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Short Communication

Combined dynamic stiffness matrix and precise time integration method for transient forced vibration response analysis of beams

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Abstract

A method has been developed for determining the transient response of a beam. The beam is divided into several continuous Timoshenko beam elements. The overall dynamic stiffness matrix is assembled in turn. Using Leung's equation, we derive the overall mass and stiffness matrices which are more suitable for response analysis than the overall dynamic stiffness matrix. The forced vibration of the beam is computed by the precise time integration method. Three illustrative beams are discussed to evaluate the performance of the current method. Solutions calculated by the finite element method and theoretical analysis are also enumerated for comparison. In these examples, we have found that the current method can solve the forced vibration of structures with a higher precision.

1. Introduction

The dynamic stiffness method can be considered as an improvement of the dynamic transfer matrix method and finite element method (FEM). It includes the frequency-dependent shape function from the solution of the governing differential equations. It also has the advantages of FEM, which is more suitable for complex problems than the transfer matrix method. The dynamic stiffness method was developed in the 1940s by Kolousěk [1], who used the exact displacement method to derive the dynamic stiffness matrix of the Bernoulli–Euler beam. Since then, many scholars have mentioned the continuous element and dynamic stiffness matrix. Åkesson reviewed research before 1976 [2]. Recent progress and advances have been discussed by Leung [3]. To solve the dynamic responses of the structures, the modal analysis method is commonly used as the theoretical analysis method for many simple boundary conditions. Åkesson [2], Leung [3], Chen et al. [4], Hong and Kim [5], and Liu et al. [6,7] used the modal analysis method and dynamic stiffness method for forced transient response analysis of many kinds of structures.

In this paper, the dynamic stiffness matrix and precise time integration method are combined to analyze the forced vibration of the beam. The method may be extended to solve the transient response of complex structures in the future. The dynamic transfer matrices of bending deformation Timoshenko beam elements are derived. Using the transform matrix, the overall dynamic stiffness matrix is assembled. We use Leung's

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Nomenclature

A	cross-section area
$\mathbf{d}$	displacement vector
E	Young's modulus of elasticity
$\mathbf{F}$	force vector
G	shear modulus of elasticity
I_z	moments of inertia with respect to z-axis
k	shear shape factor
$\mathbf{K}$	overall stiffness matrix
$\mathbf{K}(\boldsymbol{\omega})$	overall dynamic stiffness matrix
l	length of the beam
M_z	bending moment with respect to z-axis
$\mathbf{M}$	total mass matrix
Q_y	shear force in y direction
t	time
$\mathbf{T}$	coordinates transformation matrix
$\mathbf{T_{xy}}$	dynamic transfer matrix of bending vibration in x–y plane
y	total deflection in y direction
x, y, z	rectangular axes (origin at point O)
θ_z	bending slope with respect to z-axis
μ	mass per unit area
v	Poisson's ratio
ρ	mass per unit volume
ω	circular frequency

Subscript

0	state at $x = 0$
l	state at $x = l$

equation [8] to derive the overall mass and stiffness matrices, which are independent of frequencies. Then the precise time integration method is used to derive the transient response of the systems. Finally, three kinds of beam are used as numerical examples to verify the proposed method.

2. Transfer matrices of bending vibration of the Timoshenko beam

The governing motion equations of the Timoshenko beam are [9]:

$$M_z = EI_z \frac{\partial \theta_z}{\partial x}, \tag{1}$$

$$Q_y = -GAk\gamma\left(\frac{\partial y}{\partial x} - \theta_z\right), \tag{2}$$

$$EI_z \frac{\partial^2 \theta_z}{\partial x^2} - GAk\left(\theta_z - \frac{\partial y}{\partial x}\right) - \rho I_z \frac{\partial^2 \theta_z}{\partial t^2} = 0, \tag{3}$$

$$GAk\left(\frac{\partial \theta_z}{\partial x} - \frac{\partial^2 y}{\partial x^2}\right) + \mu \frac{\partial^2 y}{\partial t^2} = 0, \tag{4}$$

where y is the total deflection in the y direction, θ_z the bending slope with respect to the z-axis, k the shear shape factor of the beam, I_z the moment of inertia with respect to the z-axis, and $\mu = \rho A$ the mass per unit area.

