Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 308 (2007) 337–342
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

Hopf bifurcation analysis of the Lev Ginzburg equation

Michael Bellamy, Ronald E. Mickens*

Department of Physics, Clark Atlanta University, P.O. Box 1744, Atlanta, GA 30314, USA

Received 10 May 2006; received in revised form 26 January 2007; accepted 27 June 2007
Available online 4 September 2007

Abstract

The Lev Ginzburg equation is shown to have a unique limit-cycle solution under certain conditions on the parameters appearing in this second-order, nonlinear differential equation. This conclusion follows from the application of the Hopf bifurcation theorem.

The class of differential equations having velocity-dependent frequencies take the form:

$$\ddot{x} + f(\dot{x})x = 0, \tag{1}$$

with the requirement that $f(0) > 0$. The general properties of the solutions to Eq. (1), illustrated with particular examples, were discussed by Mickens and Deft [1]. It was found that for a given $f(\dot{x})$ some solutions were periodic, while others became unbounded. This result follows from the fact that Eq. (1) can be written in terms of the phase-space variables, $(x, y = \dot{x})$, as

$$y\frac{\mathrm{d}y}{\mathrm{d}x} + f(y)x = 0. \tag{2}$$

Integrating this equation gives the following relation:

$$\int \frac{y\mathrm{d}y}{f(y)} + \frac{x^2}{2} = \text{constant}. \tag{3}$$

The first term, on the left-side, can be considered a generalized kinetic energy expression, $T(y)$ and this allows Eq. (3) to be expressed as

$$T(y) + \frac{x^2}{2} = \text{constant}. \tag{4}$$

One consequence of the relationship given by Eq. (4) , which is a first-integral of Eq. (1) , is that it indicates the system modeled by Eq. (1) which has features similar to those of a conservative system [1,2]. In particular, this means that while Eq. (1) can have periodic solutions, none of them are limit-cycles [1]. In other words, all the fixed-points corresponding to periodic solutions are either linear or nonlinear centers.

*Corresponding author. Tel.: +1 404 880 6923; fax: +1 404 880 6258.
E-mail address: rohrs@math.gatech.edu (R.E. Mickens).

0022-460X/$ - see front matter © 2007 Published by Elsevier Ltd.
doi:10.1016/j.jsv.2007.06.073

In this Paper, a special case of a generalization of Eq. (1)

$$\ddot{x} + f(\dot{x})x = g(\dot{x}), \tag{5}$$

is studied, i.e.,

$$\ddot{x} + \alpha(1 - \beta_1\dot{x})x = (1 - \beta_1\dot{x})(\gamma + \beta\dot{x}), \tag{6}$$

where $(\alpha, \beta_1, \gamma)$ are positive parameters and β is a real value. This highly nonlinear, second-order differential equation can be derived from the work of Ginzburg [3,4]. By numerically integrating Eq. (5) , Ginzburg found that for special values of the four parameters, stable limit-cycles exist. The main purpose of this communication is to apply the Hopf bifurcation theorem to Eq. (6) and determine the conditions for the mathematical existence of limit-cycles. The reason why the mathematical analysis has to be done is related to the fact that numerical integration techniques may produce solutions not corresponding to any of the solutions of the original differential equation [5,6]. Consequently, a rigorous mathematical analysis is required to ensure that Eq. (6) has limit-cycle solutions.

To proceed, note that Eq. (6) can be written as two-coupled, first-order equations,

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = (1 - \beta_1 y)(\gamma - \alpha x + \beta y), \tag{7}$$

where, as indicated above, x and y are the two-dimensional phase-space variables. The fixed-point is

$$(\bar{x}, \bar{y}) = \left(\frac{\gamma}{\alpha}, 0\right). \tag{8}$$

The eigenvalues [7–9] for the fixed-point can be easily calculated and are given by the expression

$$\lambda_{1,2} = \left(\frac{1}{2}\right)\left[\beta \pm i\sqrt{4\alpha - \beta^2}\right], \tag{9}$$

where it is assumed that $\beta^2 < 4\alpha$. From linear stability theory [7,8], it follows that the fixed-point is unstable for $\beta > 0$, has neutral stability for $\beta = 0$, and is stable for $\beta < 0$.

For the sake of completeness, the first-order differential equation, for which its solutions give the trajectories in phase-space, i.e., $y = y(x)$, is

$$\frac{dy}{dx} = \frac{(1 - \beta_1 y)(\gamma - \alpha x + \beta y)}{y}. \tag{10}$$

Inspection of both Eqs. (6) and (10) gives the following results:

(i) A special exact solution is

$$x(t) = \left(\frac{1}{\beta_1}\right)t + x_0. \tag{11}$$

This can be verified directly from Eq. (6).

