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Short Communication

Solvability condition in multi-scale analysis of gyroscopic continua

Li-Qun Chen[a,b,*], Jean W. Zu[c]

[a]*Department of Mechanics, Shanghai University, 99 Shang Da Road, Shanghai 200444, China*
[b]*Shanghai Institute of Applied Mathematics and Mechanics, Shanghai 200072, China*
[c]*Department of Mechanical & Industrial Engineering, University of Toronto, Toronto, Ontario, Canada M5S 3G8*

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Abstract

The solvability condition is investigated for the method of multiple scales applied to gyroscopic continua. The general framework of the multi-scale analysis is proposed for a linear gyroscopic continuous system under small nonlinear time-dependent disturbances. The solvability condition is derived from the properties of the systems. The condition holds only for appropriate boundary conditions. The appropriateness of the boundary conditions can be examined for unperturbed linear systems. An example is presented to highlight the requirements on the boundary conditions.

1. Introduction

Gyroscopic continua are translating or rotating structures with distributed mass and elasticity or viscoelasticity. Gyroscopic continua include translating and rotating strings, beams, cables, membranes, plates, and shells. A skew symmetric term, the gyroscopic term, in the governing equation results in some particular characteristics of gyroscopic continua.

Many investigators addressed dynamic analysis on gyroscopic continua. Hughes and D'Eleuterio [1] developed a modal analysis approach. Wickert and Mote [2] proposed a modal analysis solution to transverse vibration of moving strings and beams under arbitrary excitations and initial conditions. Based on the transfer function formulation, Yang [3] proved some eigenvalue inclusion theorems for gyroscopic continua under pointwise, nondissipative constraints. Renshaw and Mote [4] presented a general observation to predict divergence instability for gyroscopic continua near vanishing eigenvalues. Wickert [5] calculated the first-order approximation for transient vibration of gyroscopic continua with unsteady superposed motion via the asymptotic method of Krylov, Bogoliubov, and Mitropolsky. Parker [6] presented a perturbation analysis to determine approximate eigenvalue loci and stability conditions in the vicinity of critical speeds and zero speed.

*Corresponding author. Department of Mechanics, Shanghai University, 99 Shang Da Road, Shanghai 200444, China.
Tel.: +86 21 66134972; fax: +86 21 66134463.
E-mail address: lqchen@staff.shu.edu.cn (L.-Q. Chen).

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All above mentioned investigations treated linear gyroscopic continua. Recently, much attention was paid to nonlinear gyroscopic continua. As the diversity of the nonlinear problem, there have had no general results for nonlinear gyroscopic continua. Instead, some specific nonlinear gyroscopic continuous systems were studied. Among other approaches, the method of multiple scales is a powerful tool that can be directly applied to nonlinear partial differential equations without discretization. The direct method of multiple scales was used to analyze two special classes of nonlinear gyroscopic continua, axially moving strings [7–11] and beams [12–14]. In addition to nonlinear vibration, the method of multiple scales was also applied to investigate the stability of linear parametric vibration of axially accelerating strings [15] and beams [16–20]. The key issue in the applications of the method of multiple scales is to derive the solvability condition, which can be utilized to determine stability conditions in linear parametric vibration, nonlinear frequencies in free vibration, and steady-state responses in forced or parametric vibrations.

The authors will develop the solvability condition for a general gyroscopic continuous system with weak nonlinear time-dependent disturbance. The conclusion is true only for appropriate boundary conditions, while the boundary conditions can be checked only for the unperturbed linear part of the system. An axially moving beam on an elastic foundation with hybrid ends is treated to demonstrate the examination of the boundary conditions.

2. Analysis via the method of multiple scales

Consider a gyroscopic continuous system with a weak disturbance

$$Mv_{,tt} + Gv_{,t} + Kv = \varepsilon N(x,t), \tag{1}$$

where $v(x,t)$ is the generalized displacement of the system at spatial coordinate x and time t, $(v)_{,t}$ denotes partial derivative of (v) with respect to t, M, G and K represent mass, gyroscopic and stiffness operators respectively, ε stands for a small dimensionless parameter, and $N(x,t)$ expresses a nonlinear function of x and t that may explicitly contain v and its spatial and temporal partial derivatives as well as its integral over a spatial region or a temporal interval. $N(x,t)$ is periodic in time with the period $2\pi/\omega$. M, G and K are linear, time-independent, spatial differential operators. Introduce an inner product

$$\langle f,g\rangle = \int_E f(x)\bar{g}(x)\,\mathrm{d}x, \tag{2}$$

for complex functions f and g defined in a bounded, open region E in R^n, $n = 1,2$, or 3, where the overbar denotes the complex conjugate. M and K are symmetric in the sense

$$\langle Mf,g\rangle = \langle f,Mg\rangle, \langle Kf,g\rangle = \langle f,Kg\rangle, \tag{3}$$

and G is skew symmetric in the sense

$$\langle Gf,g\rangle = -\langle f,Gg\rangle \tag{4}$$

for all functions f and g satisfying appropriate boundary conditions.