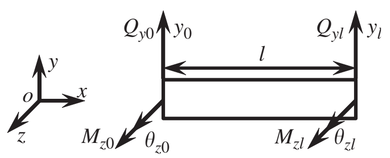

Fig. 1. End forces and displacements conditions of the Timoshenko beam element bending in x–y plane.

Using the variables-separable form and the theory of differential equations, we may obtain the results for $y(x)$, $\theta_z(x)$, $Q_y(x)$, $M_z(x)$. The end conditions for displacements and forces of the Timoshenko beam, whose length is l, are shown in Fig. 1. So,

when $x = 0$,

$$y(0) = y_0, \theta_z(0) = \theta_{z_0}, Q_y(0) = Q_{y_0}, M_z(0) = -M_{z_0}, \tag{5}$$

when $x = l$,

$$y(l) = y_l, \theta_z(l) = \theta_{z_l}, Q_y(l) = -Q_{y_l}, M_z(l) = M_{z_l}. \tag{6}$$

Using the boundary conditions mentioned above, we can obtain the relationship for the state vectors of bending vibration of the Timoshenko beam in the x–y plane from the results of $y(x)$, $\theta_z(x)$, $Q_y(x)$, $M_z(x)$,

$$\begin{Bmatrix} y_l \\ \theta_{zl} \\ Q_{yl} \\ M_{zl} \end{Bmatrix} = \mathbf{T_{xy}} \begin{Bmatrix} y_0 \\ \theta_{z0} \\ Q_{y0} \\ M_{z0} \end{Bmatrix}, \tag{7}$$

where

$$\mathbf{T_{xy}} = \begin{bmatrix} D_0 - \sigma D_2 & lD_4 & \dfrac{l^3((\beta^4+\sigma^2)D_3 - \sigma D_1)}{\beta^4 EI_z} & -\dfrac{l^2 D_2}{EI_z} \\ \dfrac{\beta^4 D_3}{l} & D_0 - \tau D_2 & \dfrac{l^2 D_2}{EI_z} & -\dfrac{l(D_4 + \sigma D_3)}{EI_z} \\ -\dfrac{\beta^4 EI_z (D_1 - \sigma D_3)}{l^3} & -\dfrac{\beta^4 EI_z D_2}{l^2} & -D_0 + \sigma D_2 & \dfrac{\beta^4 D_3}{l} \\ \dfrac{\beta^4 EI_z D_2}{l^2} & \dfrac{EI_z}{l}(\sigma C_4 + D_5) & lD_4 & -D_0 + \tau D_2 \end{bmatrix},$$

$$\lambda_1 = \sqrt{-\frac{\sigma+\tau}{2} + \sqrt{\beta^4 + (\sigma-\tau)^2/4}}, \quad \lambda_2 = \sqrt{\frac{\sigma+\tau}{2} + \sqrt{\beta^4 + (\sigma-\tau)^2/4}}, \quad \lambda = \frac{1}{\lambda_1^2 + \lambda_2^2},$$

$$\sigma = \frac{\mu\omega^2 l^2}{GAk}, \quad \tau = \frac{\mu\omega^2 l^2}{EA}, \quad \beta^4 = \frac{\mu\omega^2 l^4}{EI_z},$$

$$D_0 = \lambda(\cosh(\lambda_1)\lambda_2^2 + \cos(\lambda_2)\lambda_1^2), \quad D_1 = \lambda\left(\sinh(\lambda_1)\frac{\lambda_2^2}{\lambda_1} + \sin(\lambda_2)\frac{\lambda_1^2}{\lambda_2}\right), \quad D_2 = \lambda(\cosh(\lambda_1) - \cos(\lambda_2)),$$

$$D_3 = \lambda\left(\frac{\sinh(\lambda_1)}{\lambda_1} - \frac{\sin(\lambda_2)}{\lambda_2}\right), \quad D_4 = \lambda(\sinh(\lambda_1)\lambda_1 + \sin(\lambda_2)\lambda_2), \;\; D_5 = \lambda(\sinh(\lambda_1)\lambda_1^3 - \sin(\lambda_2)\lambda_2^3).$$

