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A stochastic optimal control strategy for partially observable nonlinear quasi-Hamiltonian systems

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Abstract

A stochastic optimal control strategy for partially observable nonlinear quasi-Hamiltonian systems is proposed. The optimal control force consists of two parts. The first part is determined by the conditions under which the stochastic optimal control problem of a partially observable nonlinear system is converted into that of a completely observable linear system. The second part is determined by solving the dynamical programming equation derived by applying the stochastic averaging method and stochastic dynamical programming principle to the completely observable linear control system. The response of the optimally controlled quasi-Hamiltonian system is predicted by solving the averaged Fokker–Planck–Kolmogorov equation associated with the optimally controlled completely observable linear system and solving the Riccati equation for the estimate errors of system states. An example is given to illustrate the procedure and effectiveness of the proposed control strategy.

1. Introduction

Stochastic optimal control is a research subject of much significance since many actual control systems such as those in engineering structures are subjected to random excitations and the system states are estimated from the measurements with random noises [1]. For a long period of time, only the linear quadratic Gaussian (LQG) control strategy was used in engineering applications. In recent years, several optimal control strategies for stochastically excited nonlinear systems have been proposed [2–7]. In these studies, the states of the controlled systems were assumed known exactly, i.e., the controlled systems are completely observable. However, the system states are actually estimated from the measurements with random noises, i.e., the controlled systems are partially observable. One basic approach to the stochastic optimal control of partially observable systems is to convert the stochastic optimal control problem of a partially observable system into that of a completely observable system using the separation principle [8–10] and then to solve the later problem. For a partially observable linear system, the converted completely observable control system is of finite dimension and it can be solved easily, e.g., by using LQG strategy. A nonlinear stochastic optimal control strategy for partially observable linear systems was proposed recently by present authors [11] based on

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the separation principle, stochastic averaging method and stochastic dynamical programming principle. For a partially observable nonlinear system, the converted completely observable control system is usually of infinite dimension and it can hardly be solved. A few years ago, Charalambous and Elliott [12,13] proved that if the nonlinearities enter the dynamics of the unobservable states and the observations as gradients of potential functions, then the partially observable nonlinear control system could be recast as a completely observable linear control system of finite dimension.

The objective of the present paper is to propose a nonlinear stochastic optimal control strategy for partially observable nonlinear quasi-Hamiltonian systems. The optimal control force is split into two parts. The first part is combined with the nonlinear terms in the control system and observation so that they are the gradients of some potential function and the control problem of the partially observable nonlinear control system is converted into that of the completely observable linear control system. The second part is determined by using our previously proposed nonlinear stochastic optimal control strategy based on the stochastic averaging method and stochastic dynamical programming principle [5,6]. The responses of the optimally controlled system are predicted by solving the averaged Fokker–Planck–Kolmogorov (FPK) equation associated with the optimally controlled completely observable linear system and by solving the Riccati equation for estimation errors. Finally, the proposed control strategy is applied to the nonlinear stochastic optimal control of a partially observable Duffing oscillator subjected to Gaussian white noise excitation to illustrate the procedure and effectiveness of the proposed control strategy.

2. Stochastic optimal control problem of partially observable nonlinear systems

Consider a controlled, stochastically excited and dissipated nonlinear Hamiltonian system governed by

$$\dot{\mathbf{Q}} = \frac{\partial H'}{\partial \mathbf{P}},$$

$$\dot{\mathbf{P}} = -\frac{\partial H'}{\partial \mathbf{Q}} - \mathbf{C}'_0 \frac{\partial H'}{\partial \mathbf{P}} + \mathbf{U} + \mathbf{K}'_0 \mathbf{W}(t), \tag{1}$$

where $\mathbf{Q}$ and $\mathbf{P}$ are n-dimensional generalized displacement and momentum vectors, respectively; $H' = H'(\mathbf{Q},\mathbf{P})$ is unperturbed Hamiltonian; $\mathbf{U} = \mathbf{U}(\mathbf{Q},\mathbf{P})$ is n-dimensional feedback control force vector; $\mathbf{C}_0{}' = \mathbf{C}_0{}'(\mathbf{Q},\mathbf{P})$ is $n \times n$-dimensional damping coefficient matrix; $\mathbf{K}_0{}' = \mathbf{K}_0{}'(\mathbf{Q},\mathbf{P})$ is $n \times m$-dimensional stochastic excitation amplitude matrix; $\mathbf{W}(t)$ is m-dimensional Gaussian white noise vector in the sense of Stratonovich with intensity matrix 2D. System (1) can be modeled as Stratonovich stochastic differential equation and then converted into Itô stochastic differential equation by adding Wong–Zakai correction terms. These terms can be split into two parts: one having the effect of modifying the conservative force vector and another modifying the damping force vector. The first part can be combined with $-\partial H'/\partial \mathbf{Q}$ to form an overall effective conservative vector $-\partial H/\partial \mathbf{Q}$ with modified Hamiltonian $H = H(\mathbf{Q},\mathbf{P})$ and with $\partial H/\partial \mathbf{P} = \partial H'/\partial \mathbf{P}$. The second part can be combined with $-\mathbf{C}_0{}'\partial H'/\partial \mathbf{P}$ to constitute an effective damping force vector $-\mathbf{C}_0 \partial H/\partial \mathbf{P}$ with $\mathbf{C}_0 = \mathbf{C}_0(\mathbf{Q},\mathbf{P})$. With these accomplished, Eq. (1) can be rewritten as

$$\mathrm{d}\mathbf{Q} = \frac{\partial H}{\partial \mathbf{P}}\,\mathrm{d}t,$$

$$\mathrm{d}\mathbf{P} = \left(-\frac{\partial H}{\partial \mathbf{Q}} - \mathbf{C}_0 \frac{\partial H}{\partial \mathbf{P}} + \mathbf{U}\right)\mathrm{d}t + \mathbf{K}_0\,\mathrm{d}\mathbf{B}(t), \tag{2}$$

where $\mathbf{B}(t)$ is m-dimensional Wiener process vector and $\mathbf{K}_0 = \mathbf{K}_0(\mathbf{Q},\mathbf{P})$ is $n \times m$-dimensional matrix with $2\mathbf{K}_0{}'\mathbf{D}\mathbf{K}_0{}'^{\mathrm{T}} = \mathbf{K}_0\mathbf{K}_0^{\mathrm{T}}$.

