70+70\mathrm{i})\sqrt{2}L^2\eta^2 & -64\mathrm{i}L^2\eta^3 + (70-70\mathrm{i})\sqrt{2}L^2\eta^2 \end{pmatrix}, \tag{140b}$$

$$V_1 = \frac{R}{32\eta L^4}\begin{pmatrix} -128\eta^5 + (252\mathrm{i}-252)\sqrt{2}\eta^4 & -128\eta^5 - (252+252\mathrm{i})\sqrt{2}\eta^4 \\ (64\mathrm{i}-64)\sqrt{2}L\eta^4 + 536\mathrm{i}L\eta^3 & -(64+64\mathrm{i})\sqrt{2}L\eta^4 - 536\mathrm{i}L\eta^3 \\ 64\mathrm{i}L^2\eta^3 + (70+70\mathrm{i})\sqrt{2}L^2\eta^2 & -64\mathrm{i}L^2\eta^3 + (70-70\mathrm{i})\sqrt{2}L^2\eta^2 \\ 32L^4\eta + (15-15\mathrm{i})\sqrt{2}L^4 & 32L^4\eta + (15+15\mathrm{i})\sqrt{2}L^4 \\ -(32+32\mathrm{i})\sqrt{2}L^3\eta^2 - 156L^3\eta & (32\mathrm{i}-32)\sqrt{2}L^3\eta^2 - 156L^3\eta \\ 64\mathrm{i}L^2\eta^3 + (198+198\mathrm{i})\sqrt{2}L^2\eta^2 & -64\mathrm{i}L^2\eta^3 + (198-198\mathrm{i})\sqrt{2}L^2\eta^2 \end{pmatrix}, \tag{140c}$$

$$V_3 = \frac{R}{16\eta L^4}\begin{pmatrix} -32\eta^5 + (63-63\mathrm{i})\eta^4 & -32\eta^5 + (63+63\mathrm{i})\eta^4 \\ (32\mathrm{i}-32)L\eta^4 - 134\mathrm{i}L\eta^3 & -(32+32\mathrm{i})L\eta^4 + 134\mathrm{i}L\eta^3 \\ 32\mathrm{i}L^2\eta^3 - (35+35\mathrm{i})L^2\eta^2 & -32\mathrm{i}L^2\eta^3 + (35\mathrm{i}-35)L^2\eta^2 \\ 32L^4\eta + (15\mathrm{i}-15)L^4 & 32L^4\eta - (15+15\mathrm{i})L^4 \\ -(32+32\mathrm{i})L^3\eta^2 + 78L^3\eta & (32\mathrm{i}-32)L^3\eta^2 + 78L^3\eta \\ 32\mathrm{i}L^2\eta^3 - (99+99\mathrm{i})L^2\eta^2 & -32\mathrm{i}L^2\eta^3 + (99\mathrm{i}-99)L^2\eta^2 \end{pmatrix}. \tag{140d}$$

Then, by defining the diagonal matrix

$$D = \begin{pmatrix} \mathrm{e}^{(1+\mathrm{i})(4-2\sqrt{2})\eta}, & 0 \\ 0, & \mathrm{e}^{(1-\mathrm{i})(4-2\sqrt{2})\eta} \end{pmatrix}, \tag{141}$$

the exterior matrices for the first and third segment are

$$\widetilde{P}_1 = \frac{1}{128R^2\lambda^2k_1^4}U_1 \cdot D \cdot V_1^{\mathrm{T}}, \tag{142a}$$

$$\widetilde{P}_3 = \frac{1}{2R^2\lambda^2k_3^4}U_3 \cdot D \cdot V_3^{\mathrm{T}}. \tag{142b}$$

The second segment has radius $2R$, so $k_2 = k_3 = E^{-1/4}/\sqrt{R}$. Replacing $k_2\sqrt{\lambda}$ with $(1+\mathrm{i})\eta/(L\sqrt{2})$ in Eq. (54) gives us

$$\widetilde{P}_2 = \frac{-1}{8\eta^4} U_2 \cdot \begin{pmatrix} \mathrm{e}^{(1+\mathrm{i})\eta} & 0 \\ 0 & \mathrm{e}^{(1-\mathrm{i})\eta} \end{pmatrix} \cdot V_2^{\mathrm{T}}. \tag{143}$$

where

$$U_2 = \begin{pmatrix} L^4 & L^4 \\ (1+\mathrm{i})L^3\eta & (1-\mathrm{i})L^3\eta \\ \mathrm{i}L^2\eta^2 & -\mathrm{i}L^2\eta^2 \\ -\eta^4 & -\eta^4 \\ (1-\mathrm{i})L\eta^3 & (1+\mathrm{i})L\eta^3 \\ \mathrm{i}L^2\eta^2 & -\mathrm{i}L^2\eta^2 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} -\eta^4 & -\eta^4 \\ (\mathrm{i}-1)L\eta^3 & -(1+\mathrm{i})L\eta^3 \\ \mathrm{i}L^2\eta^2 & -\mathrm{i}L^2\eta^2 \\ L^4 & L^4 \\ -(1+\mathrm{i})L^3\eta & (\mathrm{i}-1)L^3\eta \\ \mathrm{i}L^2\eta^2 & -\mathrm{i}L^2\eta^2 \end{pmatrix}. \tag{144a,b}$$

Putting this all together, we find that the eigenfrequencies of the system are the roots to the equation

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}^{\mathrm{T}} \cdot U_3 \cdot D \cdot V_3^{\mathrm{T}} \cdot U_2 \cdot \begin{pmatrix} \mathrm{e}^{(1+\mathrm{i})\eta} & 0 \\ 0 & \mathrm{e}^{(1-\mathrm{i})\eta} \end{pmatrix} \cdot V_2^{\mathrm{T}} \cdot U_1 \cdot D \cdot V_1^{\mathrm{T}} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0. \tag{145}$$