3. Dynamic stiffness matrix of the continuous Timoshenko beam element

Supposing the displacement vector $\mathbf{d}$ and force vector $\mathbf{F}$ of the Timoshenko beam element are:

$$\begin{aligned} \mathbf{d} &= \{y, \theta_z\}^{\mathrm{T}}, \\ \mathbf{F} &= \{Q_y, M_z\}^{\mathrm{T}}. \end{aligned} \tag{8}$$

Then the dynamic transfer matrix of this beam is

$$\begin{Bmatrix} \mathbf{d} \\ \mathbf{F} \end{Bmatrix}_l = \begin{bmatrix} \mathbf{T_{11}} & \mathbf{T_{12}} \\ \mathbf{T_{21}} & \mathbf{T_{22}} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \mathbf{F} \end{Bmatrix}_0. \tag{9}$$

To obtain the dynamic stiffness matrix, transform equations are introduced [10]:

$$\begin{aligned}
\mathbf{k}_{11} &= -\mathbf{T}_{12}^{-1}\mathrm{T}_{11},\\
\mathbf{k}_{12} &= \mathbf{T}_{12}^{-1},\\
\mathbf{k}_{21} &= \mathbf{T}_{21} - \mathbf{T}_{22}\mathbf{T}_{12}^{-1}\mathbf{T}_{11},\\
\mathbf{k}_{22} &= \mathbf{T}_{22}\mathbf{T}_{12}^{-1}.
\end{aligned} \tag{10}$$

The dynamic stiffness matrix is

$$\begin{Bmatrix} \mathbf{F}_0 \\ \mathbf{F}_l \end{Bmatrix} = \begin{bmatrix} \mathbf{k}_{11} & \mathbf{k}_{12} \\ \mathbf{k}_{21} & \mathbf{k}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{d}_0 \\ \mathbf{d}_l \end{Bmatrix}. \tag{11}$$

Supposing the transform matrix from local coordinates to global coordinates is $\mathbf{T}$. Using this transform matrix, we may change the dynamic stiffness matrix from local coordinates to global coordinates. Then the overall dynamic stiffness matrix $\mathbf{K}(\boldsymbol{\omega})$ is obtained by assembling dynamic stiffness matrices of all of the individual elements in the usual FEM.

4. Analytical motion equation of forced vibration

Basing on the dynamic stiffness matrix, the forced vibration motion equation with damping is

$$\mathbf{K}(\boldsymbol{\omega})\mathbf{q} + \mathbf{C}\dot{\mathbf{q}} = \mathbf{Q}(\mathbf{t}), \tag{12}$$

where $\mathbf{K}(\boldsymbol{\omega})$ is the overall dynamic stiffness matrix, $\mathbf{C}$ the damping matrix, $\mathbf{q}$ the vector of generalized displacement, and $\mathbf{Q}(\mathbf{t})$ the vector of generalized force.

In 1977, Richards and Leung gave the equations [8]

$$\mathbf{M} = -\frac{\partial \mathbf{K}(\boldsymbol{\omega})}{\partial \omega^2}, \quad \mathbf{K} = \mathbf{K}(\boldsymbol{\omega}) + \omega^2 \mathbf{M}. \tag{13}$$

Because the overall stiffness matrix $\mathbf{K}(\boldsymbol{\omega})$ is assembled, a numerical differentiation of Eq. (13) may be used in practical computations [11]. We may obtain

$$\mathbf{M} = -\frac{\mathbf{K}(\omega_2) - \mathbf{K}(\omega_1)}{\omega_2^2 - \omega_1^2}, \quad \mathbf{K} = \mathbf{K}(\boldsymbol{\omega}) + \omega^2 \mathbf{M}, \tag{14}$$

where $\omega_1 = \omega - \varepsilon$, $\omega_2 = \omega + \varepsilon$, with ε being a small number.