By letting $\mathbf{X} = [\mathbf{Q}^{\mathrm{T}},\mathbf{P}^{\mathrm{T}}]^{\mathrm{T}}$, Eq. (2) is rewritten as

$$\mathrm{d}\mathbf{X} = \bar{\mathbf{A}}(\mathbf{X})\,\mathrm{d}t + \bar{\mathbf{U}}\,\mathrm{d}t + \mathbf{C}_1\,\mathrm{d}\mathbf{B}(t), \tag{3}$$

where

$$\bar{\mathbf{A}}(\mathbf{X}) = \begin{bmatrix} \partial H/\partial \mathbf{P} \\ -\partial H/\partial \mathbf{Q} - \mathbf{C}_0 \partial H/\partial \mathbf{P} \end{bmatrix}, \quad \bar{\mathbf{U}} = \begin{bmatrix} 0 \\ \mathbf{U} \end{bmatrix}, \quad \mathbf{C}_1 = \begin{bmatrix} 0 \\ \mathbf{K}_0 \end{bmatrix}. \tag{4}$$

Suppose that system state $\mathbf{X}$ is estimated from the measurement with noises. The observation equation is of the form

$$\mathrm{d}\mathbf{Y} = \bar{\mathbf{D}}(\mathbf{X})\,\mathrm{d}t + \mathbf{F}\bar{\mathbf{U}}\,\mathrm{d}t + \mathbf{C}_2\,\mathrm{d}\mathbf{B}(t) + \mathbf{C}_3\,\mathrm{d}\mathbf{B}_1(t), \tag{5}$$

where $\mathbf{Y}$ is n_1-dimensional observation vector; $\bar{\mathbf{D}}(\mathbf{X})$ is n_1-dimensional function vector; $\mathbf{B}_1(t)$ is m_1-dimensional Wiener process vector; $\mathbf{F}$, $\mathbf{C}_2$ and $\mathbf{C}_3$ are $n_1 \times 2n$, $n_1 \times m$ and $n_1 \times m_1$-dimensional constant matrices, respectively. The objective of stochastic optimal control is to minimize a performance index

$$J = E\left\{ \int_0^{\mathrm{T}} L(\mathbf{X}, \mathbf{U})\,\mathrm{d}t + \Psi(\mathbf{X}(T)) \right\} \tag{6a}$$

for finite time-interval control, or

$$J' = \lim_{T \to \infty} \frac{1}{T} \int_0^{\mathrm{T}} L(\mathbf{X}, \mathbf{U})\,\mathrm{d}t \tag{6b}$$

for semi-infinite time-interval ergodic control, where $E\{\cdot\}$ denotes expectation operation; T is the terminal time of control; $L(\mathbf{X},\mathbf{U})$ is cost function, which is a continuous, differential and convex function of both $\mathbf{X}$ and $\mathbf{U}$; $\Psi(T)$ is terminal cost. Eqs. (3), (5) and (6a,b) constitute a stochastic optimal control problem of partially observable stochastically excited and dissipated nonlinear Hamiltonian system. It consists of two coupled problems of optimal filtering and optimal control.

To convert the stochastic optimal control problem of partially observable nonlinear system governed by Eqs. (3), (5) and (6a,b) into one of completely observable linear system, control force $\bar{\mathbf{U}}$ is first split into $\bar{\mathbf{U}}_1 = [\mathbf{0}, \mathbf{U}_1^{\mathrm{T}}]^{\mathrm{T}}$ and $\bar{\mathbf{U}}_2 = [\mathbf{0}, \mathbf{U}_2^{\mathrm{T}}]^{\mathrm{T}}$. $\bar{\mathbf{U}}_1$ is combined with the uncontrolled system and observation so that Eqs. (3) and (5) become

$$\mathrm{d}\mathbf{X} = [\mathbf{A}\mathbf{X} + \mathbf{G}(\mathbf{X})]\,\mathrm{d}t + \bar{\mathbf{U}}_2\,\mathrm{d}t + \mathbf{C}_1\,\mathrm{d}\mathbf{B}(t), \tag{7}$$

$$\mathrm{d}\mathbf{Y} = [\mathbf{D}\mathbf{X} + \mathbf{E}(\mathbf{X})]\,\mathrm{d}t + \mathbf{F}\bar{\mathbf{U}}_2\,\mathrm{d}t + \mathbf{C}_2\,\mathrm{d}\mathbf{B}(t) + \mathbf{C}_3\,\mathrm{d}\mathbf{B}_1(t), \tag{8}$$

where $\mathbf{A}$ and $\mathbf{D}$ are $2n \times 2n$ and $n_1 \times 2n$-dimensional constant matrices, respectively,

$$\mathbf{A} = \begin{bmatrix} \dfrac{\partial^2 \bar{H}(0)}{\partial \mathbf{Q}\,\partial \mathbf{P}} & \dfrac{\partial^2 \bar{H}(0)}{\partial \mathbf{P}^2} \\ -\dfrac{\partial^2 \bar{H}(0)}{\partial \mathbf{Q}^2} - \dfrac{\partial}{\partial \mathbf{Q}}\left(\mathbf{C}_0(0)\dfrac{\partial \bar{H}(0)}{\partial \mathbf{P}}\right) & -\dfrac{\partial^2 \bar{H}(0)}{\partial \mathbf{P}\,\partial \mathbf{Q}} - \dfrac{\partial}{\partial \mathbf{P}}\left(\mathbf{C}_0(0)\dfrac{\partial \bar{H}(0)}{\partial \mathbf{P}}\right) \end{bmatrix},$$

$$\mathbf{G}(\mathbf{X}) = \bar{\mathbf{A}}(\mathbf{X}) + \bar{\mathbf{U}}_1 - \mathbf{A}\mathbf{X}, \quad \mathbf{D} = \frac{\partial}{\partial \mathbf{X}}(\bar{\mathbf{D}}(0) + \mathbf{F}\bar{\mathbf{U}}_1(0)), \quad \mathbf{E}(\mathbf{X}) = \bar{\mathbf{D}}(\mathbf{X}) + \mathbf{F}\bar{\mathbf{U}}_1 - \mathbf{D}\mathbf{X}. \tag{9}$$

Here, $\bar{H}$ is the Hamiltonian modified by $\mathbf{U}_1$. Note that control system (7) and observation (8) contain nonlinear terms $\mathbf{G}(\mathbf{X})$ and $\mathbf{E}(\mathbf{X})$, respectively. Correspondingly, performance index (6a,b) is modified as

$$J_0 = E\left\{ \int_0^{T} L(\mathbf{X}, \mathbf{U}_2)\,\mathrm{d}t + \Psi(\mathbf{X}(T)) \right\} \tag{10a}$$

for finite time-interval control, or

$$J_0' = \lim_{T \to \infty} \frac{1}{T} \int_0^{\mathrm{T}} L(\mathbf{X}, \mathbf{U}_2)\,\mathrm{d}t \tag{10b}$$

for semi-infinite time-interval ergodic control.

