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Short Communication

Vibration tailoring of inhomogeneous rod that possesses a trigonometric fundamental mode shape

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Abstract

In this study, a special class of closed-form solutions for inhomogeneous rod is investigated. Namely, the following problem is considered: determine the distribution axial rigidity when the material density is given of an inhomogeneous rod so that the postulated fundamental trigonometric mode shape serves as an exact vibration mode. In this study, the associated semi-inverse problem is solved that results in the distributions of axial rigidity that together with a specified law of material density satisfy the governing eigenvalue problem. For comparison, the obtained closed-form solutions are contrasted with approximate solutions based on an appropriate polynomial shapes, serving as trial functions in an energy method. The obtained results are utilized for vibration tailoring, i.e. construction of the rod with a given natural frequency.

1. Introduction

Vibration of longitudinally vibrating homogeneous rods is a classic subject that is covered in nearly every textbook but inhomogeneous rods have been studied much less extensively. Conway et al. [1] considered tapered rods. Li considered cases of continuous [2] and piecewise variations of cross-sectional area [3]. Candan and Elishakoff [4] constructed several closed-form solutions for inhomogeneous rods with continuously variable modulus of elasticity. They postulated polynomial mode shapes and polynomial elastic modulus variations (see also the recent monograph by Elishakoff [5]). Ram and Elishakoff [6] recently dealt with a discrete formulation whereas new closed-form solution was derived by Raj and Sujith [7]. Trigonometric function as candidate mode shape for deriving novel closed-form solutions have also been used by Caliò and Elishakoff with reference to inhomogeneous beam–columns both for buckling [8] and vibration problems [9–11]. Specifically in Ref. [8], it has been proved that harmonic functions can constitute closed-form buckling mode of inhomogeneous column subjected to a distribution of axial compressive load. In Ref. [9], it has been shown that a trigonometric function can serve both as the vibration and the buckling mode of inhomogeneous beam–columns under different end conditions while in Refs. [10,11], the study has been extended to inhomogeneous beam–column on elastic foundations.

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This study extends Refs. [4,5] to the case of trigonometric variations of modulus of elasticity along with postulated mode shape, that also varies trigonometrically. Specifically, the problem formulation ought be understood as follows: for a given material density function, reconstruct the flexural rigidity, such that the rod has a given mode shape. For a given mass density, it turns out that the semi-inverse problem formulation yields closed-form solutions.

2. Formulation of problem

Let us consider an inhomogeneous rod of length L, cross-sectional area $A(x)$, varying modulus of elasticity $E(x)$, and varying material density $\rho(x)$. The governing differential equation of the axial dynamic behavior of such an inhomogeneous rod is given by

$$\frac{\partial}{\partial x}\left[E(x)A(x)\frac{\partial u(x,t)}{\partial x}\right]-\rho(x)A(x)\frac{\partial u^2(x,t)}{\partial t^2}=0, \tag{1}$$

where x is the axial coordinate, t the time and $u(x,t)$ the axial displacement.

For simplicity, the non-dimensional coordinate $\xi = x/L$ is introduced. Harmonic vibration is studied so that the displacement $u(x,t)$ is represented as follows:

$$u(\xi,t)=U(\xi)\mathrm{e}^{\mathrm{i}\omega t}, \tag{2}$$

where $U(\xi)$ is the postulated mode shape and ω the corresponding natural frequency which has to be determined. Upon substitution of Eqs. (2) into (1), the latter becomes

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[E(\xi)A(\xi)\frac{\mathrm{d}U(\xi)}{\mathrm{d}\xi}\right]+\rho(\xi)A(\xi)\omega^2L^2U(\xi)=0. \tag{3}$$

The semi-inverse eigenvalue problem is posed as follows: Find an inhomogeneous beam that possesses a specified harmonic mode, $U(\xi)$, that satisfies the boundary conditions and the governing dynamic equation of motion. This semi-inverse problem requires the determination of the distribution of axial rigidity, $D(\xi) = E(\xi)A(\xi)$, that together with a pre-specified law of the mass distribution, $m(\xi) = \rho(\xi)A(\xi)$ satisfy Eq. (3).

