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Optimization of non-uniformly distributed multiple tuned mass damper

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Abstract

A gradient-based method for optimizing non-uniformly distributed multiple tuned mass damper (MTMD) is presented in this paper. By solving an optimization problem with multiple objectives, optimized non-uniformly distributed MTMDs are obtained. Then the dynamic characteristics, effectiveness, robustness and redundancy of MTMDs are investigated in detail. Without restrictive assumptions such as uniformly distributed frequency, identical damping ratio, the optimized non-uniformly MTMDs obtained here can be considered as the "true" optimal ones. Unlike the references [L. Zuo, S.A. Nayfeh, Optimization of the individual stiffness and damping parameters in multiple-tuned-mass–damper system, *Journal of Vibration and Acoustics—Transactions of the ASME* 127(1) (2005) 77–83; N. Hoang, P. Warnitchai, Design of multiple tuned mass damper by using a numerical optimizer, *Earthquake Engineering and Structural Dynamics* 34(2) (2005) 125–144], the maximum displacement or frequency response of the main structure is chosen as the objective function in the present paper, because the maximum displacement is more concerned than the root-mean-square response sometimes. Using the presented method, the errors of estimate of the parameters of the structure and the MTMD can be taken into account quantitatively in the design procedure of the MTMD. It is demonstrated that the MTMDs designed in this paper are more effective than the traditional optimal uniformly distributed MTMDs whose frequency spacing, stiffness or mass and damping are under some restrictions for simplicity. And the MTMDs designed by accounting for possible errors of estimate are more robust than those without consideration of errors.

1. Introduction

The tuned mass damper (TMD) is an energy dissipation device, which suppresses structural vibration by transferring some of the structural vibration energy to the TMD and dissipates the energy through the damping of the TMD. The TMD has many advantages, such as simplicity, reliability, effectiveness and low cost. The first application of TMD dates back to 1909 [3]. Since Den Hartog [4] proposed an optimal design for a TMD property under harmonic conditions, the TMD optimization has been further studied for various types of excitations. A majority of those efforts were devoted to developing the design procedure of optimizing the TMD parameters [5–11].

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However, single tuned mass damper (STMD) is sensitive to the frequency ratio between the TMD and the structure and the damping ratio of the TMD. The effectiveness of STMD is reduced significantly due to the mistuning or off-optimum damping. As a result, the use of more than one TMDs with different dynamic characteristics has been proposed by Xu and Igusa [12] to improve the effectiveness and robustness. The multiple tuned mass dampers (MTMDs) for controlling the structural vibration consist of a large number of small TMDs whose natural frequencies are distributed around the natural frequency of a controlled mode of the structure.

Xu and Igusa [12] studied the case that the multiple substructures with light damping and equally spaced over a frequency range can be more effective and more robust than a single TMD with equal total mass when the system is excited by a wideband random excitation. Igusa and Xu [13] introduced an analysis and design method of the MTMD for a wide-band input, and a closed-form expression for the design parameters was presented. The fundamental characteristics of MTMD under harmonically forced oscillation were analytically investigated by Yamaguchi and Harnpornchai [14]. Abe and Fujino [15] applied the perturbation technique to study the model characteristics of the combined system and the efficiency of the MTMD. Abe and Igusa [16] also investigated the effectiveness of multiple oscillators in reducing the response of structures with closely spaced natural frequencies.

The procedure of designing both TMD to control a particular structural mode and MTMD to optimally control a multi-degree-of-freedom (mdof) structure were summarized by Rana and Soong [17]. Jangid [18] conducted a parametric study to investigate the effectiveness of MTMD on reducing the response of torsionally coupled system and concluded that the effectiveness of MTMD, designed for an asymmetric system by ignoring the effect of the torsional coupling, is overestimated. Later, Li and Qu [19] also performed numerical simulations to accurately estimate the dynamic characteristics of the MTMD for asymmetric structures subject to ground acceleration. In the simulations, the effectiveness and robustness of the MTMD strategies with different layouts are also investigated. Joshi and Jangid [20] studied optimum parameters of MTMD for suppressing the dynamic response of the structure subjected to base excitation which is modeled as a stationary white noise random process. Jangid [21], Bakre and Jangid [22] investigated optimal parameters of the MTMD for undamped and damped systems to harmonic base excitation using numerical searching techniques. Curve-fitting schemes were also carried out to find out the closed-form expressions of the optimized parameters of the MTMD.

Gu et al. [23] studied the buffeting control of the Yangpu Bridge using the MTMD. Lin et al. [24] investigated the applicability of MTMD to suppress train-induced vibration on bridges. Li [25,26], Li and Liu [27,28] compared many types of MTMDs under various restrictive assumptions on the mass, stiffness and damping of dampers. Zuo and Nayfeh [29] formulated the problem of designing a mdof TMD attached to a mdof primary system as a decentralized static-output feedback problem. The descent-subgradient method was proposed to maximize the minimal damping of modes over a prescribed frequency range in order to obtain the optimal parameters of the mdof TMD.

All above parametric studies about the MTMD are based upon certain constraints such as identical damping ratio, uniformly distributed frequency spacing, uniform mass distribution or uniform stiffness to simplify the optimization process. Consequently, a few scholars paid their attention to the effectiveness and robustness of non-uniform MTMD to explore the different possible combinations of parameters to make the MTMD more effective and robust. The dynamic characteristics and performance of the MTMD with non-uniform mass distribution or non-uniform frequency distribution under random loading were further analyzed by Kareem and Kline [30]. Park and Reed [31] examined the performance of uniformly and linearly distributed MTMD by assessing the effectiveness and robustness of the structure-MTMD system and considering the effects of redundancy under harmonic excitation. However, the studies in Refs. [30,31] not only impose some constraints on frequency and mass distribution, but also set the damping ratios of TMDs to be equal.

Recently, Zuo and Nayfeh [1] used a gradient-based algorithm to directly optimize the individual stiffness and damping parameters of the TMDs when mass distribution among the TMDs had been given. They found that neither the optimal damping ratios nor the frequency spaces are identical. The numerical examples in their paper also suggested that the mass distribution slightly influenced optimization result. Hoang and Warnitchai [2] proposed a new method of optimizing MTMD based on the Davidon–Fletcher–Powell method.

In this method, the parameters except the masses of TMDs are treated as unconstrained optimization variables. Analytical expressions of the quadratic performance function and the gradient are explicitly evaluated to avoid numerical errors and speed up the convergence.

Authors of Refs. [1,2] investigated the optimization of unrestrictive MTMD to minimize root-mean-square responses of the structure. However, the maximum displacement of the structure is usually more concerned by engineers. In the present paper, an optimization procedure of MTMD is introduced. The MTMDs added to a single-degree-of-freedom (sdof) structure under harmonic forces are optimized for various combinations of parameters of the structure and MTMD. The effectiveness and robustness of the optimal non-uniformly distributed MTMD are discussed in comparison with those of the optimal STMD and the optimal uniformly distributed MTMD.

