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Short Communication

Damped quadratic and mixed-parity oscillator response using Krylov–Bogoliubov method and energy balance

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Abstract

Approximate analytic solutions of linearly damped quadratic and mixed parity nonlinear oscillators are obtained using the Krylov–Bogoliubov method and considering energy balance during the oscillatory motion. Difference in the system characteristic during the positive and negative half cycles of the oscillations is incorporated by considering two equivalent auxiliary equations of the system. The amplitudes of oscillations during the positive motion and negative motion are obtained using principle of energy balance. Solutions obtained through numerical simulation of the system are compared with the analytic expressions.

1. Introduction

Undamped oscillators having quadratic and mixed-parity type of nonlinearities in stiffness have been investigated by many researchers [1–5]. Structures showing different stiffness in tension and compression and those constructed in asymmetric form can be modeled by these nonlinearities. For example, the eardrum is best modeled by quadratic nonlinear oscillator while the nonlinear free vibrations of thin-laminated plates are best modeled by mixed-parity nonlinear oscillators [1].

1.1. Quadratic nonlinear oscillator

Mickens in Ref. [1] represented the undamped quadratic nonlinear oscillator in nondimensional form as

$$\ddot{y} + y + \varepsilon y^2 = 0, \quad y(0) = A, \quad \dot{y}(0) = 0, \tag{1}$$

where dot ($\dot{}$) represent the differentiation with respect to nondimensional time parameter τ. He analyzed the system with weak quadratic nonlinearity using different methods like Lindstedt–Poincare perturbation method, Krylov–Bogoliubov (KB) method and the Krylov–Bogoliubov–Mitropolsky (KBM) method. A brief description of the solutions obtained by these methods is given below. The first-order approximation by any of these methods is unable to incorporate the effect of quadratic nonlinearity on the oscillation frequency.

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The second-order approximation by Lindstedt Poincare perturbation method gives the following result:

$$y(\theta,\varepsilon) = y_0(\theta) + \varepsilon y_1(\theta) + \varepsilon^2 y_2(\theta) + \cdots \tag{2}$$

$$= A\cos\theta + \varepsilon\left(\frac{A^2}{6}\right)(-3 + 2\cos\theta + \cos 2\theta)\ldots$$
$$+ \varepsilon^2\left(\frac{A^3}{3}\right)\left[-1 + \left(\frac{29}{48}\right)\cos\theta + \left(\frac{1}{3}\right)\cos 2\theta + \left(\frac{1}{16}\right)\cos 3\theta\right] + O(\varepsilon^3), \tag{3}$$

where $\theta = \omega(\varepsilon)\tau$, and

$$\omega(\varepsilon) = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \tag{4}$$

$$= 1 - \varepsilon^2\left(\frac{5A^2}{12}\right) + O(\varepsilon^3). \tag{5}$$

The KB method which is the first-order averaging technique is not able to incorporate the effect of small quadratic nonlinearity in the analytic expression governing the response of the system [1].

KBM method is used for second-order approximation and the response of the system is as follows:

$$y = A_0\cos\left[\left(1 - \frac{5\varepsilon^2 A_0^2}{12}\right)\tau + \theta_0\right] + \varepsilon\left(\frac{A_0^2}{6}\right)\left\{\cos\left[2\left(1 - \frac{5\varepsilon^2 A_0^2}{12}\right)\tau + 2\theta_0\right] - 3\right\}, \tag{6}$$

where A_0 and θ_0 are constants. These results show that the effect of quadratic nonlinearity on fundamental frequency of oscillation appears in the second-order approximation. Similar finding was also reported by Nayfeh in Ref. [5].

Hu in Ref. [2] studied the solution by the Lindstedt–Poincare perturbation method and presented an improved solution using harmonic balance method. He analyzed the positive and negative displacement part of oscillatory motion separately for better accuracy. He showed that the first-order approximate solution by harmonic balance method is more accurate than the second-order approximate solution obtained by Lindstedt–Poincare perturbation method. The solution is as given below:

$$y = A\cos\left(\sqrt{1 + \frac{8\varepsilon A}{3\pi}}\right)\tau \quad \text{for } y>0, \tag{7}$$

$$y = B\cos\left(\sqrt{1 - \frac{8\varepsilon B}{3\pi}}\right)\tau \quad \text{for } y<0, \tag{8}$$

$$B = \frac{1}{4\varepsilon}\left[3 + 2\varepsilon A - 3\sqrt{1 - \frac{4}{3}\varepsilon A(1+\varepsilon A)}\right]. \tag{9}$$

The amplitude of oscillation B in the negative direction was obtained by integrating the conservative system Eq. (1) and then solving the resulting cubic algebraic equation.