3. Converted stochastic optimal control problem of completely observable linear systems

According to the separation principle [8–10], the stochastic optimal control problem of partially observable system (7), (8) and (10a,b) can be converted into one of completely observable system governed by

$$\mathrm{d}p = K^{*}p\,\mathrm{d}t + (\mathbf{D}\hat{\mathbf{X}} + \mathbf{E})^{\mathrm{T}} p\mathbf{C}^{-1}\,\mathrm{d}\mathbf{Y} - \sum_{i} \frac{\partial}{\partial \hat{X}_i}(\mathbf{C}_1 p \mathbf{C}_2^{\mathrm{T}})_i \mathbf{C}^{-1}\,\mathrm{d}\mathbf{Y}, \tag{11}$$

$$p(\hat{\mathbf{X}}, 0|\mathbf{Y}) = p_0(\hat{\mathbf{X}}|\mathbf{Y}),$$

$$J_1 = E\left\{\int_0^T \mathrm{d}t \int_{\hat{\mathbf{X}}} L(\hat{\mathbf{X}}, \mathbf{U}_2)p(\hat{\mathbf{X}}, t|\mathbf{Y})\,\mathrm{d}\hat{\mathbf{X}} + \int_{\hat{\mathbf{X}}(T)} \Psi(\hat{\mathbf{X}})p(\hat{\mathbf{X}}, T|\mathbf{Y})\,\mathrm{d}\hat{\mathbf{X}}\right\} \tag{12a}$$

for finite time-interval control, or

$$J_1 = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t \int_{\hat{\mathbf{X}}} L(\hat{\mathbf{X}}, \mathbf{U}_2)p(\hat{\mathbf{X}}|\mathbf{Y})\,\mathrm{d}\hat{\mathbf{X}} \tag{12b}$$

for semi-infinite time-interval ergodic control, where $p = p(\hat{\mathbf{X}}, t|\mathbf{Y})$ is the unnormalized conditional probability density of system state estimation $\hat{\mathbf{X}}$ for given observation $\mathbf{Y}(\tau)$, $0 \leqslant \tau \leqslant t$; $\mathbf{C} = \mathbf{C}_2\mathbf{C}_2^{\mathrm{T}} + \mathbf{C}_3\mathbf{C}_3^{\mathrm{T}}$; K^{*} is the formal adjoint of operator K defined by

$$K\psi = \frac{1}{2}\mathrm{tr}\left(\mathbf{C}_1\mathbf{C}_1^{\mathrm{T}}\frac{\partial^2\psi}{\partial\hat{\mathbf{X}}^2}\right) + (\mathbf{A}\hat{\mathbf{X}} + \mathbf{G} + \bar{\mathbf{U}}_2)^{\mathrm{T}}\frac{\partial\psi}{\partial\hat{\mathbf{X}}} \tag{13}$$

in which $\psi = \psi(\hat{\mathbf{X}})$ is an arbitrary function and $\mathrm{tr}(\cdot)$ denotes the trace of a square matrix. Eq. (11) is the so-called Duncan–Mortensen–Zakai (DMZ) stochastic partial differential equation for p. Eqs. (11) and (12a,b) constitute a stochastic optimal control problem of completely observable system. Usually, it is a very difficult problem since it is of infinite dimension.

To make the converted stochastic optimal control problem of completely observable system of finite dimension, according to Charalambous and Elliott [12,13], assume that initial system state $\hat{\mathbf{X}}(0)$ has the following probability density:

$$p_0(\hat{\mathbf{X}}|\mathbf{Y}) = \frac{1}{\sqrt{(2\pi)^n|\boldsymbol{\sigma}_0|}}\, e^{-(\hat{\mathbf{X}}-\mathbf{m}_0)^{\mathrm{T}}\boldsymbol{\sigma}_0^{-1}(\hat{\mathbf{X}}-\mathbf{m}_0)/2} \times e^{\phi(\hat{\mathbf{X}},0)}, \tag{14}$$

where $\mathbf{m}_0$ and $\boldsymbol{\sigma}_0$ are constant vector and symmetric positive-definite matrix, respectively; and the nonlinear terms in control system (7) and observation (8) have potential function $\phi(\hat{\mathbf{X}}, t)$, i.e.,

$$\mathbf{G}(\hat{\mathbf{X}}) = \mathbf{C}_1\mathbf{C}_1^{\mathrm{T}}\frac{\partial\phi(\hat{\mathbf{X}}, t)}{\partial\hat{\mathbf{X}}}, \quad \mathbf{E}(\hat{\mathbf{X}}) = \mathbf{C}_2\mathbf{C}_1^{\mathrm{T}}\frac{\partial\phi(\hat{\mathbf{X}}, t)}{\partial\hat{\mathbf{X}}} \tag{15}$$

in which $\phi(\hat{\mathbf{X}}, t)$ satisfies the following partial differential equation

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathrm{tr}\left(\mathbf{C}_1\mathbf{C}_1^{\mathrm{T}}\frac{\partial^2\phi}{\partial\hat{\mathbf{X}}^2}\right) + \frac{1}{2}\left|\mathbf{C}_1^{\mathrm{T}}\frac{\partial\phi}{\partial\hat{\mathbf{X}}}\right|^2 + (\mathbf{A}\hat{\mathbf{X}} + \bar{\mathbf{U}}_2)^{\mathrm{T}}\frac{\partial\phi}{\partial\hat{\mathbf{X}}} = 0. \tag{16}$$

Then, by using the gauge transformation

$$\tilde{p}(\hat{\mathbf{X}}, t|\mathbf{Y}) = p(\hat{\mathbf{X}}, t|\mathbf{Y})\, e^{-\phi(\hat{\mathbf{X}},t)}, \tag{17}$$

DMZ Eq. (11) becomes

$$\mathrm{d}\tilde{p} = \tilde{K}^{*}\tilde{p}\,\mathrm{d}t + (\mathbf{D}\hat{\mathbf{X}})^{\mathrm{T}}\tilde{p}\mathbf{C}^{-1}\,\mathrm{d}\mathbf{Y} - \sum_{i}\frac{\partial}{\partial\hat{\mathbf{X}}_i}(\mathbf{C}_1\tilde{p}\mathbf{C}_2^{\mathrm{T}})_{i.}\mathbf{C}^{-1}\,\mathrm{d}\mathbf{Y}, \tag{18}$$

where $\tilde{K}^*$ is the formal adjoint of operator $\tilde{K}$ defined by

$$\tilde{K}\psi = \frac{1}{2}\mathrm{tr}\left(\mathbf{C}_1\mathbf{C}_1^{\mathrm{T}}\frac{\partial^2\psi}{\partial\hat{\mathbf{X}}^2}\right) + (\mathbf{A}\hat{\mathbf{X}} + \bar{\mathbf{U}}_2)^{\mathrm{T}}\frac{\partial\psi}{\partial\hat{\mathbf{X}}} \tag{19}$$

and the performance index (10a,b) becomes

$$\tilde{J}_1 = E\left\{\int_0^T \mathrm{d}t \int_{\hat{\mathbf{X}}} L(\hat{\mathbf{X}}, \mathbf{U}_2)\tilde{p}(\hat{\mathbf{X}}, t|\mathbf{Y})\, e^{\phi(\hat{\mathbf{X}},t)}\,\mathrm{d}\hat{\mathbf{X}} + \int_{\hat{\mathbf{X}}(T)} \Psi(\hat{\mathbf{X}})\tilde{p}(\hat{\mathbf{X}}, T|\mathbf{Y})e^{\phi(\hat{\mathbf{X}},T)}\,\mathrm{d}\hat{\mathbf{X}}\right\} \tag{20a}$$

for finite time-interval control, or

$$\tilde{J}_1' = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t \int_{\hat{\mathbf{X}}} L(\hat{\mathbf{X}}, \mathbf{U}_2)\tilde{p}(\hat{\mathbf{X}}|\mathbf{Y})\, e^{\phi(\hat{\mathbf{X}})}\,\mathrm{d}\hat{\mathbf{X}} \tag{20b}$$

for semi-infinite time-interval ergodic control. It is seen from the comparison of Eqs. (18), (19) and Eqs. (11), (13) that in the former equations the nonlinear terms $\mathbf{G}(\mathbf{X})$ and $\mathbf{E}(\mathbf{X})$ have been deleted. Thus, Eqs. (18)–(20a,b) constitute a stochastic optimal control problem of completely observable linear system.