The axial rigidity $D(\xi)$ and the mass distribution $m(\xi)$ are represented as follows:

$$D(\xi)=D_o+D_1\sin(\varphi\pi\xi)+D_2\cos(\varphi\pi\xi), \tag{4}$$

$$m(\xi)=m_o+m_1\sin(\gamma\pi\xi)+m_2\cos(\gamma\pi\xi), \tag{5}$$

where D_o, D_1, D_2, m_o, m_1, m_2 are constants, while φ and γ are real numbers. There are several alternatives: (1) The posed semi-inverse problem may have no solution, or (2) it may possess multiple solutions, or (3) it may possess a unique solution. It will be shown that for a pre-specified distribution of material density, the solution turns out to be a unique one.

Let us consider rods whose ends are either fixed or free. At the fixed end, the boundary condition reads

$$U=0, \tag{6}$$

whereas at the free end, the boundary condition is

$$\mathrm{d}U/\mathrm{d}\xi=0. \tag{7}$$

In this study, the differential equation (1) will be solved in the closed-form for three different boundary conditions corresponding to the fixed–fixed rod, fixed–free rod, and free–free rod.

3. Fixed–fixed inhomogeneous rod

We first consider a fixed–fixed inhomogeneous rod for which the sinusoidal vibration mode that is possessed by the uniform rod [12], is postulated:

$$U(\xi)=\sin(\pi\xi). \tag{8}$$

Furthermore, the following expression of the axial rigidity and of the given mass distribution, according to the general forms (4) and (5), are considered:

$$D(\xi) = D_o[1 + \alpha \cos(\pi\xi)], \tag{9}$$

$$m(\xi) = m_o[1 + \beta \cos(\pi\xi)]. \tag{10}$$

The coefficients m_o and β are given, we must find D_o and α. In view of Eqs. (9),(10), the left-hand side differential equation (3), denoted R can be re-written as

$$R = \{-\pi^2 D_o[1 + 2\alpha \cos(\pi\xi)] + L^2 m_o[1 + \beta \cos(\pi\xi)]\omega^2\} \sin(\pi\xi). \tag{11}$$

With the view to obtaining a closed-form solution, we set $\alpha = \beta/2$, in order to be able to write the latter expression as a product

$$[-\pi^2 D_o + L^2 m_o \omega^2][1 + \beta \cos(\pi\xi)] \sin(\pi\xi) = 0 \tag{12}$$

leading to the following natural frequency:

$$\omega^2 = \pi^2 D_o / m_o L^2. \tag{13}$$

It is worth noticing that the inhomogeneous rod has fundamental natural frequency, which is the fundamental of the homogeneous rod with constant stiffness D_o and uniform mass distribution m_o. Since the stiffness and the mass distributions must be positive β must satisfy the inequality $-1 < \beta < 1$.

Note that the value D_o is not specified; therefore, we conclude that D_o can be any positive number, thus we arrive at infinite amount of solutions. In Section 7, we will demonstrate how to obtain the unique solution to the posed problem.

4. Fixed–free inhomogeneous rod

Proceeding as in the previous case the vibration mode of the uniform rod under the same boundary condition is postulated:

$$U(\xi) = \sin(\pi\xi/2). \tag{14}$$

The axial rigidity and the mass distribution are considered in the form

$$D(\xi) = D_o[1 + \alpha \cos(\pi\xi/2)], \tag{15}$$

$$m(\xi) = m_o[1 + \beta \cos(\pi\xi/2)]. \tag{16}$$

Again, m_o and β are given, the problem consists in determining D_o and α.

By setting $\alpha = \beta/2$, and substituting expressions (15) and (16) into the differential equation (3) we obtain

$$\left[-\frac{\pi^2}{4} D_o + L^2 m_o \omega^2\right]\left[1 + \beta \cos\left(\frac{\pi}{2}\xi\right)\right] \cos(\pi\xi) = 0 \tag{17}$$

leading to the following natural frequency:

$$\omega^2 = \pi^2 D_o / 4 m_o L^2. \tag{18}$$

As in the previous case, also for the fixed–free boundary conditions, the inhomogeneous rod shares the same fundamental natural frequency of the homogeneous one. Since the stiffness and the mass distributions must be positive β, must satisfy the inequality $\beta > -1$.