1.2. Mixed parity nonlinear oscillator

Mickens in Ref. [1] represented the undamped mixed-parity nonlinear oscillator in its nondimensional form as:

$$\ddot{y} + y + \alpha y^2 + \beta y^3 = 0, \quad y(0) = A, \quad \dot{y}(0) = 0. \tag{10}$$

He analyzed the oscillator using Lindstedt–Poincare perturbation method with proper scaling. The solution obtained was

$$\begin{aligned} y(\theta, A) = {} & A\cos\theta + \left(\frac{\alpha A^2}{6}\right)[-3 + 2\cos\theta + \cos 2\theta]\ldots \\ & + \left(\frac{A^3}{3}\right)\left[-\alpha^2 + \left(\frac{174\alpha^2 - 27\beta}{288}\right)\cos\theta + \left(\frac{\alpha^2}{3}\right)\cos 2\theta + \left(\frac{2\alpha^2 + 3\beta}{32}\right)\cos 3\theta\right]\ldots \\ & + O(A^3), \end{aligned} \tag{11}$$

where $\theta = \omega\tau$, and

$$\omega(A) = 1 + A^2\left(\frac{9\beta - 10\alpha^2}{24}\right) + O(A^3), \quad \text{where} \quad 0 < A \ll 1. \tag{12}$$

Hu in Ref. [3] studied the above solution and presented improved solution using harmonic balance method. He nondimensionalized the equation in a particular way and obtained the following nondimensional form of the equation

$$\ddot{y} + y + \varepsilon y^2 + y^3 = 0, \quad y(0) = A > 0, \quad \dot{y}(0) = 0. \tag{13}$$

He analyzed the positive-displacement and negative-displacement part of oscillatory motion separately for better accuracy. He showed that for large amplitude of oscillations the first-order approximate solution by harmonic balance method is more accurate than the second-order approximate solution by Lindstedt–Poincare perturbation method. The solution is as given below

$$y = A\cos\left(\sqrt{1 + \frac{8\varepsilon A}{3\pi} + \frac{3A^2}{4}}\right)\tau \quad \text{for} \quad y > 0, \tag{14}$$

$$y = B\cos\left(\sqrt{1 - \frac{8\varepsilon B}{3\pi} + \frac{3B^2}{4}}\right)\tau \quad \text{for} \quad y < 0. \tag{15}$$

The amplitude B in the negative direction was obtained in terms of A by integrating the conservative Eq. (13) and then solving the resulting fourth-order algebraic equation.

Sun et al. in Ref. [6] suggested modified Lindstedt–Poincare method, in order to obtain the solutions with higher accuracy for strongly mixed-parity nonlinear oscillators. Linear and constant terms with an undetermined parameter are introduced in the system equation and later the optimal value of the parameter is determined. The analytic expressions of the frequencies and corresponding periodic solutions are obtained. The solutions obtained through the modified Lindstedt–Poincare method represent the system response better than those obtained through classical Lindstedt–Poincare method.

The study of relevant literature shows that most of the previous works were related to the study of periodic solutions of undamped oscillators having quadratic and mixed-parity type of nonlinearities in stiffness of the system. In the present work linearly damped oscillators having quadratic and mixed-parity types of stiffness nonlinearities are considered. The transient solution of the damped oscillator is obtained using the KB method. The positive displacement and negative displacement part of the oscillatory motion are considered separately by considering two equivalent auxiliary equation of the system. The amplitude of oscillations during the positive motion and negative motion are obtained using the principle of energy balance. The analytic solution and the solution obtained through numerical simulation using fourth-order Runge–Kutta method are compared.