Furthermore, by using Eq. (9), Eqs. (15) and (16) can be rewritten as

$$\left(\frac{\partial\bar{H}}{\partial\hat{\mathbf{Q}}} + \mathbf{C}_0\frac{\partial\bar{H}}{\partial\hat{\mathbf{P}}}\right)_N = -\mathbf{K}_0\mathbf{K}_0^{\mathrm{T}}\frac{\partial\phi}{\partial\hat{\mathbf{P}}}, \quad (\bar{\mathbf{D}}(\hat{\mathbf{X}}) + \mathbf{F}\bar{\mathbf{U}}_1)_N = \mathbf{C}_2\mathbf{K}_0^{\mathrm{T}}\frac{\partial\phi}{\partial\hat{\mathbf{P}}}, \tag{21}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathrm{tr}\left(\mathbf{K}_0\mathbf{K}_0^{\mathrm{T}}\frac{\partial^2\phi}{\partial\hat{\mathbf{P}}^2}\right) + \frac{1}{2}\left|\mathbf{K}_0^{\mathrm{T}}\frac{\partial\phi}{\partial\hat{\mathbf{P}}}\right|^2 + \left(\frac{\partial\bar{H}}{\partial\hat{\mathbf{P}}}\right)^{\mathrm{T}}\frac{\partial\phi}{\partial\hat{\mathbf{Q}}} - \left(\frac{\partial\bar{H}}{\partial\hat{\mathbf{Q}}} + \mathbf{C}_0\frac{\partial\bar{H}}{\partial\hat{\mathbf{P}}}\right)_L^{\mathrm{T}}\frac{\partial\phi}{\partial\hat{\mathbf{P}}} + \mathbf{U}_2^{\mathrm{T}}\frac{\partial\phi}{\partial\hat{\mathbf{P}}} = 0, \tag{22}$$

where $(\cdot)_N$ and $(\cdot)_L$ represent nonlinear and linear terms, respectively. For stationary potential $\phi(\hat{\mathbf{X}})$, the first term in Eq. (22) vanishes. Since Eqs. (18)–(20a,b) represent a stochastic optimal control problem of completely observable linear system, they can be formulated equivalently as

$$\mathrm{d}\hat{\mathbf{X}} = (\mathbf{A}\hat{\mathbf{X}} + \bar{\mathbf{U}}_2)\,\mathrm{d}t + (\mathbf{R}_C\mathbf{D}^{\mathrm{T}} + \mathbf{C}_1\mathbf{C}_2^{\mathrm{T}})\mathbf{C}^{-1}\,\mathrm{d}\mathbf{V}_I, \tag{23}$$

$$\mathrm{d}\mathbf{V}_I = \mathrm{d}\mathbf{Y} - \mathbf{D}\hat{\mathbf{X}}\,\mathrm{d}t, \tag{24}$$

$$J_2 = E\left\{\int_0^T L_2(\hat{\mathbf{X}}, \mathbf{U}_2)\,\mathrm{d}t + \Psi_2(\hat{\mathbf{X}}(T))\right\} \tag{25a}$$

for finite time-interval control, or

$$J_2' = \lim_{T\to\infty}\frac{1}{T}\int_0^T L_2(\hat{\mathbf{X}}, \mathbf{U}_2)\,\mathrm{d}t \tag{25b}$$

for semi-infinite time-interval ergodic control, where $\mathbf{V}_I$ is n_1-dimensional innovation process vector; $\mathbf{R}_C$ is the covariance matrix of state estimation error $\tilde{\mathbf{X}} = \mathbf{X} - \hat{\mathbf{X}}$, which has Gaussian probability density

$$p(\tilde{\mathbf{X}}) = \frac{1}{\sqrt{(2\pi)^n|\mathbf{R}_C|}}e^{-\tilde{\mathbf{X}}^{\mathrm{T}}\mathbf{R}_C^{-1}\tilde{\mathbf{X}}/2}. \tag{26}$$

Covariance $\mathbf{R}_C$ satisfies the following differential Riccati equation:

$$\dot{\mathbf{R}}_C = \mathbf{A}\mathbf{R}_C + \mathbf{R}_C\mathbf{A}^{\mathrm{T}} - (\mathbf{R}_C\mathbf{D}^{\mathrm{T}} + \mathbf{C}_1\mathbf{C}_2^{\mathrm{T}})\mathbf{C}^{-1}(\mathbf{D}\mathbf{R}_C + \mathbf{C}_2\mathbf{C}_1^{\mathrm{T}}) + \mathbf{C}_1\mathbf{C}_1^{\mathrm{T}} \tag{27a}$$

for finite time-interval control, or algebraic Riccati equation

$$\mathbf{A}\mathbf{R}_C + \mathbf{R}_C\mathbf{A}^{\mathrm{T}} - (\mathbf{R}_C\mathbf{D}^{\mathrm{T}} + \mathbf{C}_1\mathbf{C}_2^{\mathrm{T}})\mathbf{C}^{-1}(\mathbf{D}\mathbf{R}_C + \mathbf{C}_2\mathbf{C}_1^{\mathrm{T}}) + \mathbf{C}_1\mathbf{C}_1^{\mathrm{T}} = 0 \tag{27b}$$

for semi-infinite time-interval ergodic control.

4. Optimal control law

Let $\hat{\mathbf{X}} = [\hat{\mathbf{Q}}^{\mathrm{T}}, \hat{\mathbf{P}}^{\mathrm{T}}]^{\mathrm{T}}$. Eqs. (23)–(25a,b) describe a stochastic optimal control problem of completely observable, stochastically excited and dissipated linear Hamiltonian system with Hamiltonian $\hat{H} = \hat{H}(\hat{\mathbf{Q}}, \hat{\mathbf{P}})$. If the dissipation, excitation intensity and control are of the same small order, then Eqs. (23)–(25a,b) constitute a stochastic optimal control problem of quasi-linear Hamiltonian system and our previously proposed nonlinear stochastic optimal control strategy for quasi-integrable Hamiltonian systems [5,6] can be applied to this control problem. Specifically, by applying the stochastic averaging method [14] to system (23), the following averaged Itô stochastic differential equations is obtained:

$$\mathrm{d}\hat{\mathbf{H}} = \left[\mathbf{m}(\hat{\mathbf{H}}) + \left\langle \left(\frac{\partial \hat{\mathbf{H}}}{\partial \hat{\mathbf{P}}}\right)^{\mathrm{T}} \mathbf{U}_2 \right\rangle\right] \mathrm{d}t + \boldsymbol{\sigma}(\hat{\mathbf{H}})\,\mathrm{d}\mathbf{B}_3(t), \tag{28}$$

where $\hat{\mathbf{H}} = [\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_n]^{\mathrm{T}}$ and $\hat{H}_i$ is the ith modal energy of the controlled linear system; $\langle \cdot \rangle$ denotes averaging operation; $\mathbf{B}_3(t)$ is standard Wiener process vector; $\mathbf{m}(\hat{\mathbf{H}})$ and $\boldsymbol{\sigma}(\hat{\mathbf{H}})$ are, respectively, drift vector and diffusion matrix with elements