Using the overall mass and stiffness matrices at ω, we can transform Eq. (12) into

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{Q}(\mathbf{t}) \tag{15}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the overall stiffness, damping, and stiffness matrices, respectively.

5. Precise time integration method

The precise time integration method is a highly precise method. This method cannot only give precise numerical results, but also has an explicit integral scheme and unconditional stability [12]. Assume that

$$\mathbf{p} = \mathbf{M}\dot{\mathbf{x}} + \mathbf{C}\mathbf{x}/2. \tag{16}$$

The second-order differentials in Eq. (15) can be transformed into the first-order form as

$$\dot{\mathbf{v}} = \mathbf{H}\mathbf{v} + \mathbf{f}, \tag{17}$$

where

$$\mathbf{v} = \{\mathbf{q}, \mathbf{p}\}^{\mathrm{T}}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{B} & \mathbf{G} \end{bmatrix}, \quad \mathbf{f} = \{0, \mathbf{Q}(\mathbf{t})^{\mathrm{T}}\}^{\mathrm{T}}, \quad \mathbf{q} = \mathbf{x},$$

$$\mathbf{A} = -\mathbf{M}^{-1}\mathbf{C}/2, \quad \mathbf{B} = \mathbf{C}\mathbf{M}^{-1}\mathbf{C}/4 - \mathbf{K}, \quad \mathbf{G} = -\mathbf{C}\mathbf{M}^{-1}/2, \quad \mathbf{D} = \mathbf{M}^{-1}.$$

The traditional beam is a linear time-invariant system. Considering the dual Eq. (17), we may obtain its general solution:

$$\mathbf{v} = e^{\mathbf{H}t}\mathbf{v_0} + \int_0^t e^{\mathbf{H}(t-\xi)}\mathbf{f}(\xi)\,d\xi. \tag{18}$$

Suppose that the time step is $\tau = t_{k+1} - t_k$, then Eq. (18) becomes

$$\mathbf{v}_{k+1} = \mathbf{T}\mathbf{v}_k + \int_{t_k}^{t_{k+1}} e^{\mathbf{H}(t_{k+1}-\xi)}\mathbf{f}(\xi)\,d\xi, \tag{19}$$

where $\mathbf{T} = \exp(\mathbf{H}\tau)$. When $\mathbf{T}$ is going to be computed, we should divide the time step τ into $\Delta t = \tau/m = \tau/2^N$. When $N = 20$, $m = 2^N = 1\,048\,576$. Because τ is a small time interval, $\Delta t = \tau/m$ is an extremely small time interval [12].

Assume that $\mathbf{T_a} = (\mathbf{H}\,\Delta t)(\mathbf{I} + \mathbf{H}\,\Delta t)/2)$. Execute the cycle

$$\text{for } (i = 0;\ \ i<N;\ \ i++)\{\mathbf{T_a} = 2\mathbf{T_a} + \mathbf{T_a} \times \mathbf{T_a}\}. \tag{20}$$

After the cycle, we may obtain $\mathbf{T}$:

$$\mathbf{T} = \mathbf{I} + \mathbf{T_a}. \tag{21}$$

Suppose that the inhomogeneous term is linear within the time step (t_k, t_{k+1}) [12], i.e.

$$\mathbf{f}(t) = \mathbf{r_0} + \mathbf{r_1} \times (t - t_k). \tag{22}$$

Substituting Eq. (22) into Eq. (19) gives the precise time integration equation:

$$\mathbf{v_{k+1}} = \mathbf{T} \times [\mathbf{v_k} + \mathbf{H}^{-1}(\mathbf{r_0} + \mathbf{H}^{-1}\mathbf{r_1})] - \mathbf{H}^{-1}(\mathbf{r_0} + \mathbf{H}^{-1}\mathbf{r_1} + \mathbf{r_1}\tau). \tag{23}$$

6. Numerical examples

Numerical results are presented to demonstrate the current method. Three beam models are used as numerical examples to test the current algorithm. In these examples, damping is neglected. The first example is a simple supported beam forced by a step force. The second example is a simple supported beam forced by a sinusoidal force. The third example is a stepped beam forced by a sinusoidal force. An IBM ThinkCentre A50 8176-KCB, which has one 2.80 GHz Intel Pentium 4 processor and 248 megabytes of physical memory, is used. The operating system is Microsoft Windows XP SP2.