In the KB method, a nonlinear systems is represented as

$$\ddot{y} + y = \varepsilon F(y, \dot{y}). \tag{16}$$

The method assumes the solution and its derivative for small nonlinearity in the following form:

$$y = A(\tau)\cos\,[\tau + \theta(\tau)], \tag{17}$$

$$\dot{y} = -A(\tau)\sin\,[\tau + \theta(\tau)]. \tag{18}$$

With these assumptions the instantaneous amplitude $A(\tau)$ and $\theta(\tau)$ are calculated by the following relations [1].

$$\dot{A} = -\left(\frac{\varepsilon}{2\pi}\right)\int_0^{2\pi} F(A\cos\psi, -A\sin\psi)\sin\psi \,\mathrm{d}\psi, \tag{19}$$

$$\dot{\theta} = -\left(\frac{\varepsilon}{2\pi A}\right)\int_0^{2\pi} F(A\cos\psi, -A\sin\psi)\cos\psi \,\mathrm{d}\psi. \tag{20}$$

2. Response of the damped quadratic and mixed-parity nonlinear oscillator

The linearly damped mixed-parity nonlinear oscillator is analyzed here. The damped quadratic nonlinear oscillator is considered as the special case of the mixed-parity nonlinear oscillator in which cubic stiffness term is absent. The free vibrations of the oscillator, initiated by imparting some initial velocity, can be modeled as

$$mx'' + cx' + k_1x + k_2x^2 + k_3x^3 = 0, \quad x(0) = 0, \quad x'(0) = X'_0, \tag{21}$$

where dash ($'$) represents the differentiation with respect to time t. The above equation is converted to nondimensional form by defining the following parameters.

$$\frac{k_1}{m} = \omega_n^2, \quad \frac{c}{2m\omega_n} = \zeta, \quad \tau = \omega_n t, \quad x = \frac{X'_0}{\omega_n}y,$$

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}\tau}, \quad \varepsilon_2 = \frac{k_2}{k_1}\frac{X'_0}{\omega_n}, \quad \varepsilon_3 = \frac{k_3}{k_1}\left(\frac{X'_0}{\omega_n}\right)^2,$$

Using the above parameters, the Eq. (21) can be represented in the following nondimensional form

$$\ddot{y} + 2\zeta\dot{y} + y + \varepsilon_2y^2 + \varepsilon_3y^3 = 0, \quad y(0) = 0 \quad \dot{y}(0) = 1. \tag{22}$$

The stiffness characteristic of the system is different during positive and negative displacement due to the presence of ε_2y^2 term in the system equation. To analyze the motion in positive and negative direction separately by KB technique, following two equivalent auxiliary equations are considered

$$\ddot{y} + 2\zeta\dot{y} + y + \varepsilon_2y|y| + \varepsilon_3y^3 = 0 \quad \text{when} \quad y>0, \tag{23}$$

$$\ddot{y} + 2\zeta\dot{y} + y - \varepsilon_2y|y| + \varepsilon_3y^3 = 0 \quad \text{when} \quad y<0. \tag{24}$$

The positive displacement motion is governed by the Eq. (23). Analyzing the equation for its instantaneous amplitude $A(\tau)$ and $\dot{\theta}(\tau)$ using KB method we get

$$A(\tau) = A_{pi}\mathrm{e}^{-\zeta\tau}, \tag{25}$$

$$\dot{\theta}(\tau) = \frac{4\varepsilon_2A(\tau)}{3\pi} + \frac{3\varepsilon_3A^2(\tau)}{8}. \tag{26}$$

Therefore, the instantaneous frequency is given by

$$\omega_{ipi}(t) = \left(1 + \frac{4\varepsilon_2A(\tau)}{3\pi} + \frac{3\varepsilon_3A^2(\tau)}{8}\right). \tag{27}$$

For small damping the average frequency (ω_{pi}) during the positive half-motion can be approximated by

$$\omega_{pi} = \left(1 + \frac{4\varepsilon_2Y_{pi}}{3\pi} + \frac{3\varepsilon_3Y_{pi}^2}{8}\right), \tag{28}$$

where Y_{pi} is amplitude of oscillation during ith positive motion. Therefore, the response can be represented by

$$y_{pi}(\tau) = A_{pi}\mathrm{e}^{-\zeta\tau}\cos[\omega_{pi}\tau + \theta_{pi}] \tag{29}$$

where A_{pi} and θ_{pi} are constants and depend on the conditions at the beginning of the motion.

The negative displacement motion is governed by the Eq. (24). The analysis in the similar way as performed for positive displacement motion provides the average frequency during jth negative displacement as

$$\omega_{nj} = \left(1 - \frac{4\varepsilon_2 Y_{nj}}{3\pi} + \frac{3\varepsilon_3 Y_{nj}^2}{8}\right), \tag{30}$$

where Y_{nj} is amplitude of oscillation during jth negative motion. Therefore, the response can be represented by

$$y_{nj}(\tau) = A_{nj}\mathrm{e}^{-\zeta\tau} \cos\left[\omega_{nj}\tau + \theta_{nj}\right], \tag{31}$$

where A_{nj} and θ_{nj} are constants and depend on the conditions at the beginning of the motion.