$$\begin{aligned} m_i(\hat{\mathbf{H}}) = \Bigg\langle &-\sum_{j,k=1}^{n} \bar{c}_{jk} \frac{\partial \hat{H}_i}{\partial \hat{P}_j}\frac{\partial \hat{H}}{\partial \hat{P}_k} + \int_{-\infty}^{0} \sum_{k,l=1}^{n_1} \sum_{j=1}^{n} \sum_{r,s=1}^{2n} \left[\left(\frac{\partial \hat{H}_j}{\partial \hat{X}_s} f_{sl}\right)_{t+\tau} \frac{\partial}{\partial \hat{H}_j} \left(\frac{\partial \hat{H}_i}{\partial \hat{X}_r} f_{rk}\right)_t \right. \\ &\left. + \left(\frac{\partial \hat{\theta}_j}{\partial \hat{X}_s} f_{sl}\right)_{t+\tau} \frac{\partial}{\partial \hat{\theta}_j} \left(\frac{\partial \hat{H}_i}{\partial \hat{X}_r} f_{rk}\right)_t \right] R_{kl}(\tau)\,\mathrm{d}\tau \Bigg\rangle, \end{aligned} \tag{29}$$

$$\sigma_{i\cdot}(\hat{\mathbf{H}})\sigma_{j\cdot}(\hat{\mathbf{H}}) = \left\langle \int_{-\infty}^{\infty} \sum_{k,l=1}^{n_1} \sum_{r,s=1}^{2n} \left(\frac{\partial \hat{H}_j}{\partial \hat{X}_s} f_{sl}\right)_{t+\tau} \left(\frac{\partial \hat{H}_i}{\partial \hat{X}_r} f_{rk}\right)_t R_{kl}(\tau)\,\mathrm{d}\tau \right\rangle \tag{30}$$

in which $\bar{c}_{jk}$ is the damping coefficient dependent on $\mathbf{A}$ in Eq. (9); $\hat{\theta}_j$ the generalized phase process; f_{rk} the element of matrix $(\mathbf{R}_C\mathbf{D}^{\mathrm{T}} + \mathbf{C}_1\mathbf{C}_2^{\mathrm{T}})\mathbf{C}^{-1}$; $R_{kl}(\tau)$ the correlation function of $\mathbf{V}_I(t)$. Eq. (28) implies that $\hat{\mathbf{H}}(t)$ is a controlled diffusion process vector. Correspondingly, performance index (25) is modified into

$$J_3 = E\left\{ \int_0^{\mathrm{T}} \left\langle L_3(\hat{\mathbf{H}}, \mathbf{U}_2) \right\rangle \mathrm{d}t + \Psi_3(\hat{\mathbf{H}}(T)) \right\} \tag{31}$$

for finite time-interval control, or

$$J_4 = \lim_{T\to\infty} \frac{1}{T} \int_0^{\mathrm{T}} \left\langle L_3(\hat{\mathbf{H}}, \mathbf{U}_2) \right\rangle \mathrm{d}t \tag{32}$$

for semi-infinite time-interval ergodic control.

By applying the stochastic dynamical programming principle [15,16] to the control problem of averaged system (28) and (31) or (32), a dynamical programming equation can be established. For performance index (31), it is

$$\begin{aligned} \frac{\partial V_1}{\partial t} = -\min_{\mathbf{U}_2} \Bigg\{ &\left\langle L_3(\hat{\mathbf{H}}, \mathbf{U}_2) \right\rangle + \left[\mathbf{m}(\hat{\mathbf{H}}) + \left\langle \left(\frac{\partial \hat{\mathbf{H}}}{\partial \hat{\mathbf{P}}}\right)^{\mathrm{T}} \mathbf{U}_2 \right\rangle\right]^{\mathrm{T}} \frac{\partial V_1}{\partial \hat{\mathbf{H}}} \\ &+ \frac{1}{2} \mathrm{tr}\left(\boldsymbol{\sigma}(\hat{\mathbf{H}})\boldsymbol{\sigma}^{\mathrm{T}}(\hat{\mathbf{H}}) \frac{\partial^2 V_1}{\partial \hat{\mathbf{H}}^2} \right) \Bigg\} \end{aligned} \tag{33}$$

and for performance index (32), it is

$$\lambda = \min_{\mathbf{U}_2}\left\{ \left\langle L_3(\hat{\mathbf{H}},\mathbf{U}_2)\right\rangle + \left[\mathbf{m}(\hat{\mathbf{H}}) + \left\langle \left(\frac{\partial \hat{\mathbf{H}}}{\partial \hat{\mathbf{P}}}\right)^{\mathrm{T}} \mathbf{U}_2 \right\rangle\right]^{\mathrm{T}} \frac{\partial V_2}{\partial \hat{\mathbf{H}}} + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}(\hat{\mathbf{H}})\boldsymbol{\sigma}^{\mathrm{T}}(\hat{\mathbf{H}})\frac{\partial^2 V_2}{\partial \hat{\mathbf{H}}^2}\right)\right\}. \tag{34}$$

In Eqs. (33) and (34), V_1 and V_2 are called the value function, and λ is a constant representing optimal average cost. The optimal control law is determined by minimizing the right-hand side of Eq. (33) or (34) with respect to $\mathbf{U}_2$, i.e.,

$$\frac{\partial L_3}{\partial \mathbf{U}_2} + \frac{\partial \hat{\mathbf{H}}}{\partial \hat{\mathbf{P}}}\frac{\partial V}{\partial \hat{\mathbf{H}}} = 0. \tag{35}$$

Let L_3 be quadratic with respect to $\mathbf{U}_2$, i.e.,

$$L_3(\hat{\mathbf{H}},\mathbf{U}_2) = g(\hat{\mathbf{H}}) + \mathbf{U}_2^{\mathrm{T}}\mathbf{R}\mathbf{U}_2, \tag{36}$$

where $g(\hat{\mathbf{H}})\geqslant 0$; $\mathbf{R}$ is a symmetric positive-definite matrix. Then optimal control law is of the form

$$\mathbf{U}_2^* = -\frac{1}{2}\mathbf{R}^{-1}\frac{\partial \hat{\mathbf{H}}}{\partial \hat{\mathbf{P}}}\frac{\partial V}{\partial \hat{\mathbf{H}}}, \tag{37}$$

which depends on the derivatives of value function V_1 or V_2 with respect to $\hat{\mathbf{H}}$. Substituting Eq. (37) into Eq. (33) or (34) yields the following final dynamical programming equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}\frac{\partial^2 V}{\partial \hat{\mathbf{H}}^2}\right) + \mathbf{m}^{\mathrm{T}}\frac{\partial V}{\partial \hat{\mathbf{H}}} - \frac{1}{4}\left\langle \left(\frac{\partial \hat{\mathbf{H}}}{\partial \hat{\mathbf{P}}}\frac{\partial V}{\partial \hat{\mathbf{H}}}\right)^{\mathrm{T}} R^{-1}\frac{\partial \hat{\mathbf{H}}}{\partial \hat{\mathbf{P}}}\frac{\partial V}{\partial \hat{\mathbf{H}}}\right\rangle + g(\hat{\mathbf{H}}) = 0 \tag{38}$$