Most of previous studies examine the robustness of MTMDs after the design of MTMDs. Yet, the possible errors of parameters are quantitatively considered in the design procedure in this paper. The method presented in this paper is very suitable to design optimal MTMDs when the errors of estimate of the parameters of the structure and manufacturing errors of MTMD occur. According to the optimized result, the errors of estimate of tuning frequency ratios must be taken into account in the design procedure of MTMD, otherwise the effectiveness of MTMD is overestimated. It is also demonstrated that the MTMDs designed in this paper are more effective and robust in controlling the vibration of the structures whose parameters are not precisely estimated.

2. Modeling of structure-MTMD system

Consider a sdof structure with MTMD as shown in Fig. 1. A set of TMDs, each of which consists of a mass, a spring and a viscous damper are attached to the main structure to suppress structural vibration. The natural frequencies of the TMDs are tuned to a frequency range near to the natural frequency of the main structure.

2.1. Equation of motion

The responses of the combined structure-MTMD system shown in Fig. 1 satisfy the following equation of motion:

$$M\ddot{X} + C\dot{X} + KX = F, \tag{1}$$

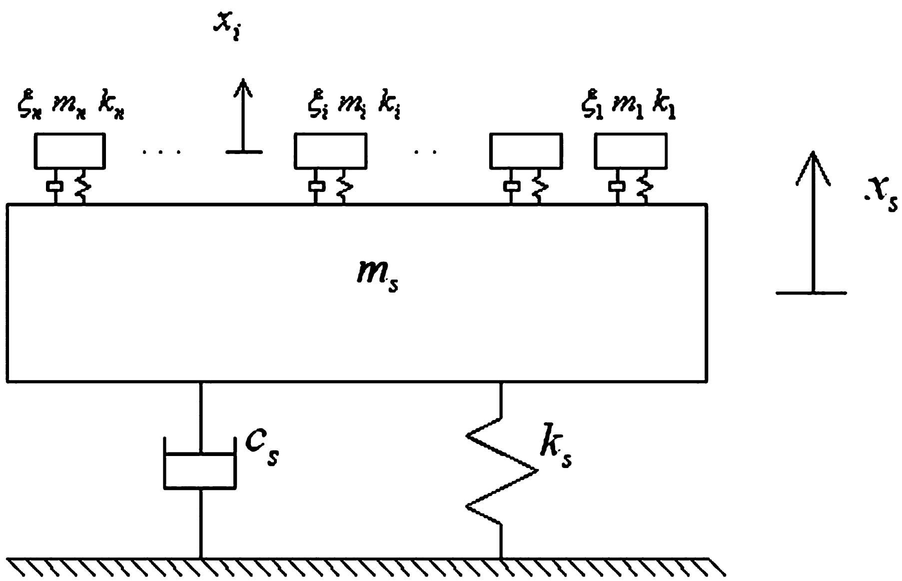

Fig. 1. Configuration of the structure-MTMD system.

where X is the displacement vector which consists of the displacement of the structure x_s and the displacements of the n TMDs $x_k(k = 1, \ldots, n)$. That is

$$X = [x_s \quad x_1 \quad x_2 \quad \cdots \quad x_n]^{\mathrm{T}}, \tag{2}$$

where M, C and K are the coefficient matrices of the mass, damping and stiffness, respectively, having the following forms:

$$M = \mathrm{diag}[m_s \quad m_1 \quad m_2 \quad \cdots \quad m_n], \tag{3}$$

$$C = \begin{bmatrix} c_s + \sum_{k=1}^{n} c_k & -c_1 & -c_2 & \cdots & -c_n \\ -c_1 & c_1 & 0 & \cdots & 0 \\ -c_2 & 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_n & 0 & 0 & \cdots & c_n \end{bmatrix}, \tag{4}$$

$$K = \begin{bmatrix} k_s + \sum_{k=1}^{n} k_k & -k_1 & -k_2 & \cdots & -k_n \\ -k_1 & k_1 & 0 & \cdots & 0 \\ -k_2 & 0 & k_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_n & 0 & 0 & \cdots & k_n \end{bmatrix}. \tag{5}$$

When the structure-MTMD system is excited by external force acting on the main structure or base excitation, the vector F can be written as

$$F = f(t)[1 \quad \Theta\mu_1 \quad \Theta\mu_2 \quad \cdots \quad \Theta\mu_n]^{\mathrm{T}}, \tag{6}$$

where m_s, c_s and k_s are the mass, damping and stiffness coefficients of the main structure; m_k, c_k and k_k are the mass, damping and stiffness coefficients of the kth damper, respectively; $f(t)$ is the external force acting on the main structure; $\mu_k = m_k/m_s$ is the mass ratio of the kth damper; $\Theta = 0$ for main structure excitation and $\Theta = 1$ for base excitation.

2.2. Dynamic magnification factor of structural response

In order to obtain the dynamic magnification factor (DMF) of the structural response, the harmonically forced vibration of the combined system is first studied. The force vector F is represented by

$$F = f_0 e^{i\omega t}[1 \quad \Theta\mu_1 \quad \Theta\mu_2 \quad \cdots \quad \Theta\mu_n]^{\mathrm{T}}, \tag{7}$$

where f_0 is the force amplitude and ω is the frequency of the external force.

Then the harmonic solution can be assumed as

$$X = e^{i\omega t}[X_s \quad X_1 \quad X_2 \quad \cdots \quad X_n]^{\mathrm{T}}. \tag{8}$$

Substituting Eqs. (2)–(8) into Eq. (1), the displacement amplitude of the structure X_s can be written as

$$X_s = \frac{f_0}{k_s} \frac{\mathrm{Re}(z_2) + \mathrm{Im}(z_2)\mathrm{i}}{\mathrm{Re}(z_1) + \mathrm{Im}(z_1)\mathrm{i}}, \tag{9}$$

where

$$\mathrm{Re}(z_1) = 1 - \beta^2 - \beta^2 \sum_{k=1}^{n} \mu_k \frac{\left(\frac{\beta_k^2}{\beta^2} - 1\right) + 4\xi_k^2}{\left(\frac{\beta_k}{\beta} - \frac{\beta}{\beta_k}\right)^2 + 4\xi_k^2},$$

$$\mathrm{Im}(z_1) = 2\beta\xi_s + \beta^2 \sum_{k=1}^{n} \mu_k \frac{\frac{2\xi_k\beta}{\beta_k}}{\left(\frac{\beta_k}{\beta} - \frac{\beta}{\beta_k}\right)^2 + 4\xi_k^2},$$

$$\mathrm{Re}(z_2) = 1 + \Theta\mu + \Theta \sum_{k=1}^{n} \mu_k \frac{1 - \frac{\beta^2}{\beta_k^2}}{\left(\frac{\beta_k}{\beta} - \frac{\beta}{\beta_k}\right)^2 + 4\xi_k^2},$$

$$\mathrm{Im}(z_2) = -\Theta \sum_{k=1}^{n} \mu_k \frac{\frac{2\xi_k\beta}{\beta_k}}{\left(\frac{\beta_k}{\beta} - \frac{\beta}{\beta_k}\right)^2 + 4\xi_k^2}$$

in which $\mu = (\sum m_k)/m_s$ is the total mass ratio of MTMD; $\beta = \omega/\omega_s$ is the frequency ratio between the external force and the structure; $\beta_k = \omega_k/\omega_s$ is the frequency ratio between the kth TMD and the structure; ω, ω_s and ω_k are the frequencies of the external force, the structure and the kth TMD, respectively.