In the present work, we are analysing the system excited by imparting initial velocity. However, the methodology, with proper definition of nondimensional parameters, is applicable to the systems with initial conditions $x(0) = X_0$ and initial velocity $x'(0) = 0$ as well. For nondimensionalizing such systems, we can define the nondimesional parameters as $x = X_0 y_1$, $\varepsilon_{2a} = k_2 X_0/k_1$ and $\varepsilon_{3a} = k_3 X_0^2/k_1$. This results in the nondimensionalized system equation as

$$\ddot{y}_1 + 2\zeta\dot{y}_1 + y_1 + \varepsilon_{2a} y_1^2 + \varepsilon_{3a} y_1^3 = 0, \quad y_1(0) = 1 \quad \dot{y}_1(0) = 0. \tag{32}$$

A representative response of damped mixed-parity nonlinear oscillator is shown in Fig. 1. Further analysis of oscillatory motion is described with reference to this figure.

2.1. Positive half-cycle

Let us say the motion 1-2-3 in Fig. 1 as 1st positive half-displacement of the oscillation, therefore the response (29) can be written as

$$y_{p1}(\tau) = A_{p1}\mathrm{e}^{-\zeta\tau} \cos\left[\omega_{p1}\tau + \theta_{p1}\right], \quad \tau_1 \leqslant \tau \leqslant \tau_3, \tag{33}$$

where τ_i represent the time at ith instance during the motion. Here $\tau_1 = 0$. Our nondimensionalized model of the system has the following initial conditions.

$$y_{p1}(\tau_1) = 0, \tag{34}$$

$$\dot{y}_{p1}(\tau_1) = v_1, \quad \text{where} \quad v_1 = 1 \tag{35}$$

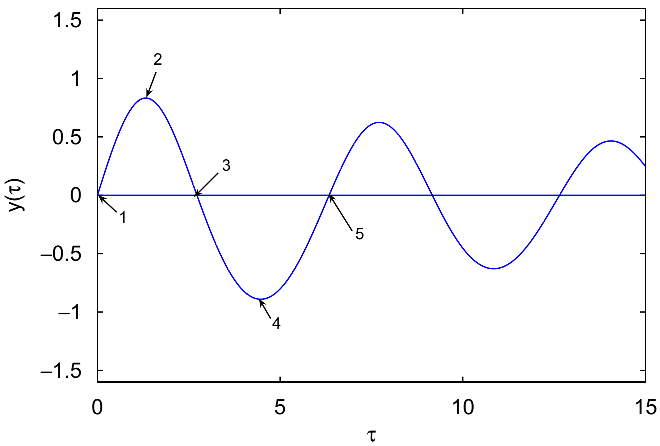

Fig. 1. A representative response of a linearly damped mixed-parity nonlinear oscillator.

Solving the above two equations for the unknown constants A_{p1} and θ_{p1}, we get the response of the system as

$$y_{p1}(\tau) = \frac{v_1}{\omega_{p1}} e^{-\zeta(\tau-\tau_1)} \sin[\omega_{p1}(\tau-\tau_1)], \quad \tau_1 \leqslant \tau \leqslant \tau_3. \tag{36}$$

The velocity during the motion is obtained by differentiating the above equation

$$\dot{y}_{p1}(\tau) = v_1 e^{-\zeta(\tau-\tau_1)} \cos[\omega_{p1}(\tau-\tau_1)] - \zeta \frac{v_1}{\omega_{p1}} e^{-\zeta(\tau-\tau_1)} \sin[\omega_{p1}(\tau-\tau_1)]. \tag{37}$$