6.1. Example 1: a simple supported beam forced by a step force

Fig. 2 shows a simple supported beam, with square cross-section. The basic member properties are length $l = 10$ m, cross-section $A = 0.01\,\text{m}^2$, Young's modulus $E = 2.058 \times 10^{11}\,\text{N m}^{-2}$, Poisson's ratio $v = 0.3$, shear shape factor $k = 5/6$, and density $\rho = 7.860 \times 10^3\,\text{kg m}^{-3}$.

As the first example, there is a step external force $F(t)$ of 300 N at the middle of the beam. When the FEM is used for the transient vibration analysis of the beam, there are at least six common beam elements to obtain high precision. The Newmark time integration method is used, and the time step is 0.01 s. The theoretical

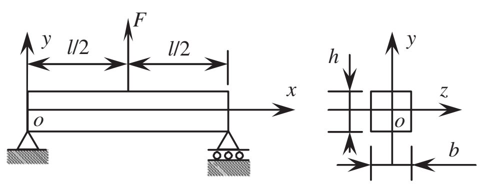

Fig. 2. Sketch map of a simple supported beam.

formulation of this problem is [13]

$$y(x,t) = \frac{2Fl^3}{\pi^4 EI_z} \sum_{i=}^{1,3,5\ldots} (-1)^{(i-1)/2} \frac{1}{i^4} \sin\frac{i\pi x}{l}(1-\cos\omega_i t), \tag{24}$$

where

$$\omega_i = i^2\pi^2\sqrt{EI_z/\rho A l^4}, \tag{25}$$

x and ω are the position and critical frequency of the external force, respectively.

Using the current method, we divide the beam into two continuous Timoshenko beam elements. The overall mass and stiffness matrices are obtained at the basic frequencies ω_1. Using the precise time integration method, we may also obtain the displacement at the mid point. For the solution, the time interval $\tau = 0.01$ s. Fig. 3 shows the vertical response at the mid point of the beam for the first 10 s. The comparisons show that the results of the three methods are similar. The current method requires fewer elements than the FEM. The performance of the current method compared to FEM is presented in Table 1. The analysis time with the current method is about two times faster than FEM.

6.2. Example 2: a simple supported beam forced by a sinusoidal force

The second example is the simple supported beam forced by a sinusoidal external force $F(t) = F_0\sin(30t)$ N at the middle of the beam, where $F_0 = 300$ N. The geometric and material properties of the beam are similar to the first example. The FEM is similar to the first example. The theoretical formulation of this problem is [13]

$$y(x,t) = \sum_{i=1}^{\infty} \frac{2F_0}{\rho A l(\omega_i^2-\omega^2)} \sin\frac{i\pi x}{l}\left(\sin\omega t - \frac{\omega}{\omega_i}\sin\omega_i t\right), \tag{26}$$

where ω_i refers to Eq. (25).

Using the current method, we may also obtain the displacement at the mid point. The basic parameters are also the same as in the first example. The vertical response at the mid point of the beam for the first 10 s is shown in Fig. 4. The comparisons show that the results of the current method are coincident with the theoretical solutions, and better than the FEM results. So the current method may be suitable for solving the