in the case of finite time-interval control, or

$$\frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}\frac{\partial^2 V}{\partial \hat{\mathbf{H}}^2}\right) + \mathbf{m}^{\mathrm{T}}\frac{\partial V}{\partial \hat{\mathbf{H}}} - \frac{1}{4}\left\langle \left(\frac{\partial \hat{\mathbf{H}}}{\partial \hat{\mathbf{P}}}\frac{\partial V}{\partial \hat{\mathbf{H}}}\right)^{\mathrm{T}} R^{-1}\frac{\partial \hat{\mathbf{H}}}{\partial \hat{\mathbf{P}}}\frac{\partial V}{\partial \hat{\mathbf{H}}}\right\rangle + g(\hat{\mathbf{H}}) = \lambda \tag{39}$$

in the case of semi-infinite time-interval ergodic control. Since the diffusion matrix in Eq. (38) or (39) is non-singular, Eq. (38) or (39) has a classical solution [16], i.e., continuous and smooth solution, which can be obtained by using conventional numerical technique. Thus, the second part of stochastic optimal control force, $\mathbf{U}_2^*$, can be obtained by solving Eq. (38) or (39) and then by substituting the resultant $\partial V/\partial\hat{\mathbf{H}}$ into Eq. (37). The total optimal control force is then $\mathbf{U}^* = \mathbf{U}_1 + \mathbf{U}_2^*$.

5. Performance of proposed control strategy

To evaluate the performance of the proposed stochastic optimal control strategy for partially observable nonlinear quasi-Hamiltonian systems, the response of the optimally controlled system is first predicted. The response consists of optimally controlled response estimation $\hat{\mathbf{X}}$ and response estimation error $\tilde{\mathbf{X}}$. The statistics of $\hat{\mathbf{X}}$ is obtained as follows: substituting $\mathbf{U}_2^*$ in Eq. (37) with solution $\partial V/\partial\hat{\mathbf{H}}$ from Eq. (38) or (39) into Eq. (28), and averaging the terms involving $\mathbf{U}_2^*$ yield the following averaged Itô stochastic differential equation for optimally controlled system

$$d\hat{\mathbf{H}} = \bar{\mathbf{m}}(\hat{\mathbf{H}})dt + \boldsymbol{\sigma}(\hat{\mathbf{H}})d\mathbf{B}_3(t), \tag{40}$$

where

$$\bar{\mathbf{m}}(\hat{\mathbf{H}}) = \mathbf{m}(\hat{\mathbf{H}}) + \left\langle \left(\frac{\partial \hat{\mathbf{H}}}{\partial \hat{\mathbf{P}}}\right)^{\mathrm{T}} \mathbf{U}_2^* \right\rangle. \tag{41}$$

The FPK equation associated with Itô Eq. (40) is

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial \hat{H}_i}[\bar{m}_i(\hat{\mathbf{H}})p] + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial \hat{H}_i \partial \hat{H}_j}\left[\sum_{l=1}^{n_1} \sigma_{il}(\hat{\mathbf{H}})\sigma_{jl}(\hat{\mathbf{H}})p\right]. \tag{42}$$

For semi-infinite time-interval ergodic control problem, a stationary solution to Eq. (42) can be obtained as follows:

$$p(\hat{\mathbf{H}}) = C_p e^{-\varphi(\hat{\mathbf{H}})}, \tag{43}$$

where C_p is a normalization constant and probability potential

$$\varphi(\hat{\mathbf{H}}) = \int_0^{\hat{\mathbf{H}}} \left(\frac{\partial \varphi}{\partial \hat{\mathbf{H}}}\right)^{\mathrm{T}} d\hat{\mathbf{H}}, \tag{44a}$$

$$\frac{\partial \varphi}{\partial \hat{H}_i} = \sum_{j=1}^{n} [\boldsymbol{\sigma}(\hat{\mathbf{H}})\boldsymbol{\sigma}^{\mathrm{T}}(\hat{\mathbf{H}})]_{ij}^{-1} \left\{ \sum_{k=1}^{n} \frac{\partial [\boldsymbol{\sigma}(\hat{\mathbf{H}})\boldsymbol{\sigma}^{\mathrm{T}}(\hat{\mathbf{H}})]_{jk}}{\partial \hat{H}_k} - 2\bar{m}_j(\hat{\mathbf{H}}) \right\} \tag{44b}$$

in which $[\boldsymbol{\sigma}(\hat{\mathbf{H}})\boldsymbol{\sigma}^{\mathrm{T}}(\hat{\mathbf{H}})]_{ij}^{-1}$ is the element of inverse matrix $[\boldsymbol{\sigma}(\hat{\mathbf{H}})\boldsymbol{\sigma}^{\mathrm{T}}(\hat{\mathbf{H}})]^{-1}$. The mean square values of estimated generalized displacements and momenta of the optimally controlled system can be calculated as follows:

$$E[\hat{Q}_i^2] = \int_0^{\infty} \left\langle \hat{Q}_i^2 \right\rangle p(\hat{\mathbf{H}})d\hat{\mathbf{H}}, \quad E[\hat{P}_i^2] = \int_0^{\infty} \left\langle \hat{P}_i^2 \right\rangle p(\hat{\mathbf{H}})d\hat{\mathbf{H}}. \tag{45}$$

The mean square values of the errors of estimated generalized displacements and momenta can be obtained from solving Riccati Eq. (27a,b) for $\mathbf{R}_C$, i.e.,

$$E[\tilde{Q}_i^2] = (\mathbf{R}_C)_{ii}, \quad E[\tilde{P}_i^2] = (\mathbf{R}_C)_{n+i,n+i}. \tag{46}$$

Thus, the mean square generalized displacements and momenta of the optimally controlled system are

$$E[Q_i^2] = E[\hat{Q}_i^2] + E[\tilde{Q}_i^2], \quad E[P_i^2] = E[\hat{P}_i^2] + E[\tilde{P}_i^2]. \tag{47}$$

The mean Hamiltonian $E[H_C]$ of the optimally controlled system can be obtained from $E[Q_i^2]$ and $E[P_i^2]$. The mean Hamiltonian $E[H_{UC}]$ of the uncontrolled system can be obtained by applying the stochastic averaging method for quasi-Hamiltonian systems [14] directly to Eq. (2) without control force **U** if the dissipation and stochastic excitation are weak. The first criterion for evaluating the effectiveness of the presently proposed control strategy is defined as the percentage reduction in mean Hamiltonian, i.e.,

$$K_1 = \frac{E[H_{UC}] - E[H_C]}{E[H_{UC}]} \times 100\%. \tag{48}$$

For further examine the performance of the proposed control strategy, Eqs. (3) and (6a,b) are treated as a stochastic optimal control problem of completely observable system. Then our previously proposed nonlinear stochastic optimal control strategy for completely observable quasi-Hamiltonian systems can be applied to evaluate the mean Hamiltonian $E[H_F]$. In this case, the total mean Hamiltonian $E[H_{UF}]$ is $E[H_F]$ plus the contribution from measurement error. Thus, the second criterion for evaluating the effectiveness of the proposed control strategy is defined as the percentage reduction in mean Hamiltonian:

$$K_2 = \frac{E[H_{UF}] - E[H_C]}{E[H_{UF}]} \times 100\%. \tag{49}$$

Higher values of K_1 and K_2 imply better effectiveness of the proposed control strategy.