At instance 3, the time τ_3 is $\left[\tau_1 + (\pi/\omega_{p1})\right]$. Therefore, the velocity at instance 3 is

$$\dot{y}_{p1}(\tau_3) = -v_1 e^{-\zeta\pi/\omega_{p1}}. \tag{38}$$

The response during the positive half of the oscillation is represented by the Eq. (36) in which ω_{p1} depends on the amplitude of oscillation Y_{p1}, which is unknown at present. The amplitude of oscillation Y_{p1} and the frequency ω_{p1} is calculated by applying energy balance between instance 1 and instance 2 of the motion. The energy balance equation can be written as

$$E2 - E1 + EL12 = 0, \tag{39}$$

where $E1$ and $E2$ are the total energy of the system at instances 1 and 2, respectively while $EL12$ is energy loss due to viscous friction during the motion 1–2. The total energy at instance 1 is kinetic energy and is given by $(v_1^2/2)$ while the total energy at instance 2 is potential energy and is given by $\left(\frac{Y_{p1}^2}{2} + \varepsilon_2 \frac{Y_{p1}^3}{3} + \varepsilon_3 \frac{Y_{p1}^4}{4}\right)$.

Therefore, we have

$$\frac{Y_{p1}^2}{2} + \varepsilon_2 \frac{Y_{p1}^3}{3} + \varepsilon_3 \frac{Y_{p1}^4}{4} - \frac{v_1^2}{2} + EL12 = 0, \tag{40}$$

$$\omega_{p1} = 1 + \frac{4\varepsilon_2 Y_{p1}}{3\pi} + \frac{3\varepsilon_3 Y_{p1}^2}{8}, \tag{41}$$

$$\tau_2 = \tau_1 + \frac{1}{\omega_{p1}} \tan^{-1}\left(\frac{\omega_{p1}}{\zeta}\right), \tag{42}$$

$$EL12 = \int_{\tau_1}^{\tau_2} 2\zeta \left[\dot{y}_{p1}(\tau)\right]^2 d\tau, \tag{43}$$

where τ_2 is the time when velocity is zero and amplitude attains the maximum value. The set of above four Eqs. (40)–(43) are solved for unknown quantities ω_{p1} and Y_{p1} through standard numerical methods. Eq. (40) is first solved numerically for Y_{p1} using some approximate guess value for $EL12$. Then ω_{p1}, τ_2 and $EL12$ are calculated using the Eqs. (41), (42) and (43) respectively. Using the value of $EL12$ obtained, Eqs. (40)–(43) are solved iteratively till the desired accuracy is obtained.

2.2. Negative half-cycle

For the motion 3-4-5 in Fig. 1 (1st negative half-displacement of oscillation), the response (31) is written as

$$y_{n1}(\tau) = A_{n1} e^{-\zeta\tau} \cos[\omega_{n1}\tau + \theta_{n1}], \quad \tau_3 \leqslant \tau \leqslant \tau_5. \tag{44}$$

Initially, at time τ_3 the displacement is zero and the velocity is given by the Eq. (38); hence the initial conditions for the motion are

$$y_{n1}(\tau_3) = 0, \tag{45}$$

$$\dot{y}_{n1}(\tau_3) = v_3 \tag{46}$$

Solving the above two equations for the unknown constants A_{n1} and θ_{n1}, we get the response of the system as

$$y_{n1}(\tau) = \frac{v_3}{\omega_{n1}} \mathrm{e}^{-\zeta(\tau-\tau_3)} \sin\left[\omega_{n1}(\tau-\tau_3)\right], \ \ \tau_3 \leqslant \tau \leqslant \tau_5. \tag{47}$$

The velocity during the motion is obtained by differentiating the above equation

$$\dot{y}_{n1}(\tau) = v_3 \mathrm{e}^{-\zeta(\tau-\tau_3)} \cos\left[\omega_{n1}(\tau-\tau_3)\right] - \zeta \frac{v_3}{\omega_{n1}} \mathrm{e}^{-\zeta(\tau-\tau_3)} \sin\left[\omega_{n1}(\tau-\tau_3)\right]. \tag{48}$$

At instance 5, the time τ_5 is $\left[\tau_3 + (\pi/\omega_{n1})\right]$. Therefore, the velocity at instance 5 is

$$\dot{y}_{n1}(\tau_5) = -v_3 \mathrm{e}^{-\zeta\pi/\omega_{n1}}. \tag{49}$$

The response during the negative half of the oscillation is represented by the Eq. (47) where ω_{n1} depends on the amplitude of oscillation Y_{n1}, which is unknown at present. The amplitude of oscillation Y_{n1} and the frequency ω_{n1} are calculated by applying energy balance between instance 3 and instance 4 of the motion. The energy balance equation can be written as