The method of multiple scales will be employed to solve Eq. (1) without discretization. A uniform approximation is sought in the form

$$v(x,t) = v_0(x,T_0,T_1) + \varepsilon v_1(x,T_0,T_1) + O(\varepsilon^2), \tag{5}$$

where $T_0 = t$, $T_1 = \varepsilon t$, and $O(\varepsilon^2)$ denotes the term with the same order as ε^2 or higher. Substitution of Eq. (5) into Eq. (1) yields

$$Mv_{0,T_0T_0} + Gv_{0,T_0} + Kv_0 = 0, \tag{6}$$

$$Mv_{1,T_0T_0} + Gv_{1,T_0} + Kv_1 = N_1(x,T_0,T_1), \tag{7}$$

where $N_1(x,T_0,T_1)$ stands for a nonlinear function of x, T_0 and T_1, which usually depends explicitly on v_0 and its derivatives and integrals. In addition, $N_1(x,T_0,T_1)$ is periodic in T_0 with the period $2\pi/\omega$.

Separation of variables leads to the solution of Eq. (6) as

$$v_0(x, T_0, T_1) = \sum_{j=1}^{\infty} A_j(T_1)\varphi_j(x)\,\mathrm{e}^{\mathrm{i}\omega_j T_0} + \mathrm{cc}, \tag{8}$$

where A_j denotes a complex function to be determined later, φ_j and ω_j represents, respectively, the complex modal function and the natural frequency defined by

$$-\omega_j^2 M\varphi_j + \mathrm{i}\omega_j G\varphi_j + K\varphi_j = 0 \tag{9}$$

and the boundary conditions, and cc stands for the complex conjugate of all preceding terms on the right side of an equation.

If ω approaches a linear combination of natural frequencies of system (6), the summation parametric response may occur. A detuning parameter σ is introduced to quantify the deviation of ω from the combination, and ω is described by

$$\omega = \sum_{j=1}^{\infty} c_j\omega_j + \varepsilon\sigma, \tag{10}$$

where c_j are real constants that are not all zero and only a finite of them are not zero. To investigate the summation parametric response, substitution of Eqs. (8) and (10) into Eq. (9) leads to

$$Mv_{1,T_0T_0} + Gv_{1,T_0} + Kv_1 = \sum_{j=1}^{\infty} F_j(x, T_1)\,\mathrm{e}^{\mathrm{i}\omega_j T_0} + \mathrm{NST} + \mathrm{cc}, \tag{11}$$

where $F_j(x,T_1)$ ($j = 1,2,\ldots$) are complex functions dependent explicitly on A_j (T_1) and their temporal derivatives as well as φ_j (x) and their spatial derivatives and integrals.

3. Solvability condition

To derive the solvability condition, assume that the solution of Eq. (11) take the following form:

$$v_1(x, T_0, T) = \sum_{j=1}^{\infty} \psi_j(x, T_1)\,\mathrm{e}^{\mathrm{i}\omega_j T_0} + u(x, T_0, T) + \mathrm{cc}, \tag{12}$$

where $u(x,T_0,T_1)$ stands for all non-secular terms in the solution. According to Eq. (12), $\psi_j(x,T_1)$ is with the same boundary conditions as $\varphi_j(x)$. Substitution of Eq. (12) into Eq. (11) and then equalization of coefficients of $\mathrm{e}^{\mathrm{i}\omega_j T_0}$ in the resulting equation give

$$-\omega_j^2 M\psi_j + \mathrm{i}\omega_j G\psi_j + K\psi_j = F_j(x, T_1). \tag{13}$$

Thus for the complex modal function $\varphi_j(x)$,

$$\left\langle F_j(x, T_1), \varphi_j \right\rangle = \left\langle -\omega_j^2 M\psi_j + \mathrm{i}\omega_j G\psi_j + K\psi_j, \varphi_j \right\rangle. \tag{14}$$

Application of the distribution law to Eq. (14) yields

$$\left\langle F_j(x, T_1), \varphi_j \right\rangle = -\omega_j^2\left\langle M\psi_j, \varphi_j \right\rangle + \mathrm{i}\omega_j\left\langle G\psi_j, \varphi_j \right\rangle + \left\langle K\psi_j, \varphi_j \right\rangle. \tag{15}$$