6. Example

As an example for the application of the proposed stochastic optimal control strategy, consider the stochastic optimal control of a partially observable Duffing oscillator under stochastic excitation governed by

$$\ddot{X}_1 + c\dot{X}_1 + aX_1 + bX_1^3 = e\xi(t) + u, \tag{50}$$

where X_1 is displacement; c, a and b are constants representing damping coefficient, linear stiffness and nonlinear intensity, respectively; e is excitation amplitude; $\xi(t)$ is a Gaussian white noise with unit intensity; u is feedback control force. Assume that c, e^2 and u are of the same small order. Letting $X_1 = Q$ and $\dot{X}_1 = P$, Eq. (50) is rewritten as

$$\mathrm{d}\mathbf{X} = \bar{\mathbf{A}}(\mathbf{X})\mathrm{d}t + \bar{\mathbf{U}}\mathrm{d}t + \mathbf{C}_1\mathrm{d}B(t), \tag{51}$$

where $B(t)$ is unit Wiener process and

$$\mathbf{X} = \begin{bmatrix} Q \\ P \end{bmatrix}, \quad \bar{\mathbf{A}} = \begin{bmatrix} P \\ -aQ - bQ^3 - cP \end{bmatrix}, \quad \bar{\mathbf{U}} = \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad \mathbf{C}_1 = \begin{bmatrix} 0 \\ e \end{bmatrix}. \tag{52}$$

Suppose that the system velocity is observed with noise. The observation equation is

$$\dot{Y} = \dot{X}_1 + e_1\xi_1(t), \tag{53}$$

where Y is the observation, e_1 the amplitude of observation error, $\xi_1(t)$ the Gaussian white noise with unit intensity independent of $\xi(t)$. Eq. (53) can be rewritten as the following Itô stochastic differential equation:

$$\mathrm{d}Y = P\,\mathrm{d}t + e_1\,\mathrm{d}B_1(t), \tag{54}$$

where $B_1(t)$ is unit Wiener process independent of $B(t)$. For ergodic control, the performance index is

$$J = \lim_{T\to\infty} \frac{1}{T}\int_0^{\mathrm{T}} L(Q, P, u)\,\mathrm{d}t. \tag{55}$$

Eqs. (51), (54) and (55) constitute a stochastic optimal control problem of partially observable nonlinear quasi-Hamiltonian system. Note that only control system contains nonlinear terms.

Split control force u into u_1 and u_2. u_1 satisfies the first equation of Eq. (21) and Eq. (22), i.e.,

$$u_1 - b\hat{Q}^3 = e^2\frac{\partial\phi}{\partial\hat{P}}, \tag{56}$$

$$\frac{\partial\phi}{\partial t} + \frac{e^2}{2}\frac{\partial^2\phi}{\partial\hat{P}^2} + \frac{e^2}{2}\left(\frac{\partial\phi}{\partial\hat{P}}\right)^2 + (u_2 - a\hat{Q} - c\hat{P})\frac{\partial\phi}{\partial\hat{P}} + \hat{P}\frac{\partial\phi}{\partial\hat{Q}} = 0, \tag{57}$$

where $\hat{Q}$ and $\hat{P}$ are the estimations of Q and P, respectively. One solution to Eqs. (56) and (57) is $\phi = 0$ and $u_1 = b\hat{Q}^3$. It is seen from Eqs. (51) and (52) that in this case u_1 plays the role of feedback linearization. Then, the stochastic optimal control problem of partially observable nonlinear quasi-Hamiltonian system is converted into the following stochastic optimal control problem of completely observable linear system

$$\mathrm{d}\hat{\mathrm{Q}} = \hat{P}\,\mathrm{d}t + (R_{C12}/e_1^2)\,\mathrm{d}V_I, \tag{58a}$$

$$\mathrm{d}\hat{P} = (-a\hat{Q} - c\hat{P} + u_2)\,\mathrm{d}t + \left(R_{C22}/e_1^2\right)\mathrm{d}V_I, \tag{58b}$$

$$J_2 = \lim_{T\to\infty} \frac{1}{T}\int_0^{\mathrm{T}} L_2(\hat{Q}, \hat{P}, u_2)\,\mathrm{d}t, \tag{59}$$

where R_{C12} and R_{C22} are the elements of covariance matrix $\mathbf{R}_C$ for estimation errors $\tilde{Q}$ and $\tilde{P}$. The stationary covariance matrix $\mathbf{R}_C$ is obtained from solving the following algebraic Riccati equation:

$$\mathbf{A}\mathbf{R}_C + \mathbf{R}_C\mathbf{A}^{\mathrm{T}} - \mathbf{R}_C\mathbf{E}_2\mathbf{R}_C/e_1^2 + e^2\mathbf{E}_2 = 0, \tag{60}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -a & -c \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{61}$$

By applying the stochastic averaging method to Eq. (58a,b), the following averaged Itô equation for the estimated Hamiltonian of the optimally controlled linear system is derived:

$$\mathrm{d}\hat{H} = \left[m(\hat{H}) + \langle \hat{P}u_2 \rangle\right]\mathrm{d}t + \sigma(\hat{H})\,\mathrm{d}B_3(t), \tag{62}$$

where $\hat{H} = \hat{P}^2/2 + a\hat{Q}^2/2$; the drift and diffusion coefficients are

$$m(\hat{H}) = \frac{1}{2e_1^2}\left(aR_{C12}^2 + R_{C22}^2\right) - c\hat{H},$$
$$\sigma^2(\hat{H}) = \frac{1}{e_1^2}\left(aR_{C12}^2 + R_{C22}^2\right)\hat{H}. \tag{63}$$

For ergodic control problem with performance index (32), the dynamical programming equation is

$$\min_{u_2}\left\{\langle L_3(\hat{H}, u_2)\rangle + \left[m(\hat{H}) + \langle \hat{P}u_2\rangle\right]\frac{\partial V}{\partial \hat{H}} + \frac{1}{2}\sigma^2(\hat{H})\frac{\partial^2 V}{\partial \hat{H}^2}\right\} = \lambda. \tag{64}$$

Let

$$L_3(\hat{H}, u_2) = g(\hat{H}) + Ru_2^2, \quad g(\hat{H}) = s_0 + s_1\hat{H} + s_2\hat{H}^2. \tag{65}$$

The optimal control law for u_2 is obtained from minimizing the left-hand side of Eq. (64) with respect to u_2 as follows:

$$u_2^* = -\frac{1}{2R}\frac{\partial V}{\partial \hat{H}}\hat{P}. \tag{66}$$

Substituting Eq. (66) into Eq. (64) to replace u_2 yields the final dynamical programming equation

$$\frac{1}{2}\sigma^2(\hat{H})\frac{\partial^2 V}{\partial \hat{H}^2} + m(\hat{H})\frac{\partial V}{\partial \hat{H}} - \frac{\hat{H}}{4R}\left(\frac{\partial V}{\partial \hat{H}}\right)^2 + g(\hat{H}) = \lambda. \tag{67}$$