The DMF of the structural response is finally obtained as follows:

$$\mathrm{DMF} = \left[\frac{\mathrm{Re}^2(z_2) + \mathrm{Im}^2(z_2)}{\mathrm{Re}^2(z_1) + \mathrm{Im}^2(z_1)}\right]^{1/2}. \tag{10}$$

3. Parametric optimization

3.1. Objective function of optimization

As obtained above, the DMF is a continuous and differentiable function of frequency ratio β, tuning frequency ratio β_k, mass ratio μ_k, damping ratio ξ_k and structural damping ratio ξ_s. Although the partial derivatives$\partial\mathrm{DMF}/\partial\beta_k$ and $\partial\mathrm{DMF}/\partial\xi_k$ cannot be expressed analytically in this case, they can be solved numerically.

Given a set of parameters of MTMD $\left\{\beta_1 \quad \beta_2 \quad \cdots \quad \beta_n \quad \xi_1 \quad \xi_2 \quad \cdots \quad \xi_n\right\}$, structural damping ratio ξ_s, total mass ratio μ and number of MTMD n, the β, which make the DMF curve reach local peaks can be obtained by solving the following equation:

$$f(\beta, \beta_k, \xi_k, \mu_k, \xi_s) = \frac{\partial\mathrm{DMF}}{\partial\beta} = 0. \tag{11}$$

Generally, β in Eq. (11) have no more than $2n+1$ solutions. Correspondingly, the DMF curve has no more than $n+1$ local maximum values and n local minimum values.

Then $\partial\beta/\partial\beta_k$ and $\partial\beta/\partial\xi_k$ can also be obtained by solving

$$\begin{cases} \dfrac{\partial f}{\partial \beta_k} = 0 \\ \dfrac{\partial f}{\partial \xi_k} = 0 \end{cases} \quad k = 1, \ldots, n. \tag{12}$$

After obtaining β, $\partial\beta/\partial\beta_k$ and$\partial\beta/\partial\xi_k$, the partial derivatives$\partial\mathrm{DMF}/\partial\beta_k$ and $\partial\mathrm{DMF}/\partial\xi_k$can directly computed. Then the following gradient matrix is established as follows:

$$G_{(n+1)\times(2n)} = \begin{bmatrix} \dfrac{\partial DMF_1}{\partial\beta_1} & \cdots & \dfrac{\partial DMF_1}{\partial\beta_n} & \dfrac{\partial DMF_1}{\partial\xi_1} & \cdots & \dfrac{\partial DMF_1}{\partial\xi_n} \\ \dfrac{\partial DMF_2}{\partial\beta_1} & \cdots & \dfrac{\partial DMF_2}{\partial\beta_n} & \dfrac{\partial DMF_2}{\partial\xi_1} & \cdots & \dfrac{\partial DMF_2}{\partial\xi_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial DMF_{n+1}}{\partial\beta_1} & \cdots & \dfrac{\partial DMF_{n+1}}{\partial\beta_n} & \dfrac{\partial DMF_{n+1}}{\partial\xi_1} & \cdots & \dfrac{\partial DMF_{n+1}}{\partial\xi_n} \end{bmatrix}, \tag{13}$$

where DMF_i denotes the ith local peak value of the DMF curve.

In the present paper, the goal of the optimization is to find a combination of the parameters of the TMDs to minimize the maximum value of DMF of the structure-MTMD system.

3.2. Gradient method

3.2.1. Searching direction

If an additional column whose entries are the values of DMF_i is attached to the gradient matrix G, the new matrix is obtained as

$$P = \begin{bmatrix} \dfrac{\partial DMF_1}{\partial\beta_1} & \cdots & \dfrac{\partial DMF_1}{\partial\beta_n} & \dfrac{\partial DMF_1}{\partial\xi_1} & \cdots & \dfrac{\partial DMF_1}{\partial\xi_n} & \mathrm{DMF}_1 \\ \dfrac{\partial DMF_2}{\partial\beta_1} & \cdots & \dfrac{\partial DMF_2}{\partial\beta_n} & \dfrac{\partial DMF_{n+1}}{\partial\beta_n} & \cdots & \dfrac{\partial DMF_2}{\partial\xi_n} & \mathrm{DMF}_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial DMF_{n+1}}{\partial\beta_1} & \cdots & \dfrac{\partial DMF_{n+1}}{\partial\beta_n} & \dfrac{\partial DMF_{n+1}}{\partial\xi_1} & \cdots & \dfrac{\partial DMF_{n+1}}{\partial\xi_n} & \mathrm{DMF}_{n+1} \end{bmatrix}. \tag{14}$$

Then remove the ith row of G, if

$$\mathrm{DMF}_{\max} - \mathrm{DMF}_i > \varepsilon, \tag{15}$$

where $\mathrm{DMF}_{\max}$ is the maximum value of DMF_i, ε in this paper is set to 10^{-4}.

The removal process will yield a new matrix:

$$J = \begin{bmatrix} J_1 \\ J_2 \\ \vdots \\ J_k \end{bmatrix}, \tag{16}$$

where J_k denotes the gradient vector corresponding to the kth local peak of DMF curve which satisfies the condition in inequality (15).

This new matrix is referred to as the searching matrix. In general, the dimension of the matrix J becomes larger when the combination of parameters of MTMD approaches the optimal one.

The steepest descent method [32,33] is used to find out the searching direction here. If the dimension of J is 1, the searching direction is directly obtained as

$$S = -J/\|J\|. \tag{17}$$

Otherwise, the matrix J is rewritten in the following form:

$$J' = \begin{bmatrix} \cos\alpha_{1,1} & \cos\alpha_{1,2} & \cdots & \cos\alpha_{1,2n} \\ \cos\alpha_{2,1} & \ddots & \cdots & \cos\alpha_{2,2n} \\ \vdots & \vdots & \ddots & \vdots \\ \cos\alpha_{k,1} & \cos\alpha_{k,2} & \cdots & \cos\alpha_{k,2n} \end{bmatrix}, \tag{18}$$

where

$$\alpha_{i,j} = \arccos\left(\frac{J_{i,j}}{\|J_i\|}\right), \tag{19}$$

where $\|J_i\|$ denotes the norm of vector J_i.