$$E4 - E3 + EL34 = 0, \tag{50}$$

where $E3$ and $E4$ are the total energy of the system at instances 3 and 4 respectively while $EL34$ is energy loss due to viscous friction during the motion 3–4. The total energy at instance 3 is kinetic energy and is given by $(v_3^2/2)$ while the total energy at instance 4 is potential energy and is given by $\left(\frac{Y_{n1}^2}{2} - \varepsilon_2 \frac{Y_{n1}^3}{3} + \varepsilon_3 \frac{Y_{n1}^4}{4}\right)$. Therefore, we have

$$\frac{Y_{n1}^2}{2} - \varepsilon_2 \frac{Y_{n1}^3}{3} + \varepsilon_3 \frac{Y_{n1}^4}{4} - \frac{v_3^2}{2} + EL34 = 0, \tag{51}$$

$$\omega_{n1} = 1 - \frac{4\varepsilon_2 Y_{nj}}{3\pi} + \frac{3\varepsilon_3 Y_{nj}^2}{8}, \tag{52}$$

$$\tau_4 = \tau_3 + \frac{1}{\omega_{n1}} \tan^{-1}\left(\frac{\omega_{n1}}{\zeta}\right), \tag{53}$$

$$EL34 = \int_{\tau_3}^{\tau_4} 2\zeta\left[\dot{y}_{n1}(\tau)\right]^2 \mathrm{d}\tau, \tag{54}$$

where τ_4 is the time when the displacement is maximum. Unknown parameters ω_{n1} and Y_{n1} are obtained by solving the set of above four equations, (51)–(54) iteratively through standard numerical methods. First iteration starts with the assumption of a reasonable value of $EL34$ in Eq. (51) and it continues till the desired accuracy is achieved.

The analytical solution for the complete oscillatory motion can be obtained by applying the above procedure for positive and negative displacement repeatedly.

3. Illustration

The system response obtained through numerical simulation and analytical method are compared. Two response parameters are considered for this purpose, the amplitude of oscillation and the average frequency of oscillation during the positive and negative half-cycles. Comparison is shown in Figs. 2–3 and Tables 1–4. The amplitude Y_{p1} and equivalent frequency ω_{p1} for the positive half-cycle and amplitude Y_{n1} and equivalent frequency ω_{n1} for negative half-cycle are obtained analytically as described in Section 2. The corresponding quantities Y_{p1_s}, ω_{p1_s}, Y_{n1_s} and ω_{n1_s} are obtained through numerical simulation. The percentage error is denoted by e e.g. $e_{y_{p1}}$.

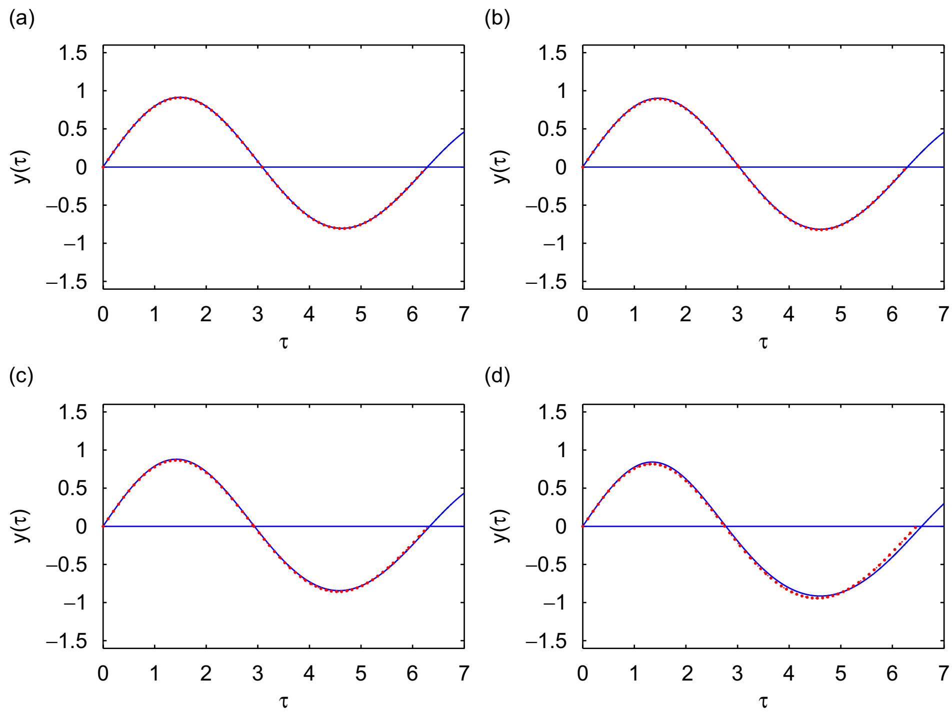

Fig. 2. Response of the linearly damped quadratic nonlinear system for $\zeta = 0.05$, numerical simulation (solid line), analytic solution (dotted line) (a) $\varepsilon_2 = 0.05$, (b) $\varepsilon_2 = 0.1$, (c) $\varepsilon_2 = 0.2$, (d) $\varepsilon_2 = 0.4$.