The optimal control force component u_2^* is obtained from solving Eq. (67) and then substituting the resultant $\partial V/\partial \hat{H}$ into Eq. (66). The total optimal control force is then

$$u^* = u_1 + u_2^* = b\hat{X}_1^3 - \frac{1}{2R}\frac{\partial V}{\partial \hat{H}}\hat{\dot{X}}_1. \tag{68}$$

The response of the optimally controlled system can be predicted by solving the FPK equation associated with averaged Itô Eq. (62) as described in the last section. The response of the uncontrolled system is obtained by applying the stochastic averaging method to Eq. (51) without control force u. Then the first performance criterion K_1 can be evaluated according to Eq. (48). E$[H_{UF}]$ can be obtained by treating Eqs. (51) and (55) as a stochastic optimal control problem of completely observable quasi-Hamiltonian system, by applying our previously proposed nonlinear stochastic optimal control strategy for completely observable quasi-Hamiltonian systems [5,6] and by adding the contribution from observation error. Thus, the second performance criterion K_2 can be evaluated according to Eq. (49). Note that in calculating $E[H_C]$ both the

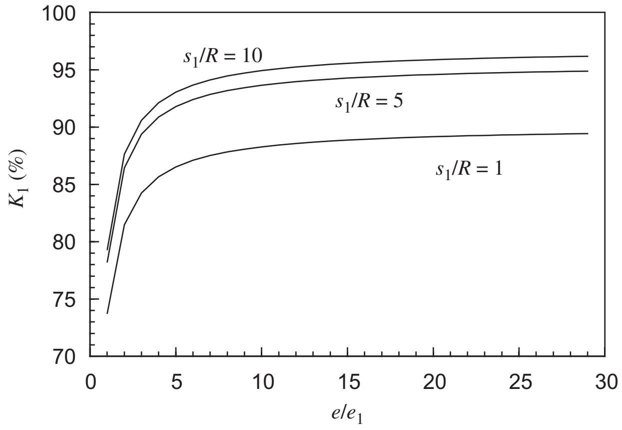

Fig. 1. K_1 as function of e/e_1 with s_1/R as parameter.

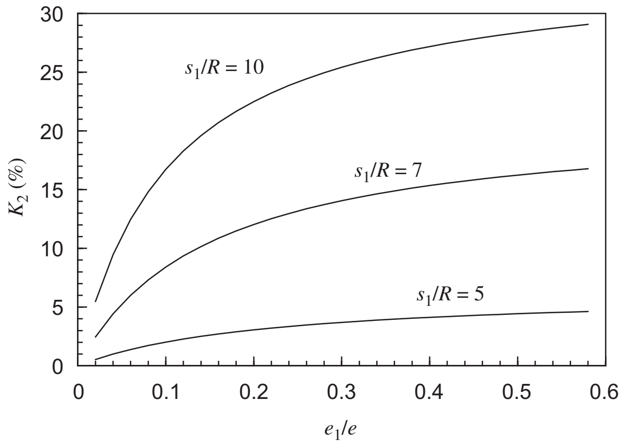

Fig. 2. K_2 as function of e_1/e with s_1/R as parameter.

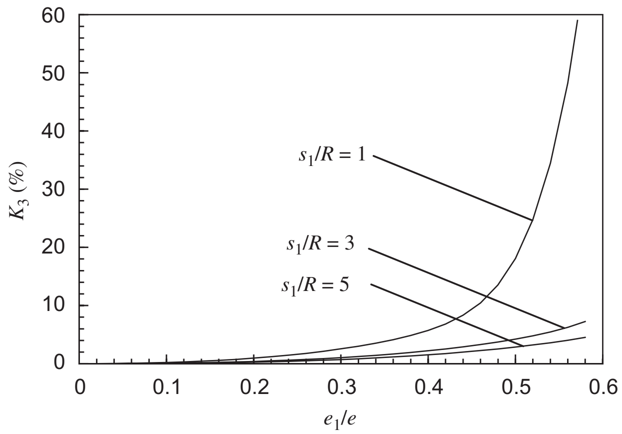

Fig. 3. K_3 as function of e_1/e with s_1/R as parameter ($b/a = 0.16$).

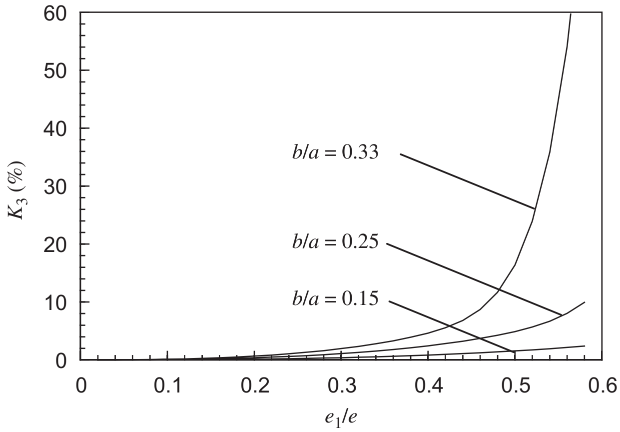

Fig. 4. K_3 as function of e_1/e with b/a as parameter ($s_1/R = 10$).

effects of estimation error on u_1 and u_2 are taken into account. If u_1 is taken without considering this effect, then mean Hamiltonian $E[H_{SF}]$ will be less than $E[H_C]$. This leads to the following third performance criterion:

$$K_3 = \frac{E[H_C] - E[H_{SF}]}{E[H_C]} \times 100\%. \tag{69}$$

For parameter values $c = 0.1$, $b/a = 0.16$, $s_0 = s_2 = 0$, some numerical results for K_1, K_2 and K_3 as functions of e/e_1 or e_1/e are shown in Figs. 1–4. It is seen from Fig. 1 that the proposed control strategy is very effective even for large observation noise. It is also seen from Fig. 1 that as the ratio of excitation intensity to observation noise increases, e.g., $e/e_1 > 5$, K_1 approaches a constant, the value of which depends on s_1/R. Fig. 2 shows that K_2 is small only $e_1/e \to 0$. This implies that the proposed control strategy has good filtering-control effectiveness and the original control problem can be treated as a stochastic optimal control problem of completely observable system only when observation noise is very small. Figs. 3 and 4 indicate that the observation noise has significant effect on u_1 when e_1/e, R/s_1 and nonlinearity are large. This effect has been taken into account in the proposed control strategy.

7. Conclusions

In the present paper, a stochastic optimal control strategy for partially observable nonlinear quasi-Hamiltonian systems has been proposed. The control force consists of two parts. The first part is used to convert the original control problem into that of completely observable linear quasi-Hamiltonian system and is determined by the condition that the combinations of it and nonlinear terms in system and observation equations are the gradients of some potential function. The second part is used to reduce the response of converted controlled system and is determined by using our previously proposed nonlinear stochastic optimal control strategy for completely observable quasi-Hamiltonian systems. It has been shown through applying the proposed control strategy to a partially observable Duffing oscillator under stochastic excitation that the proposed control strategy is very effective even for large observation noise and strong nonlinearity.

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