Then the searching direction is computed as follows:

$$S = \left\{ -\cos\left(\frac{\sum_{i=1}^{k}\alpha_{i,1}}{k}\right), \quad -\cos\left(\frac{\sum_{i=1}^{k}\alpha_{i,2}}{k}\right), \quad \cdots, \quad -\cos\left(\frac{\sum_{i=1}^{k}\alpha_{i,2n}}{k}\right) \right\}. \tag{20}$$

3.2.2. *Outline of the gradient method*

In this paper, the parameters of the optimal uniformly distributed MTMD are chosen as the initial condition for the iteration.

Suppose that initial solution P_k has been obtained:

(i) Evaluate the gradient matrix G in order to yield searching matrix J.
(ii) Compute the searching direction S_k using Eq. (17) or (20).
(iii) Search along the line through P_k in the direction S_k. Setting a step size h, we will arrive at a point P_{k+1}.
(iv) Construct the next point $P_{k+1} = P_k + hS_k$.
(v) Perform the termination test for optimization.

Repeat the process.

3.2.3. *Termination condition*

The optimization in the present paper is a multiple objective programming problem. The "space of tradeoffs" among the objectives DMFs should be searched. To be optimal, however, a point must be efficient. A point in the feasible region is efficient if and only if is not possible to move feasibly from it to decrease an objective without increasing at least one of the others [34].

Thus, the termination condition is no longer as follows:

$$\frac{\partial \mathrm{DMF}}{\partial \beta_k} = \frac{\partial \mathrm{DMF}}{\partial \xi_k} = 0 \quad (\text{for certain local peak in which } k \in [1, n+1]). \tag{21}$$

Otherwise, the termination condition is that the following non-homogeneous linear equation has no solution:

$$Jx = b, \tag{22}$$

where J is the searching matrix obtained above, x is an incremental vector along the searching direction, b is the incremental vector of local peaks' values of DMF curve. In order to minimize the maximum value of local peaks, the elements of vector b should be non-positive and not all zero.

Eq. (22) has no solution if and only if [35,36]

$$\text{rank}(J) < \text{length}(J), \tag{23}$$

$$\text{rank}(J) < \text{rank}(J|b), \tag{24}$$

where rank() is the rank of the corresponding matrix; length() denotes the number of rows of the corresponding matrix; the matrix $(J|b)$ is the augmented matrix of the system $Jx = b$.

In this case, the vectors J_k are linearly dependent. However, it is generally a too strict criterion for numerical evaluation. In practice, a "near-optimal" solution will typically be satisfied. In the present paper, the condition that for any direction the vector $b = Jx$ cannot be non-positive and not all zero when the step size is small enough (for example 10^{-6}) is considered as the termination condition.

4. Consideration of the error of estimate

The previous studies often design the TMD or MTMD without taking into account errors of estimate of the parameters of the structure–damper system, and then verify their robustness. Using the method stated above, the optimal non-uniformly distributed MTMD under some uncertainties of the parameters of the structure and TMDs can be easily obtained.

To interpret the difference of the design procedures of MTMD for the uncertain condition, the case that the upper bound of error of estimate of the structural natural frequency is 5% is considered without any loss of generality. Herein, 11 conditions in which the errors of estimate are $\pm 5\%$, $\pm 4\%$, $\pm 3\%$, $\pm 2\%$, $\pm 1\%$, 0 are considered.

The gradient vectors with different errors of estimate are calculated, respectively, and then those searching matrices are combined together. Other steps are almost as same as the procedure in the foregoing section.

5. Optimization results and discussion

The optimal parameters for the single TMD, which were analytically determined by Warburton [9], are used in this paper. And the optimal parameters for the uniformly distributed MTMD can be obtained by using linear search techniques such as quadratic interpolation [37,38].

The hypothesis that the masses of optimal MTMD are continuously distributed over a frequency range near the structural natural frequency is adopted in this paper. All mass ratios of the TMDs are reasonably assumed to be equal. And when the number of TMDs is large enough, the configuration of the MTMD approaches to that of the "true" optimal one.

5.1. *Configurations of optimal non-uniformly distributed MTMD*

Here, the frequency density of the mass distribution is defined as

$$\rho(\beta_i) = \frac{2\mu_i}{\beta_{i+1} - \beta_{i-1}} \quad (i = 1, \ldots, n). \tag{25}$$

To make the $\rho(\beta_1)$ and $\rho(\beta_n)$ well defined, we also assign $\beta_0 = 2\beta_1 - \beta_2$ and $\beta_{n+1} = 2\beta_n - \beta_{n-1}$ [1,13]. This function represents the mass density of the MTMD with respect to the normalized natural frequency.

The optimal configurations of MTMD for minimizing the DMF under harmonic force are as shown in Figs. 2 and 3. Numerical results indicate that neither tuning frequency spacings nor damping ratios of MTMD are identical for optimal design. As the number of TMDs increases, the normalized frequency range $(\beta_n - \beta_1)$ increases and the damping ratios decrease. The optimal damping ratios of individual TMDs obtained here does not coincide with those of Refs. [1,2], because the objectives of optimization are different. Fig. 4 shows that the median frequency tuning ratio is less than unity as same as the case of uniformly distributed MTMD. And the TMDs are more closely spaced in the center of the frequency range.

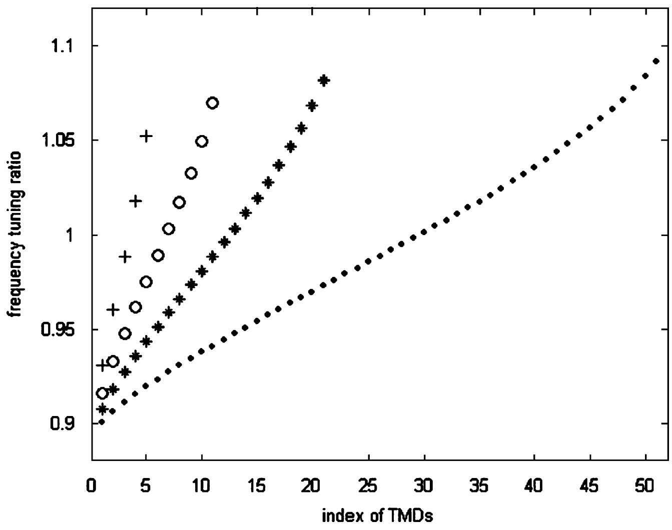

Fig. 2. Frequency tuning ratios of the optimized non-uniformly distributed MTMD for: $\mu = 0.01$, $\xi_s = 0.02$, +, $n = 5$; ○, $n = 11$; ✱, $n = 21$; •, $n = 51$.