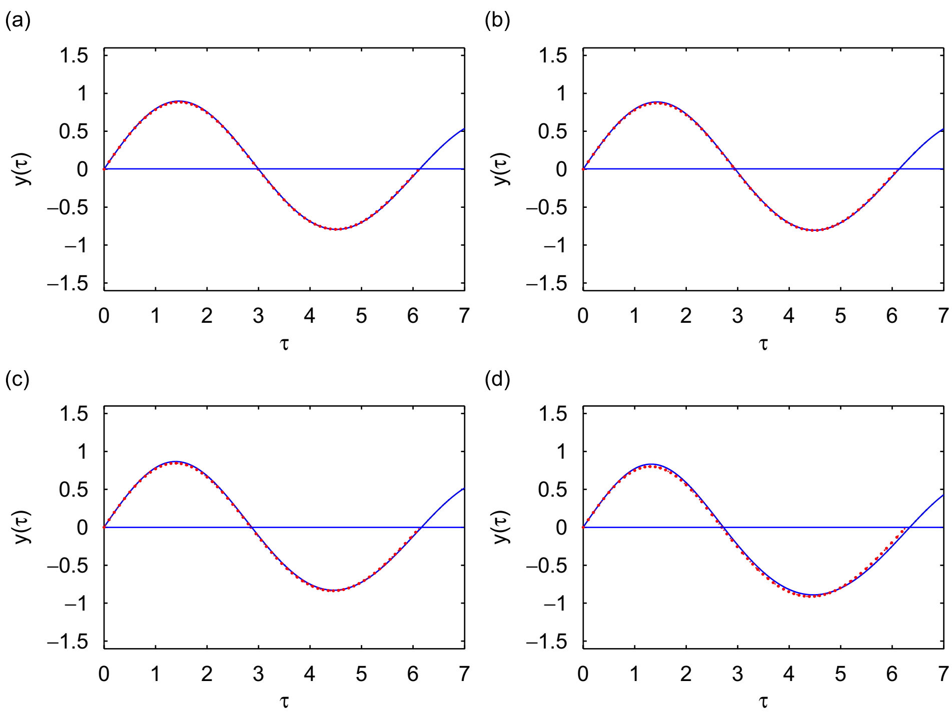

Fig. 3. Response of the linearly damped mixed-parity nonlinear system for $\zeta = 0.05$ and $\varepsilon_3 = 0.1$, numerical simulation (solid line) analytic solution (dotted line) (a) $\varepsilon_2 = 0.05$, (b) $\varepsilon_2 = 0.1$, (c) $\varepsilon_2 = 0.2$, (d) $\varepsilon_2 = 0.4$.

Table 1
Comparison between analytical and simulated response for quadratic oscillator: positive half of the oscillation $\zeta = 0.05$

ε_2	Y_{p1_s}	Y_{p1}	$e_{y_{p1}}$	ω_{p1_s}	ω_{p1}	$e_{\omega_{p1}}$
0.05	0.9137	0.9093	0.548	1.0180	1.0194	−0.138
0.10	0.9017	0.8940	0.944	1.0365	1.0383	−0.174
0.20	0.8797	0.8658	1.648	1.0711	1.0748	−0.345
0.40	0.8427	0.8173	2.779	1.1325	1.1435	−0.971

Table 2
Comparison between analytical and simulated response for quadratic oscillator : negative half of the oscillation $\zeta = 0.05$

ε_2	Y_{n1_s}	Y_{n1}	$e_{y_{n1}}$	ω_{n1_s}	ω_{n1}	$e_{\omega_{n1}}$
0.05	−0.8035	−0.8061	−0.285	0.9817	0.9829	−0.122
0.10	−0.8159	−0.8220	−0.676	0.9634	0.9653	−0.197
0.20	−0.8432	−0.8565	−1.512	0.9240	0.9282	−0.455
0.40	−0.9140	−0.9423	−3.358	0.8283	0.8443	−1.932