(ii) Whenever a trajectory, $y = y(x)$, crosses the x-axis, it does so with unbounded slope. This is a consequence of the fact that the null-cline [8], corresponding to $dy/dx = \pm\infty$, is located at $y = 0$.

(iii) The $dy/dx = 0$ null-clines [8] are

$$y_0^{(1)}(x) = \frac{1}{\beta_1}, \quad y_0^{(2)}(x) = \left(\frac{1}{\beta}\right)(\alpha x - \gamma). \tag{12}$$

This result, along with (i), implies that the (x, y) phase-plane is divided into two distinct regions:

$$\text{(I) } x(0) = \text{arbitrary}, \quad y(0) > \left(\frac{1}{\beta_1}\right), \tag{13}$$

$$\text{(II) } x(0) = \text{arbitrary}, \quad y(0) < \left(\frac{1}{\beta_1}\right), \tag{14}$$

Trajectories originating in either region remain in that region. The curve (in phase-space) $y(x) = (1/\beta_1)$ is the boundary between the two regions. Since the boundary curve corresponds to the special solution given by

Eq. (11) and since two distinct solutions cannot intersect each other, because of the existence and uniqueness theorems for differential [9,10] the above conclusions are immediately seen to be correct.

(iv) The above discussion indicates that if oscillatory solutions exist, they must do so only in region II. Note that the fixed-point is in region II and a limit-cycle must always contain in its interior a fixed-point [8,9].

To continue, the Hopf bifurcation theorem has to be introduced [6–8]. Consider a two-dimensional system

$$\frac{dx}{dt} = F(x, y, \lambda), \quad \frac{dy}{dt} = G(x, y, \lambda), \tag{15}$$

where λ is a parameter. Let Eqs. (15) have an isolated fixed-point at $(\bar{x}(\lambda), \bar{y}(\lambda))$, where the dependence of the location of the fixed-point on λ is indicated. Let the eigenvalues of the Jacobian matrix, $J(\lambda)$, be

$$r_{1,2}(\lambda) = a(\lambda) \pm ib(\lambda), \tag{16}$$

where

$$J(\lambda) = \begin{vmatrix} F_1(\lambda) & F_2(\lambda) \\ G_1(\lambda) & G_2(\lambda) \end{vmatrix}, \tag{17}$$

with

$$F_1(\lambda) \equiv \left.\frac{\partial F}{\partial x}\right|_{(\bar{x},\bar{y})}, \quad F_2(\lambda) \equiv \left.\frac{\partial F}{\partial y}\right|_{(\bar{x},\bar{y})}, \tag{18a}$$

$$G_1(\lambda) \equiv \left.\frac{\partial G}{\partial x}\right|_{(\bar{x},\bar{y})}, \quad G_2(\lambda) \equiv \left.\frac{\partial G}{\partial y}\right|_{(\bar{x},\bar{y})}. \tag{18b}$$

Let the fixed-point $(\bar{x}(\lambda), \bar{y}(\lambda))$ be asymptotically stable for $\lambda < 0$, unstable for $\lambda > 0$, and let $a(0) = 0$. Further, let

$$\left.\frac{da(\lambda)}{d\lambda}\right|_{\lambda=0} > 0, \quad b(0) \neq 0. \tag{19}$$

The Hopf bifurcation theorem states that under these conditions, for sufficiently small $|\lambda|$, an isolated closed trajectory exists for either $\lambda > 0$ or $\lambda < 0$. If $(\bar{x}(0), \bar{y}(0))$ is locally stable, then a stable limit-cycle exists about $(\bar{x}(\lambda), \bar{y}(\lambda))$ for sufficiently small $\lambda > 0$.

Examination of Eqs. (7), (8) and (9) shows that the bifurcation parameter λ should be identified with β, i.e., $\lambda = \beta$, and

$$a(\beta) = \frac{\beta}{2}, \quad b(\beta) = (\sqrt{4\alpha - \beta^2})/2. \tag{20}$$

Noting that

$$a(0) = 0, \quad \left.\frac{da}{d\beta}\right|_{\beta=0} = \frac{1}{2} > 0, \quad b(0) = \sqrt{\alpha}, \tag{21}$$

it follows that all the conditions hold for application of the Hopf bifurcation theorem. Thus, the general conclusion is that the Lev Ginzburg equation can have a stable limit-cycle solution if β is positive and sufficiently small. However, the following observations can be made:

(a) The fixed-point, $(\bar{x}, \bar{y})$, does not depend explicitly on β, i.e.,

$$\bar{x}(\beta) = \frac{\gamma}{\alpha}, \quad \bar{y}(\beta) = 0. \tag{22}$$

(b) The Hopf bifurcation theorem implies that there exists some constant, β_c, such that a stable limit-cycle occurs for

$$0 < \beta < \beta_c. \tag{23}$$

However, the theorem provides no guidance as to how to calculate and/or estimate β_c. It should be noted that the Hopf bifurcation theorem provides only a local result for the creation of a limit-cycle. In general, the direct analytical determination of β_c for an arbitrary second-order differential equation for which the theorem applies is a major unsolved problem in the theory of bifurcations [10–12].