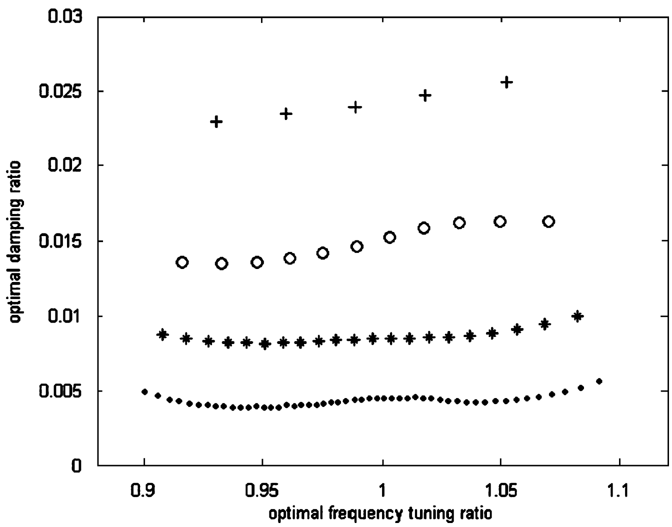

Fig. 3. Optimal damping ratios versus optimal frequency tuning ratios for: $\mu = 0.01$, $\xi_s = 0.02$, +, $n = 5$; ○, $n = 11$; ✱, $n = 21$; •, $n = 51$.

5.2. Comparison of non-uniform and uniform MTMDs

5.2.1. Effectiveness

The calculations for the structure-MTMD are conducted with various numbers of TMDs and the results are shown in Figs. 5 and 6. For comparison, the results of the structure with an optimal STMD and optimal uniform MTMD, which has equal frequency spacing and damping ratio, are also plotted. There are two distinct peaks for structure with optimal STMD. A similar trend is observed for structure with optimal non-uniformly distributed MTMD. The structure-MTMD system's DMF curves have $(n+1)$ local peaks with equal heights. The shapes of structure-MTMD system's DMF curves are appreciably different form the result of Zuo and Nayfeh [1]. In their paper, the peaks at higher frequencies tend to be suppressed more than those at lower frequencies, because the goal of optimization is to minimize the root-mean-square response.

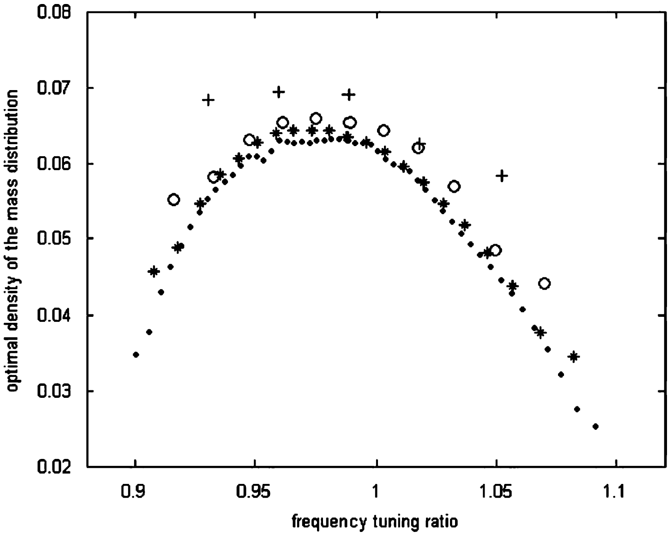

Fig. 4. Density of the mass distribution of the optimized non-uniformly distributed MTMD for: $\mu = 0.01$, $\xi_s = 0.02$, +, $n = 5$; ○, $n = 11$; ✱, $n = 21$; •, $n = 51$.

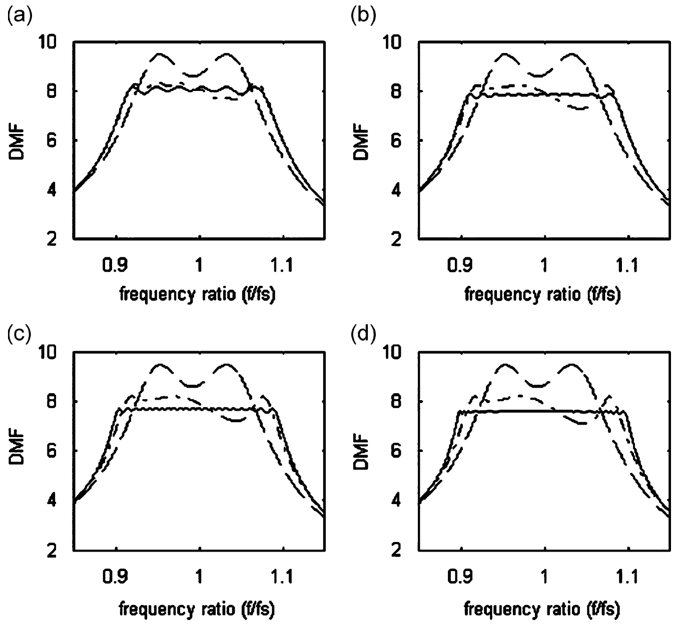

Fig. 5. DMF of the structure-MTMD system for: $\mu = 0.01$, $\xi_s = 0.02$: (a) $n = 5$, (b) $n = 11$, (c) $n = 21$, (d) $n = 51$, ————, optimal non-uniformly distributed MTMD; - - - - - -, optimal uniformly distributed MTMD; - ———, optimal STMD.

Fig. 6 shows the effectiveness of a MTMD is dependent on the total number of TMDs. When the total mass ratio is fixed, the effectiveness increases with the increasing of the total number of TMDs, but the reduction of the maximum DMF is not obvious when the number of TMDs exceeds a certain value. It is also seen that the non-uniformly distributed MTMD can achieve the same reduction of maximum DMF with a relatively small number of TMDs. It may be an advantage of non-uniformly distributed MTMD for practical use.

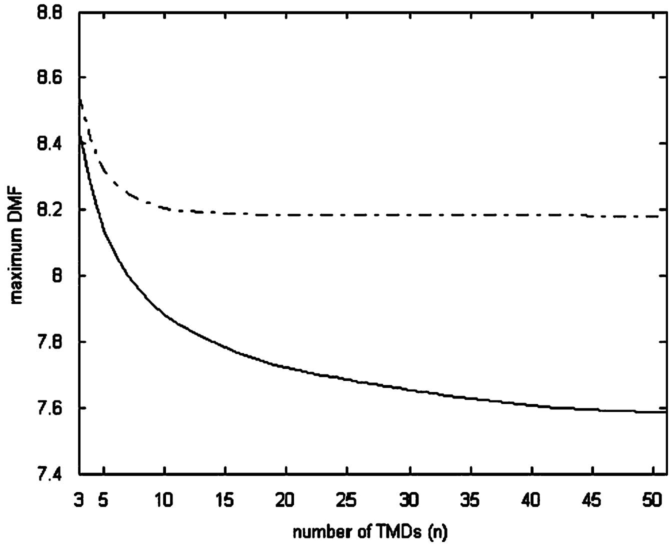

Fig. 6. Maximum DMF of the structure-MTMD system for: $\mu = 0.01$, $\xi_s = 0.02$, variable n, ————, optimal non-uniformly distributed MTMD; - – - – -, optimal uniformly distributed MTMD.