Table 3
Comparison between analytical and simulated response for mixed-parity oscillator : positive half of the oscillation $\varepsilon_3 = 0.1$, $\zeta = 0.05$

ε_2	Y_{p1_s}	Y_{p1}	$e_{y_{p1}}$	ω_{p1_s}	ω_{p1}	$e_{\omega_{p1}}$
0.05	0.8976	0.8852	1.563	1.0468	1.0494	−0.248
0.10	0.8867	0.8714	1.902	1.0639	1.0673	−0.320
0.20	0.8668	0.8459	2.521	1.0958	1.1021	−0.575
0.40	0.8325	0.8012	3.515	1.1537	1.1680	−1.239

Table 4
Comparison between analytical and simulated response for mixed-parity oscillator: negative half of the oscillation $\varepsilon_3 = 0.1$, $\zeta = 0.05$

ε_2	Y_{n1_s}	Y_{n1}	$e_{y_{n1}}$	ω_{n1_s}	ω_{n1}	$e_{\omega_{n1}}$
0.05	−0.7932	−0.7919	0.145	1.0056	1.0069	−0.129
0.10	−0.8045	−0.8063	−0.203	0.9889	0.9902	−0.131
0.20	−0.8291	−0.8372	−0.934	0.9529	0.9553	−0.252
0.40	−0.8904	−0.9119	−2.583	0.8683	0.8780	−1.117

3.1. Damped quadratic nonlinear oscillator

The damped quadratic nonlinear oscillator is considered as a special case of mixed-parity nonlinear oscillator in which the cubic stiffness nonlinearity is absent. Therefore Eq. (22) is simulated numerically through fourth-order Runge–Kutta method for four different values of ε_2, with $\varepsilon_3 = 0$. The four chosen values of ε_2 are $0.05, 0.1, 0.2$ and 0.4. A damping factor, $\zeta = 0.05$ is assumed in all simulation. The analytical response and that obtained numerically are shown in Fig. 2 and the results are compared in Tables 1–2. It is observed from Fig. 2 that analytical solutions are in good agreement with numerical results up to a low value of $\varepsilon_2 = 0.2$. The deviation between the two responses is visible at higher value of $\varepsilon_2 = 0.4$. The tabulated results show that the deviation of the analytical solution from the numerical result increases with increase in nonlinear parameter ε_2. The maximum error in average frequency is 0.455% while the maximum error in amplitude is 1.648% for $\varepsilon_2 = 0.2$. The corresponding maximum error increases to a value of 1.932% and 3.358% for $\varepsilon_2 = 0.4$.

3.2. Damped mixed-parity nonlinear oscillator

The analysis of damped mixed-parity nonlinear oscillator is carried out similarly for the damped quadratic nonlinear oscillator. A constant value of $\varepsilon_3 = 0.1$ is assumed in all the cases. The response obtained through numerical simulation and analytical method are shown in Fig. 3 and in Tables 3–4. It is observed from Fig. 3 that the analytical solutions are in good agreement with the numerical solution up to a low value of $\varepsilon_2 = 0.2$. The deviation between the two responses is visible at higher value of $\varepsilon_2 = 0.4$. The deviation of the analytical solution from the numerical solution increases with increase in nonlinear parameter ε_2. The maximum error in average frequency is 0.575% while the maximum error in amplitude is 2.521% for $\varepsilon_2 = 0.2$. The corresponding maximum error increases to 1.239% and 3.515%, respectively for $\varepsilon_2 = 0.4$.

The above comparison between numerical simulation and analytical solution by KB method along with energy balance approach shows that the analytical solution represents the system response with fairly good accuracy for lower values of nonlinear parameters, say for $\varepsilon_2 \leqslant 0.2$ and for $\varepsilon_3 \leqslant 0.1$. Modeling the positive half and negative half-cycle separately is a better way to represent the mixed-parity nonlinear oscillators. In this way the KB method which is first-order averaging method is able to incorporate the effect of quadratic nonlinearity on the response of the system. It also helps in defining the average frequency of oscillation during positive half and negative half-cycle separately through Eqs. (28) and (30), respectively. This closed form analytic expressions can be used to investigate the unknown system whose vibration signal has been recorded experimentally.

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