Using Eqs. (3) and (4) and the distribution law, one obtains

$$\left\langle F_j(x, T_1), \varphi_j \right\rangle = \left\langle \psi_j, -\omega_j^2 M\varphi_j + \mathrm{i}\omega_j G\varphi_j + K\varphi_j \right\rangle \tag{16}$$

in which the following equation is employed:

$$c\langle f, Gg\rangle = \langle f, \bar{c}Gg\rangle \tag{17}$$

for a complex constant c. Hence one concludes from Eq. (9) that

$$\left\langle F_j(x, T_1), \varphi_j \right\rangle = 0. \tag{18}$$

Eq. (18) is the solvability condition that requires the coefficient of $e^{i\omega_j T_0}$ in the right-hand side of the first-order equation to be orthogonal to jth model function of the zero order equation.

It should be noticed that the solvability condition (18) holds providing the boundary conditions are appropriate. That is, M and K are symmetric and G is skew symmetric under the boundary conditions. In a specific problem, these requirements can be checked for a given operators, boundary conditions and the modal functions. However, the examination depends only on the unperturbed linear part of the problem. Here an example is presented to demonstrate the procedure.

4. An example

A uniform axially moving beam, with linear density ρA, initial tension P_0, and flexural rigidity EI, travels at the constant mean axial speed γ_0 between two ends separated by distance l on an elastic foundation with stiffness k under some small disturbances due to nonlinearity, viscoelasticity, and excitations such as fluctuations in the axial speed or the string tension. At two ends, the axially moving beam is constrained by simple supports with torsion springs.

In this case, mass, gyroscopic and stiffness operators are, respectively,

$$M = \rho A, \quad G = 2\rho A\gamma_0 \frac{\partial}{\partial x}, \quad K = (\rho A\gamma_0^2 - P_0)\frac{\partial^2}{\partial x^2} + \mathrm{EI}\,\frac{\partial^4}{\partial x^4} + k. \tag{19}$$

For complex modal functions φ_j of the corresponding linear problem and a complex function ψ_j with the same boundary conditions, the prescribed boundary conditions are:

$$\varphi_j(0) = 0, \quad \varphi_j(l) = 0, \quad \psi_j(0) = 0, \quad \psi_j(l) = 0, \tag{20}$$

$$\varphi_j''(0) - k_1\varphi_j'(0) = 0, \quad \varphi_j''(l) - k_2\varphi'(l) = 0, \quad \psi_j''(0) - k_1\psi_j'(0) = 0, \quad \psi_j''(l) - k_2\psi_j'(l) = 0, \tag{21}$$

where k_1 and k_2 are two constants. Application of Eq. (2) directly gives

$$\left\langle M\psi_j, \varphi_j \right\rangle = \left\langle \psi_j, M\varphi_j \right\rangle. \tag{22}$$

Therefore, M is always symmetric regardless boundary conditions. Application of Eq. (2) and integration by parts yield

$$\left\langle G\psi_j, \varphi_j \right\rangle = -\left\langle \psi_j, G\varphi_j \right\rangle + 2\rho A\gamma_0(\psi_j\bar{\varphi}_j)\Big|_0^l. \tag{23}$$

Hence, G is skew symmetric for all motionless boundaries satisfying Eq. (20). Application of Eq. (2) and integration by parts repeatedly lead to

$$\left\langle K\psi_j, \varphi_j \right\rangle = \left\langle \psi_j, K\varphi_j \right\rangle + [(\rho A\gamma_0^2 - P_0)(\psi_j'\bar{\varphi}_j - \psi_j\bar{\varphi}_j') + \mathrm{EI}(\psi_j'''\bar{\varphi}_j - \psi_j''\bar{\varphi}_j' + \psi_j'\bar{\varphi}_j'' - \psi_j\bar{\varphi}_j''')]\Big|_0^l. \tag{24}$$

Thus, K is symmetric for ends with Eqs. (20) and (21).

5. Summary

The authors present the general framework of the application of the method of multiple scales to a weak nonlinear gyroscopic continuous system without discretization. The solvability condition is proved as the orthogonality of the coefficient of the resonant term in the first-order equation and the corresponding modal function of the zero order equation. The correctness of the solvability condition depends on the appropriateness of boundary conditions, which can be checked for the mass, gyroscopic and stiffness operators in the zero order equation. An axially moving beam on an elastic foundation is treated as an example to demonstrate the procedure of examining boundary conditions.

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