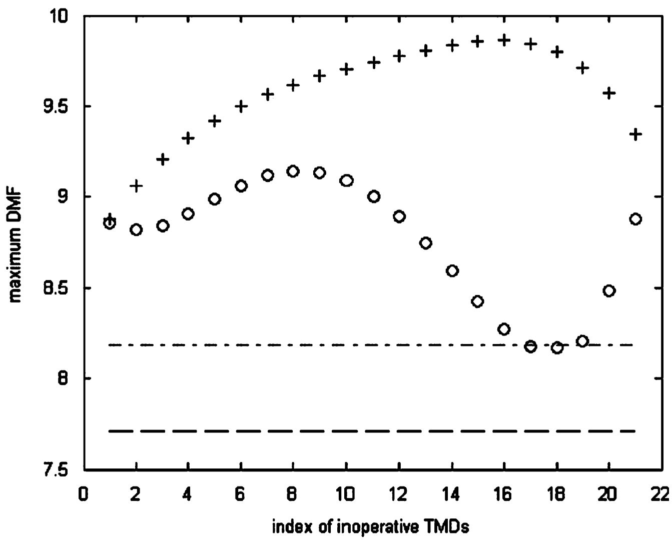

Fig. 7. Comparison of redundancy of non-uniformly distributed MTMD and uniformly distributed MTMD, +, partially inoperative optimal non-uniformly distributed MTMD; ○, partially inoperative optimal uniformly distributed MTMD; - – - – -, normal optimal uniformly distributed MTMD; - — — -, normal optimal non-uniformly distributed MTMD.

5.2.2. *Redundancy analysis*

Redundancy is defined as the ability of the system to be effective when one or more of the dampers does not function [31]. Fig. 7 indicates that the 17th and 18th TMDs of optimal uniform MTMD are redundant, removing one of them does not weaken the vibration reduction effect. However, invalidation of both the 17th and 18th TMDs results in obvious reduction of effectiveness of MTMD. In contrast, every TMD of optimal non-uniformly MTMD is essential, if any TMD is out of work, the response of structure increases rapidly. It is also noted that the non-uniform MTMD is more sensitive to the invalidation of certain TMD.

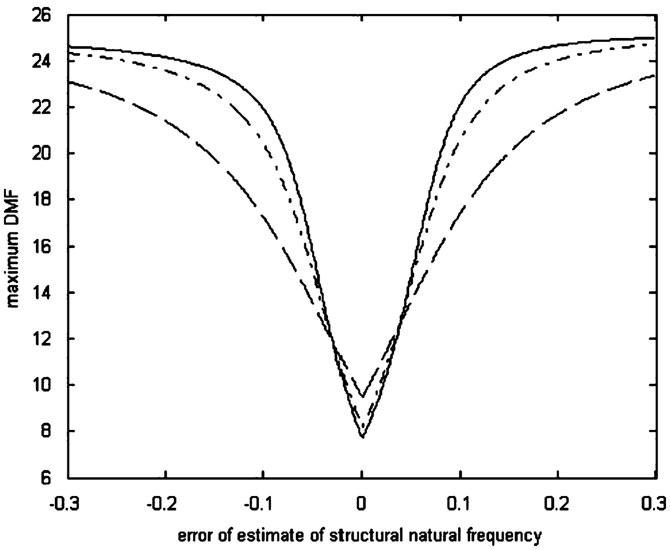

Fig. 8. Comparison of robustness of non-uniformly and uniformly distributed MTMD for the structural natural frequency, ————, optimal non-uniformly distributed MTMD; ------, optimal uniformly distributed MTMD; -———, optimal STMD.

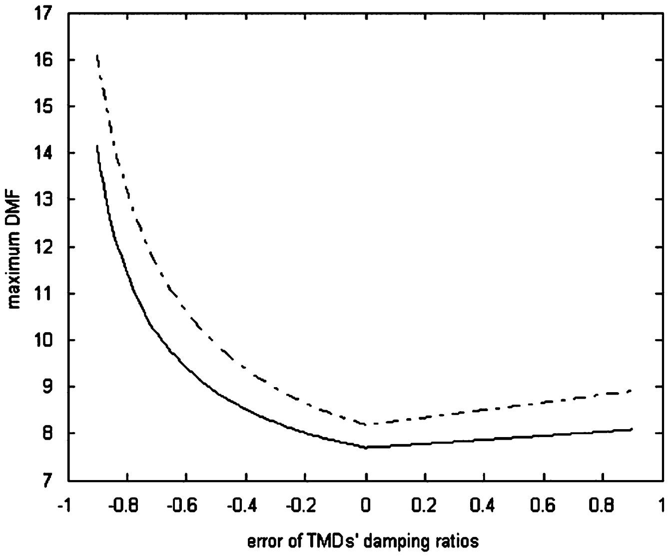

Fig. 9. Comparison of robustness of non-uniformly and uniformly distributed MTMD for TMDs' damping ratios, ————, optimal non-uniformly distributed MTMD; ------, optimal uniformly distributed MTMD.

5.2.3. Robustness of MTMD

The maximum DMF for a range of error of estimate of the natural frequency of the main structure are shown in Fig. 8. It can be seen that the effectiveness of MTMDs is better than that of the optimal STMD when the shift of the frequency tuning ratio is small. But the effectiveness of MTMDs rapidly decreases as the error of estimate of the structural frequency becomes larger. When the error is beyond 10%, the MTMD's effect of reducing vibration is obscure. Fig. 9 shows that the effectiveness of non-uniformly distributed MTMD is better than the uniformly distributed MTMDs for the same error of damping ratios of TMDs.

Table 1
Maximum DMF of the main structure with different damping ratios and various types of MTMDs

	ξ_s	STMD	Uniform MTMD with equal mass	Uniform MTMD with equal stiffness	Non-uniform MTMD
$\mu = 0.01$	0	14.2836	11.5122	11.2421	10.7881
	0.02	9.5290	8.2593	8.1256	7.8786
	0.05	6.2862	5.7424	5.6807	5.5565
$\mu = 0.05$	0	6.6408	5.6465	5.2376	5.0288
	0.02	5.4532	4.7707	4.4826	4.3260
	0.05	4.2820	3.8590	3.6746	3.5675
$\xi_s = 0.10$	0	4.9193	4.2975	3.8902	3.7362
	0.02	4.2747	3.7936	3.4790	3.3547
	0.05	3.5614	3.2218	2.9989	2.9057

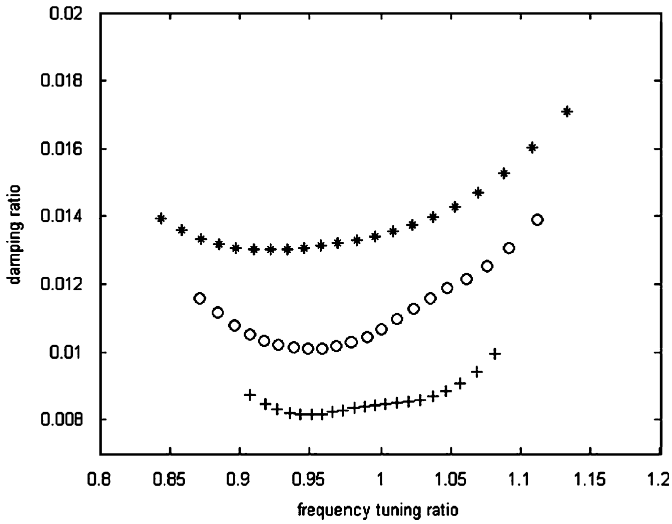

Fig. 10. Configurations of optimized non-uniformly distributed MTMDs for $\xi_s = 0.02$, $n = 21$, $+$, $\mu = 0.01$; ○, $\mu = 0.02$; ✱, $\mu = 0.03$.