Multiplying this out, we get

$$\frac{-64R^4\eta^{11}}{L^4}\mathrm{e}^{(9-4\sqrt{2})\eta} \cdot (16\eta\mathrm{e}^{\mathrm{i}(4\sqrt{2}-9)\eta} + 16\eta\mathrm{e}^{\mathrm{i}(9-4\sqrt{2})\eta} + (1+\mathrm{i})(39\sqrt{2}-53)\mathrm{e}^{\mathrm{i}(4\sqrt{2}-9)\eta} + 14\mathrm{e}^{\mathrm{i}\eta} + 14\mathrm{e}^{-\mathrm{i}\eta}$$
$$+ (1-\mathrm{i})(39\sqrt{2}-53)\mathrm{e}^{\mathrm{i}(9-4\sqrt{2})\eta} + O(1/\eta)) = 0. \tag{146}$$

Note that taking the complex conjugate of this equation shows that $\overline{\eta}$ solves the same equation as η. Since the eigenfrequencies cannot have positive real parts, we can conclude that the solutions for η are all real, producing purely complex values for λ. This is not surprising, since we did not include dissipative joints. Had we included a damper in the problem, this pattern would not appear. Nonetheless, we can take advantage of this fact to convert to trig functions.

$$28\cos(\eta) + (32\eta + 78\sqrt{2} - 106)\cos((9-4\sqrt{2})\eta) + (78\sqrt{2} - 106)\sin((9-4\sqrt{2})\eta) = 0. \tag{147}$$

This produces the solutions

$$\lambda L^2/(R\sqrt{E}) = \{0.752108\mathrm{i}, 1.98306\mathrm{i}, 5.2622\mathrm{i}, 11.3821\mathrm{i}, 17.7145\mathrm{i}, 26.5723\mathrm{i}, 37.8953\mathrm{i}, 49.372\mathrm{i}, 63.8113\mathrm{i},$$
$$80.2353\mathrm{i}, 96.9647\mathrm{i}, 116.957\mathrm{i}, 138.417\mathrm{i}, 160.501\mathrm{i}, 185.991\mathrm{i}, 212.455\mathrm{i}, 239.985\mathrm{i}, 270.896\mathrm{i},$$
$$302.363\mathrm{i}, 335.415\mathrm{i}, 371.658\mathrm{i}, 408.161\mathrm{i}, \ldots\}. \tag{148}$$

It is not hard to compute the analytical asymptotical expansion for the eigenvalues as $|\lambda| \to \infty$. The dominant term of Eq. (147) is $32\eta\cos((9-4\sqrt{2})\eta)$, so the first term in the asymptotic expansion is

$$\eta \sim \frac{(k+1/2)\pi}{(9-4\sqrt{2})} \quad \text{as } \eta \to \infty, \tag{149}$$

where k is an integer. To find the next term, we plug

$$\eta = \frac{(k+1/2)\pi + \varepsilon}{(9-4\sqrt{2})} \tag{150}$$

into Eq. (147), and keep terms of order 1. We find that

$$28\cos\left(\frac{(k+1/2)\pi}{(9-4\sqrt{2})}\right) + 32\frac{(k+1/2)\pi}{(9-4\sqrt{2})}(\varepsilon(-1)^{k+1}) + (78\sqrt{2}-106)(-1)^k = O(\varepsilon). \tag{151}$$

From this, we find that

$$\frac{\varepsilon}{9-4\sqrt{2}} \sim \frac{78\sqrt{2} - 106 + 28(-1)^k \cos((k+1/2)\pi/(9-4\sqrt{2}))}{32(k+1/2)\pi}. \tag{152}$$

This gives us the first two terms for η:

$$\eta = \frac{(k+1/2)\pi}{(9-4\sqrt{2})} + \frac{39\sqrt{2} - 53 + 14(-1)^k \cos((k+1/2)\pi/(9-4\sqrt{2}))}{16(k+1/2)\pi} + O(1/k^2). \tag{153}$$

Finally, using the fact that $\lambda L^2/(R\sqrt{E}) = \mathrm{i}\eta^2$ gives us

$$\frac{\lambda L^2}{R\sqrt{E}} \sim \mathrm{i}\left(\frac{(k+1/2)^2\pi^2}{113-72\sqrt{2}} + \frac{39\sqrt{2} - 53 + 14(-1)^k \cos((k+1/2)\pi/(9-4\sqrt{2}))}{8(9-4\sqrt{2})}\right) + O(1/k). \tag{154}$$

In conclusion, the exterior matrix method provides a powerful tool for finding the eigenvalues of any complicated structure which produces sequentially coupled fourth-order equations. Many of the proofs and results required the use of the symbolic manipulator *Mathematica* [12], and a *Mathematica* notebook available at site [25] shows how the results were derived.

Yet this paper only scratches the surface as to the applications of this method. If the tapered beam example is redone to allow $m(x)$ and $I(x)$ to be arbitrary cubic functions, then the vibrations of even a non-uniform beam such as a helicopter propeller can be analyzed by using a cubic spline on the $m(x)$ and $I(x)$ functions using only 2×2 matrices. Combination structures of both cables and beams can be analyzed as well. It is ironic that studying the higher order exterior equation (Eq. (81) or Eq. (101)) allows us to find the eigenfrequencies so much easier. It is as though the eigenfrequencies actually live in the exterior vector space, and hence the best way to study them is to go into the space where they live.

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