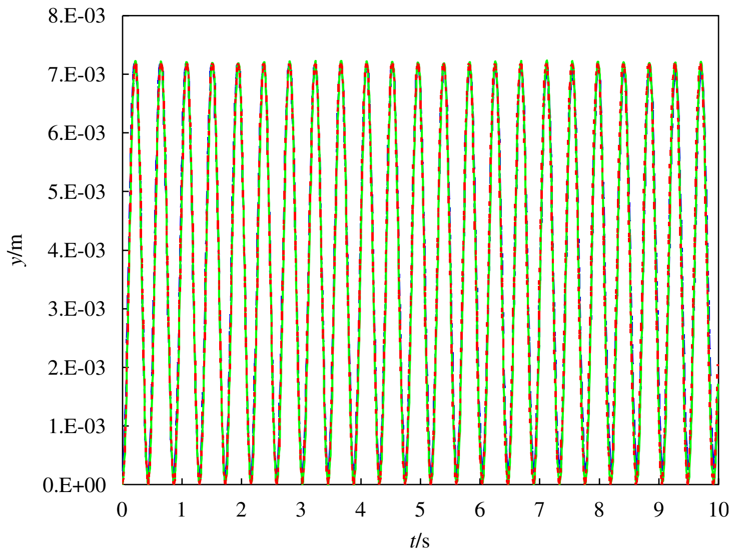

Fig. 3. Mid point vertical (y) response of a simple supported beam forced by a step force at mid point: —·–·— solution of finite element method, —— solution of current method, and ---- theory solution.

Table 1
Computation time for time analysis for example 1

Method	Current method	FEM
Computation time (s)	44.5	93.6

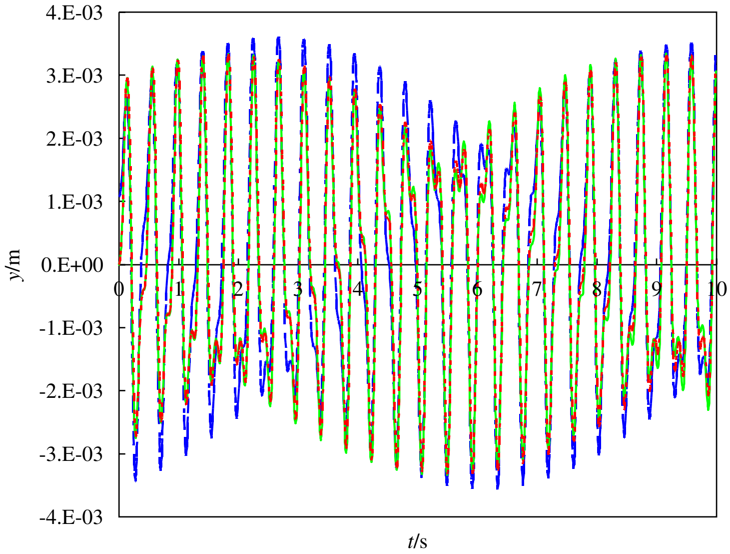

Fig. 4. Mid point vertical (y) response of the second simple supported beam forced by a sinusoidal force at mid point: —··— solution of finite element method, —— solution of current method, and ---- theory solution.

Table 2
Computation time for time analysis for example 2

Method	Current method	FEM
Computation time (s)	44.3	93.8

forced vibration of complex structures. Table 2 shows the analysis time with the current method and FEM, respectively. The current method is two times faster than FEM.

6.3. Example 3: a cantilever stepped beam forced by a sinusoidal force

The third example is a cantilever stepped beam forced by a sinusoidal external force $F(t) = F_0 \sin(30t)$ N at the free end of the beam, where $F_0 = 1000$ N. The beam is shown in Fig. 5. The material properties of the stepped beam are similar to the first example. Each segment has a square cross-section. l_i and A_i are the length and area of the ith segment ($i = 1, 2, 3$), where $l_1 = l_2 = 3$ m, $l_3 = 4$ m, $A_1 = 0.7744\,\text{m}^2$, $A_2 = 0.4096\,\text{m}^2$, and $A_3 = 0.1296\,\text{m}^2$. The FEM is similar to the first example. The stepped beam is divided into 50 beam elements. The Newmark time integration method is used, and the time step is 0.01 s. The total calculation time is 10 s.

Using the current method, we divide the stepped beam into three continuous Timoshenko beam elements. The vertical response at the free end of the beam for the first 6 s of the total calculation time (10 s) is shown in Fig. 6, which shows the results of the current method and the FEM. To illustrate the difference of the results in Fig. 6, we use FFT method to analyze the time domain responses. The responses in the frequency domain are shown in Fig. 7. The first amplitude is the forced vibration response. The second and third amplitudes are the free vibration response calculated by the current method and FEM, respectively, which are at 10.83984 and

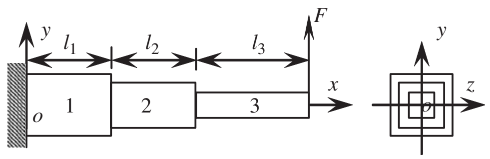

Fig. 5. Sketch map of a cantilever stepped beam.