5.2.4. Comparison of non-uniform MTMDs and uniform MTMDs with equal stiffness

Xu and Igusa [12] suggested that the manufacturing of the MTMD with uniform stiffness is simpler as compared with that with varying stiffness. The optimum parameters of the uniformly distributed MTMD with uniform stiffness under harmonic base excitation for both undamped and damped main systems have been obtained by Jangid [21] and Bakre and Jangid [22]. The values of the maximum DMF of the main structure for various damping ratios of structure and mass ratios of MTMD are given in Table 1.

Table 1 shows that the effectiveness of the optimal non-uniform MTMD is best and the effectiveness of the optimal uniform MTMD with uniform stiffness is better than that of the optimal uniform MTMD with equal mass. Thus, the uniformly distributed MTMD with uniform stiffness is not only simpler to produce, but also more effective than the uniformly distributed MTMD with uniform mass.

5.3. Effect of parameters on the configuration of optimal non-uniform MTMD

5.3.1. Effect of total mass ratio of MTMD

The structural responses with MTMDs, which have different total mass ratios, are potted in Fig. 10. With the increasing of the total mass ratio of MTMD, the frequency range of MTMD increases and damping ratios of TMDs increase.

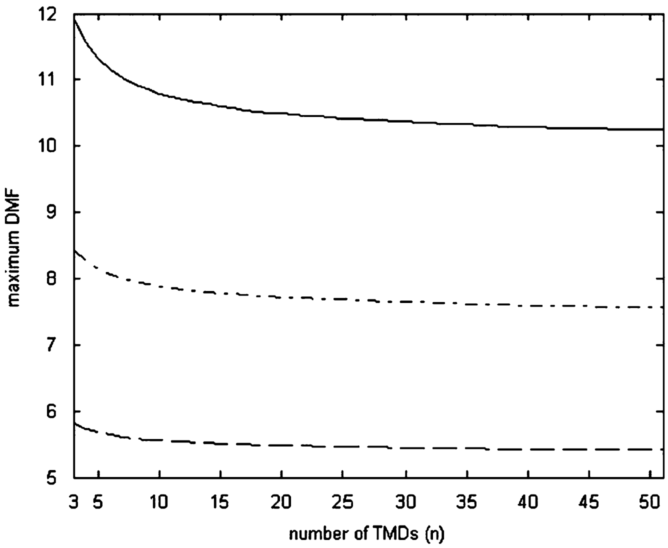

Fig. 11. Maximum DMF of the structure-MTMD system for: $\mu = 0.01$, variable n, ———, $\xi_s = 0$; - - - - -, $\xi_s = 0.02$; - ———, $\xi_s = 0.05$.

5.3.2. Effect of structural damping

The responses of the structure-MTMD system with various numbers of TMDs and structural damping ratios have been plotted in Fig. 11. For increasing number of TMDs, the effectiveness of MTMDs increases more obviously for the structure with small damping ratio. When the number of TMDs n exceeds 21, the effectiveness of MTMD nearly keeps constant.

5.4. Consideration of error of estimate

5.4.1. Effect of error of estimate of structural frequency

As stated above, the effectiveness of MTMD without considering the error of estimate of the structural frequency rapidly reduces for the structural frequency apart from its design value. However, the estimation error in the main structural natural frequency is inevitable in practice. To design robust MTMD is of primarily interest. Qualitative suggestions such as expanding the frequency range of MTMD, increasing damping ratios of TMDs are given by some previous papers, but quantitative consideration of the estimation error for MTMD's optimal design is lacking in their studies. This paper introduces a method to design optimal MTMD with quantitative consideration of certain level of error.

The result in Fig. 12 shows that although the MTMD designed for high level of error is less effective, it is more robust. Losing some effectiveness, the MTMD obtained here has an evident effect even if the error of estimate is very large. Fig. 13 suggests that increasing of number of TMDs can improve the effectiveness of MTMD, which considers estimation error.

5.4.2. Effect of error of estimate of structural damping

According to Fig. 14, the consideration of error of estimate of the structural damping has unconspicuous influence on the maximum DMF of the structural response. Although the maximum DMF is slightly reduced when the maximum error occurs, it is increased in other cases.

5.4.3. Effect of error of estimate of TMDs' damping

As presented in Figs. 9 and 15, the non-uniformly distributed MTMD is insensitive to the damping ratios of individual TMDs, except for very low values. And the influence of consideration of error shown in Fig. 15 is

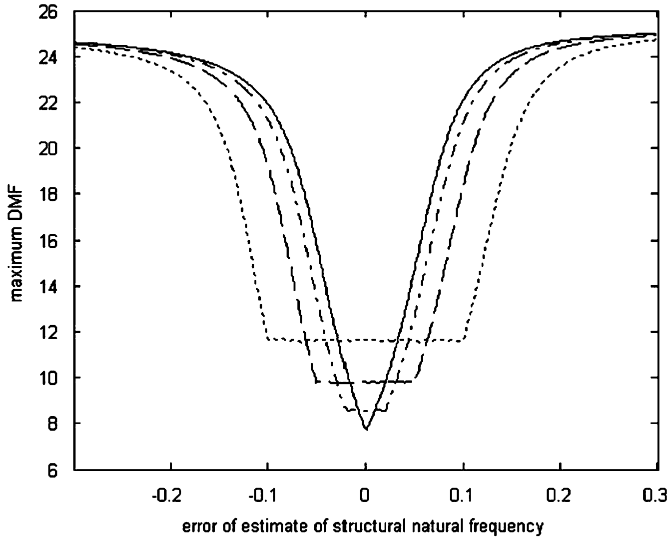

Fig. 12. Maximum DMF of the structure-MTMD system versus the error of estimate of structure's natural frequency for: $\mu = 0.01$, $\xi_s = 0.02$, $n = 21$, ————, without consideration of error; - - - - - -, consider maximum error = 0.02; - ——–, consider maximum error = 0.05; · - - - - - - - - ·, consider maximum error = 0.10.