(c) From a qualitative perspective, if initial conditions are such that $-\infty < x(0) < +\infty$ and $y(0) < (1/\beta_1)$, then for $0 < \beta < \beta_c$, the solution $x(t)$ should eventually settle in a steady-state, periodic motion of fixed amplitude and frequency. For the same set of initial conditions, but with $\beta < 0$ (and small), $x(t)$ should oscillate with a decreasing amplitude.

(d) The numerical results, given in Figs. 1 and 2, illustrate the conclusions reached in (c).

The Lev Ginzburg equation is but one example of the interesting class of differential equations of the type given by Eq. (5) . This class of equations has both velocity-dependent frequencies, $f(\dot{x})$, and forcing terms, $g(\dot{x})$. The expectation is that such equations will also provide valid mathematical models for a range of phenomena involving nonlinear oscillations. The work presented here is an initial step in this direction. However, for sufficiently small and positive β, the method of harmonic balance [9] can be used to estimate the parameters of the limit-cycle for Eq. (6). These parameters are the associated angular frequency and the amplitude of the small oscillators in a neighborhood of the fixed-point.

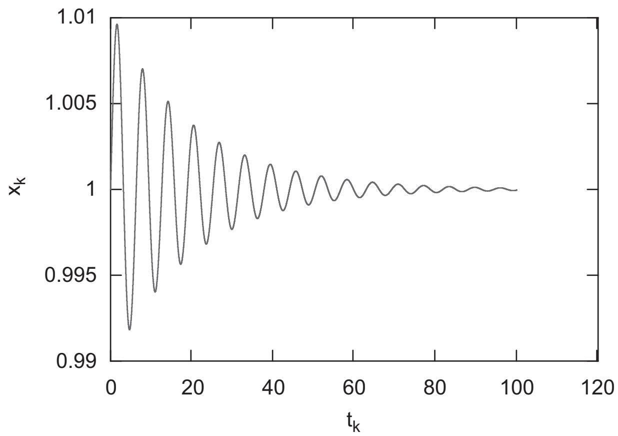

Fig. 1. Plot of $x(t)$ vs t corresponding to $x(0) = 1$, $y(0) = 0.01$, $\alpha = 1$, $\beta_1 = 1$, $\gamma = 1$, $\beta = 0.01$. The subscript k denotes samples of the variable.

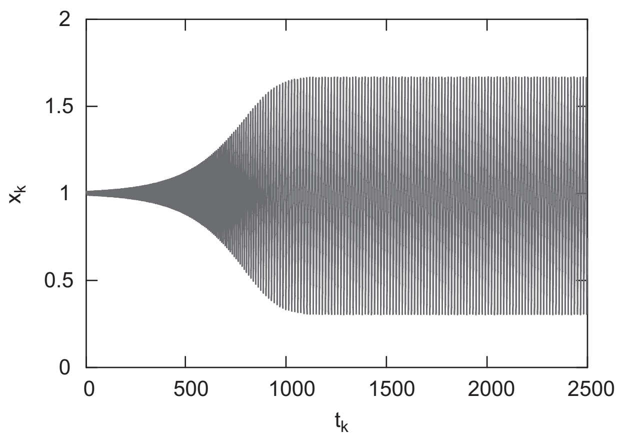

Fig. 2. Plot of $x(t)$ vs t corresponding to $x(0) = 1$, $y(0) = 0.01$, $\alpha = 1$, $\beta_1 = 1$, $\gamma = 1$, $\beta = -0.1$. The subscript k denotes samples of the variable.

First, write $x(t)$ as

$$x(t) = \bar{x} + u(t), \tag{24}$$

where $\bar{x} = \gamma/\alpha$. Substituting this result in Eq. (6) gives the following differential equation for $u(t)$

$$\ddot{u} + \alpha u = (\alpha\beta_1)u\dot{u} + \beta\dot{u} - (\beta\beta_1)\dot{u}^2, \tag{25}$$

where it is observed that the parameter γ does not appear.