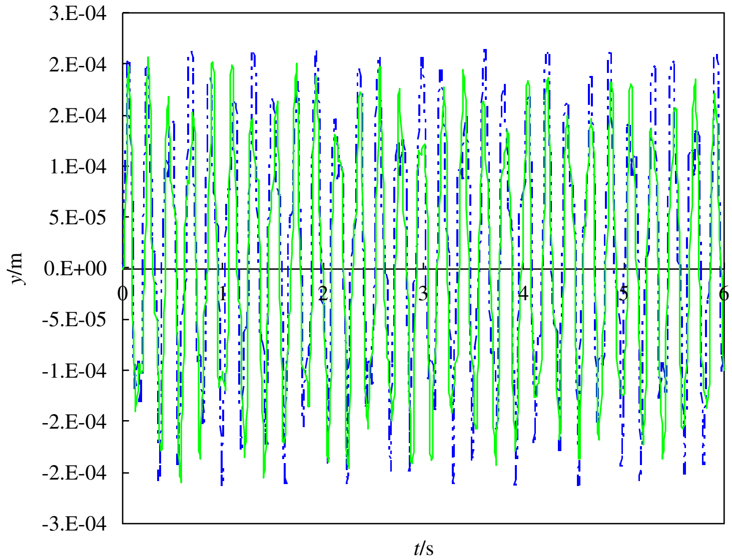

Fig. 6. Free end vertical (y) response in the time domain of the stepped beam forced by a sinusoidal force at free end: —·—·— solution of finite element method, and —— solution of current method.

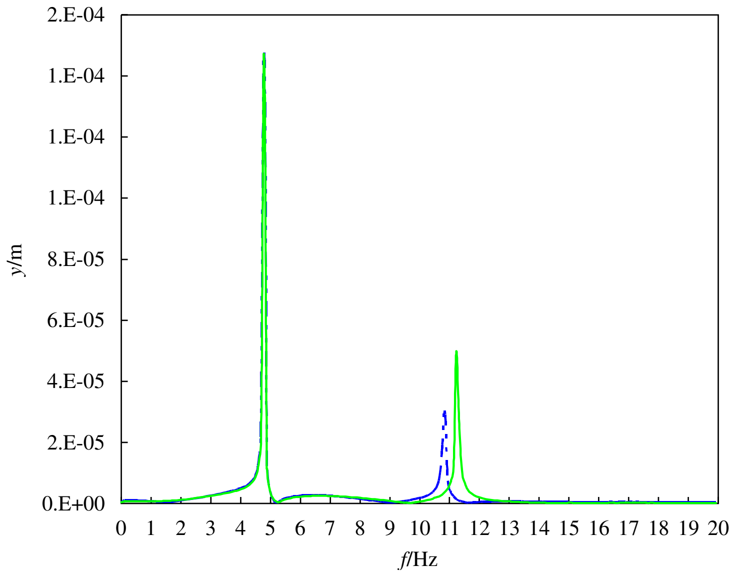

Fig. 7. Free end vertical (y) response in the frequency domain of the stepped beam forced by a sinusoidal force at free end: —·—·— solution of finite element method, and —— solution of current method.

Table 3
Computation time for time analysis for example 3

Method	Current method	FEM
Computation time (s)	55.0	94.6

11.23047 Hz. The free vibration frequency calculated by the current method is more accurate than the FEM. Table 3 shows the computation time for each method. The current method is still faster than the FEM.

7. Conclusion

In this paper, based on the dynamic stiffness matrix and precise time integration method, the dynamic responses of a beam are analyzed. This method combines the advantages of the dynamic stiffness matrix and the precise time integration method. A simple supported beam with two boundary conditions and a stepped beam with one boundary condition are used as examples. The calculations and comparisons prove that this method can be used effectively for forced vibration analysis of beams.

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