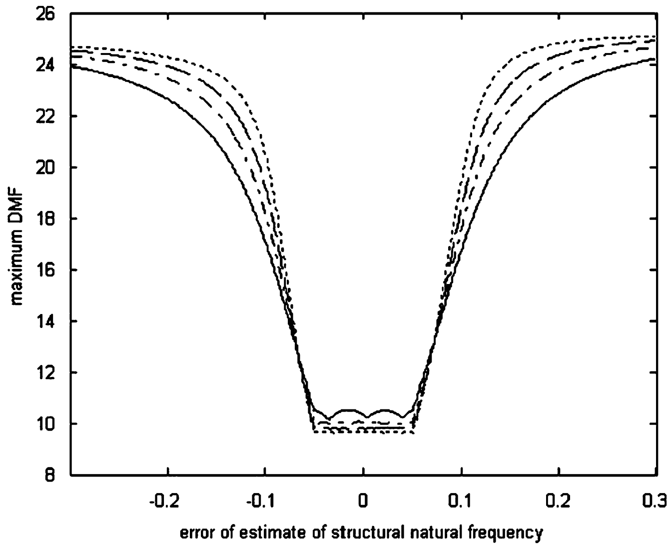

Fig. 13. Maximum DMF of the structure-MTMD system versus the error of estimate of the structure's natural frequency for: $\mu = 0.01$, $\xi_s = 0.02$, account for max error = 0.05, ————, $n = 5$; - - - - - -, $n = 11$; - ——–, $n = 21$; · - - - - - - - - ·, $n = 51$.

very limited. Therefore, for the damping values of TMDs near the design values, it is not necessary to take into account the errors of estimate of damping ratios of TMDs.

5.5. *Reduction of the maximum displacement responses to earthquake loadings*

Although the optimum parameters of the non-uniformly distributed MTMD to achieve the largest reduction for a prescribed earthquake record can be obtained using the method presented in this paper, they may be not proper for other earthquake loadings. Herein, we consider the MTMDs for minimizing the

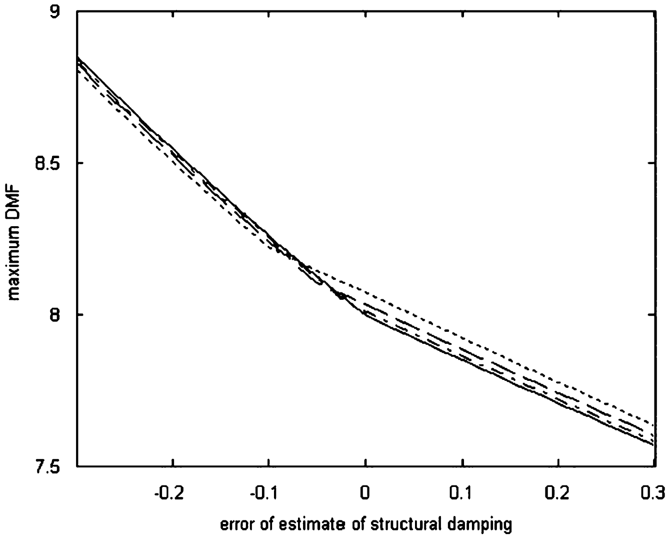

Fig. 14. Maximum DMF of the structure-MTMD system for: $\mu = 0.01$, $\xi_s = 0.02$, $n = 7$, ————, without consideration of error; - - - - - -, consider maximum error = 0.02; -———, consider maximum error = 0.05; · - - - - - - - - ·, consider maximum error = 0.10.

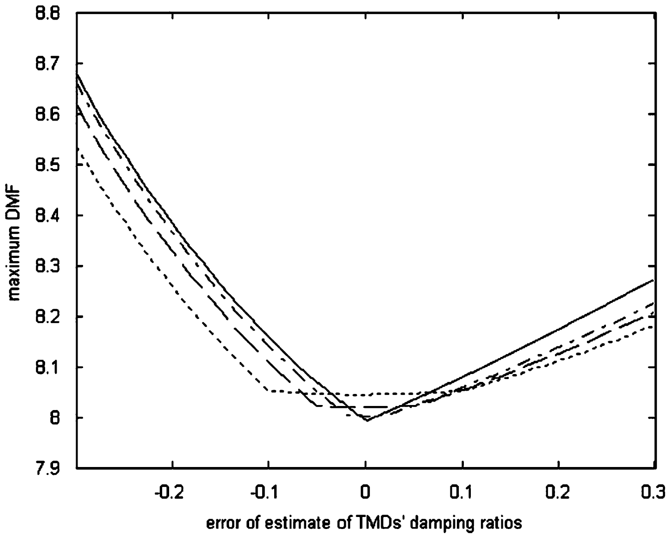

Fig. 15. Maximum DMF of the structure-MTMD system for: $\mu = 0.01$, $\xi_s = 0.02$, $n = 7$, ————, without consideration of error; - - - - - -, consider maximum error = 0.02; -———, consider maximum error = 0.05; · - - - - - - - - · consider maximum error = 0.10.

maximum response in frequency domain as the proper ones for seismic applications. The reduction ratios of different types of MTMDs under various base excitations are summarized in Table 2. It is seen that the effectiveness of the non-uniformly distributed MTMD is best for most cases and the effectiveness decreases as the damping ratio of the main structure increases.

6. Concluding remarks

In this paper, a gradient-based method for optimizing non-uniformly distributed MTMD is presented. Owing to the flexibility of the proposed method, it can be used when uncertainties of parameters of the

Table 2
Reduction ratios for various earthquake records and damping ratios of the main structure, $\mu = 0.05$

	ξ_s	STMD (%)	Uniform MTMD (%)	Non-uniform MTMD (%)
El Centro	0	36.76	43.35	44.92
	0.02	34.44	40.02	40.63
	0.05	31.81	34.74	35.02
Taft	0	61.82	59.95	59.39
	0.02	21.29	18.69	18.99
	0.05	3.24	3.87	4.01
Kobe	0	34.84	35.66	35.93
	0.02	10.39	11.68	12.03
	0.05	10.23	11.61	12.01
Tianjin	0	23.66	25.71	26.24
	0.02	10.05	12.50	13.03
	0.05	9.84	12.18	12.75

structure and TMDs exist. The proposed method has been conducted to obtain the "true" optimal MTMD for various conditions. The following results are presented:

(1) The optimal non-uniformly distributed MTMD is more effective than uniformly distributed MTMD with some restrictions on the frequency spacing and damping. Both the frequency spacings and damping ratios of TMDs are unequal. Without restrictions on the frequency spacing and damping of TMDs, the MTMD obtained in the present paper is "true" optimal MTMD for harmonic forces.
(2) The harmonic response of the structure-MTMD system optimized for maximum DMF has $(n+1)$ peaks with equal heights. And as some peaks fall, others rise. This is the common feature of the final solution of the multiple objectives optimization problem.
(3) The frequency tuning is the most important factor of the effectiveness of MTMD. The error of estimate of the structural natural frequency should be considered; otherwise the effectiveness of MTMD is overestimated. An efficient numerical algorithm of designing optimal MTMDs, which have better robustness for uncertainties in the structure's natural frequency, is also proposed.
(4) The effectiveness of non-uniformly distributed MTMD is better than the uniformly distributed MTMDs for the same error of damping ratios of TMDs and the structural frequencies near the design values.
(5) Owing to the flexibility of the proposed method, other errors of estimate can be taken into account easily, although their effects are not so apparent as that of the structural natural frequency.

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