Second, note that the right-side of Eq. (25) is quadratic in $\dot{u}$. This implies that the first-order harmonic balance approximation for Eq. (25) should be of the form

$$u(t) = A + B\cos(\omega t), \tag{26}$$

where A, B and ω are to be determined. Substituting Eq. (26) into Eq. (25) gives

$$\left(\alpha A + \frac{\beta_1\beta\omega^2 B^2}{2}\right) + B(\alpha - \omega^2)\cos\theta + [\alpha\beta_1\omega AB + \omega\beta B]\sin\theta$$
$$+ (\text{Higher} - \text{order harmonics}) \simeq 0, \tag{27}$$

where $\theta = \omega t$. Harmonic balancing gives the three relations

$$\alpha A + \frac{\beta_1\beta\omega^2 B^2}{2} = 0, \tag{28a}$$

$$B(\alpha - \omega^2) = 0, \tag{28b}$$

$$\alpha\beta_1\omega AB + \omega\beta B = 0. \tag{28c}$$

The nontrivial solutions of these equations for (ω, A, B) are

$$\omega = \sqrt{\alpha}, \quad A = \left(\frac{\beta}{\alpha\beta_1}\right), \quad B = \pm\left(\frac{2}{\alpha}\right)^{1/2}\left(\frac{1}{\beta_1}\right), \tag{29}$$

where the ($\pm$) in the B equation merely indicates two oscillations that are 180° out of phase. Picking the (+) sign for B and using the relationships, given in Eqs. (24) and 26, it follows that a first-order harmonic balance solution for $x(t)$ is

$$x(t) = \left[\left(\frac{\gamma}{\alpha}\right) - \left(\frac{1}{\alpha\beta_1}\right)\beta\right] + \left(\frac{2}{\beta_1^2\alpha}\right)^{1/2}\cos(\sqrt{\alpha}t). \tag{30}$$

Inspection of Eq. (30) leads to the following results for this level of harmonic balance application:

(i) The "center" of the limit-cycle is shifted by an amount proportional to β. Since β is assumed small, a requirement of the calculation, the "center" of the limit-cycle oscillation is close to the fixed-point of Eq. (6), but modified by a correction of $O(\beta)$.
(ii) The amplitude of the oscillation is given by B; see Eq. (29).
(iii) The angular frequency ω is equal to the oscillations of the linear part of Eq. (25).

Finally, preliminary numerical integrations of Eq. (6) indicate that β_c may be close to the value 0.0354. However, as stated above, it is not possible to analytically estimate a value for β_c based on current mathematical results involving the Hopf bifurcation theorem. It is also unlikely that the use of harmonic balance will provide any substantial guidance for the calculation of β_c since, for limit-cycle behavior, harmonic balance methods generally are equivalent to perturbation procedures for which β must be taken small.

In summary, under certain conditions the Lev Ginzburg equation has been shown to undergo a Hopf bifurcation such that a stable limit-cycle exists. The method of harmonic balance was then used to estimate the parameters, amplitude and angular frequency, of this periodic solution. Numerical experiments were consistent with these results and a (numerical) estimate was made for the critical value of β, namely $\beta_c = 0.0354$.

Acknowledgments

The work reported here was supported in part by a research grant from DOE to R.E. Mickens and Title III Funds from Clark Atlanta University to M. Bellamy. Professor Kale Oyedeji is thanked for carrying out the numerical work.

References

[1] R.E. Mickens, J. Deft, Oscillations in systems having velocity dependent frequencies, *Journal of Sound and Vibration* 223 (1999) 329–333.
[2] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, Berlin, 1978.
[3] L.R. Ginzburg, The theory of population dynamics, *Journal of Theoretical Biology* 122 (1986) 385–399.
[4] L. Ginzburg, M. Colyvan, *Ecological Orbits: How Planets Move and Populations Grow*, Oxford University Press, New York, 2004.
[5] R.E. Mickens, *Nonstandard Finite Difference Models of Differential Equations*, World Scientific, Singapore, 1994.
[6] A.M. Stuart, A.R. Humphries, *Dynamical Systems and Numerical Analysis*, Cambridge University Press, New York, 1996.
[7] E. Beltrami, *Mathematics for Dynamic Modeling*, Academic Press, Boston, 1987.
[8] H. Strogatz, *Nonlinear Dynamics and Chaos*, Addison-Wesley, Reading, MA, 1994.
[9] R.E. Mickens, *Oscillations in Planar Dynamic Systems*, World Scientific, Singapore, 1996.
[10] S.L. Ross, *Differential Equations*, Blaisdell Publishing, Waltham, MA, 1964.
[11] M. Medved', *Fundamentals of Dynamical Systems and Bifurcation Theory*, Adam Hilger, New York, 1992.
[12] S.-N. Chow, J.K. Hale, *Methods of Bifurcation Theory*, Springer, New York, 1996.