5. Free–free inhomogeneous rod

The vibration mode of the uniform free–free rod is postulated:

$$U(\xi) = \cos(\pi\xi). \tag{19}$$

Furthermore, for the inhomogeneous rod under question, the following expression of the axial rigidity and of the mass distribution are considered:

$$D(\xi) = D_o[1 + \alpha \sin(\pi\xi)], \tag{20}$$

$$m(\xi) = m_o[1 + \beta \sin(\pi\xi)]. \tag{21}$$

Substituting the expressions (20) and (21), the left-hand side of the differential equation (3) becomes

$$R = \{-\pi^2 D_o[1 + 2\alpha \sin(\pi\xi)] + L^2 m_o[1 + \beta \cos(\pi\xi)]\omega^2\} \cos(\pi\xi). \tag{22}$$

By setting $\alpha = \beta/2$ expression (22) assumes the product form and the equation of motion becomes

$$\lfloor -\pi^2 D_o + L^2 m_o \omega^2 \rfloor \lfloor 1 + \beta \sin(\pi\xi) \rfloor \cos(\pi\xi) = 0 \tag{23}$$

that leads to the natural frequency

$$\omega^2 = \frac{\pi^2 D_o}{m_o L^2}. \tag{24}$$

Since the stiffness and the mass distributions must be positive, the coefficient β must satisfy the inequality $\beta > -1$. Note that Eq. (24) coincides with Eq. (13) formally; it should be stressed that in the two cases the mode shapes differ. Again, as in preceding two cases, Eq. (24) signifies, due to arbitrariness of D_o, that we arrived at infinite amount of solutions. In Section 7, we introduce an additional condition that selects a unique solution from the infinite set.

6. Extension of the closed-form solutions to all modes

The previous obtained results can easily be extended to all modes by replacing π by $n\pi$. The only verification that must be performed is on the positive signs of the mass and stiffness distributions. In Table 1, the results of this generalization are summarized.

7. Vibration tailoring

Let the vibration tailoring consist in the requirement the natural frequency be equal a pre-specified value Ω. For the clamped–clamped rod we utilize Eq. (13); for a given value of m_o we find the value

$$D_o = \Omega^2 m_o L^2/\pi^2 \tag{25}$$

that is needed in order the rod's fundamental frequency to be equal to Ω. For the clamped–free rod, in order to achieve the same natural frequency Ω, one has to take, as Eq. (18) suggests

$$D_o = 4\Omega^2 m_o L^2/\pi^2, \tag{26}$$

whereas for the free–free rod the necessary D_o is again given by Eq. (26), although the mode shapes differ.

The solutions reported herein can be used as benchmarks. Also, in the future, when the technology allows the production of rods with any desired distribution of axial rigidity and material density along the axis of the rod, one will be able to design tailored inhomogeneous rods with pre-selected natural frequency. Thus, the methodology of solving semi-inverse problems presented in this study may represent a valuable design tool for vibration problems within the trigonometric class of inhomogeneity.

Table 1
Extension of the obtained results to all modes of vibrations

Boundary condition	Postulated mode shapes	Frequencies	Condition on the values of β
Fixed–fixed	$U(\xi) = \sin(n\pi\xi)$	$\omega^2 = (n^2\pi^2 D_o)/(m_o L^2)$	$-1 < \beta < 1$
Fixed–free	$U(\xi) = \sin(n\pi\xi/2)$	$\omega^2 = (n^2\pi^2 D_o)/(4m_o L^2)$	$-1 < \beta < 1$
Free–free	$U(\xi) = \cos(n\pi\xi)$	$\omega^2 = (n^2\pi^2 D_o)/(m_o L^2)$	$-1 < \beta < 1$

8. Conclusion

Apparently for the first time in the literature, trigonometric closed-form solutions have been derived for the natural frequencies of a rod under three sets of boundary conditions. The distributions of the axial rigidity and material density are represented in terms of trigonometric functions. The trigonometric function was also postulated for the fundamental mode shape; conditions were established for which this postulate holds.

It appears remarkable that while the trigonometric mode shapes are possessed by homogeneous rods (see the definitive textbook by Rao [12]), they can also appear in the inhomogeneous rods. The semi-inverse problem turned out to be an effective tool for determining the closed-form solutions. One should stress that we cannot analytically reconstruct both the material density and the flexural rigidity, as the definitive study of Baruch [13] demonstrates. Therefore, the obtained results ought be interpreted in the following manner: For the given material density, we are able to reconstruct the flexural rigidity that is compatible with the postulated mode shape such that the fundamental natural frequencies equal the pre-selected value. Thus formulation that was put forward in this study allowed the effective solution of the vibration-tailoring problem.

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