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Natural frequencies of rectangular membranes with partial intermediate supports

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Abstract

The natural vibrations of rectangular membranes with partial intermediate supports are solved by a direct variational method known as whole element method (WEM). It is based on the use of extended trigonometrical series of uniform convergence. Fortunately, for the case of membranes supported on the perimeter, which is the case that interests us, the simplest series that we will use is reduced, in the unitary domain, to a Fourier series of sines in both coordinate axes. The characteristic that the supports are internal and partial (instead of complete) in the membrane, gives the work one of its conditions of singularity. To the authors' knowledge, the analysis of the aforementioned case is not reported elsewhere in the literature. The proposed methodology guarantees that the frequencies found are only those related to the problem, eliminating spurious frequencies. It is demonstrated how, depending on the characteristic algorithm, it is possible to identify in an unmistakable way, spurious parameters that result when adopting this approach. It is proved that, in general, the frequency parameter of polygonal membranes does not match the square root of the parameter for frequency simple supported plates of the same shape. Evidently, this is due to the addition of intermediate supports. As has been known for the last century, without the presence of the analogy of the quadratic ratio between corresponding parameters is verified.

1. Introduction

Even though the amount of work dedicated to the study of membranes is not as large as in the case of plates, there have been many attempts to solve for the response of a homogeneous membrane of simply geometry. The Helmholtz equation is frequently encountered in various fields of engineering and physics [1–3]. It is used for analysing acoustics, wave diffraction problems, vibration of membranes, electromagnetic field, etc. Recently, thin membranes are increasingly being used for space structures applications, due to the growing requirement for reflecting surfaces in solar arrays, space radars and reflector antennas. These ultra-lightweight

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structures have become very attractive because they can meet structural requirements for space applications at a low cost. Therefore, the development and validation of analysis methods for predicting their vibration behaviour have been at the forefront of recent research activities. So, the development of a new analysis technique for the vibration of membrane structures is described here.

In the present work, the search for eigenvalues in problems ruled by Helmholtz's equation in 2D, is analysed by means of a direct method, whose minimizers are extended trigonometric series of uniform convergence, and which constitute a methodology known as the whole element method (WEM). See for example [4–6]. The problem of rectangular vibrating membranes is presented in Section 2, but due to the inclusion of partial linear intermediate supports of arbitrary directive, a contribution to the classic functional is added. The proposition, to extend the functional that we use in the variational method, is an adaptation of a similar method regarding plate vibrations [7], where said extension is justified theoretically, through the null virtual work of the unknown linear reaction of the support, when the deflection (modal shape) is null. The analysis of rectangular membranes with internal supports has been a problem of interest to engineers for over the last decades. Most theoretical analyses were limited to rectangular membranes, with continuous internal line supports in one direction. For example, the work of Vega et al. [8] describes the deduction of natural frequencies in rectangular membranes with slanted internal supports, but with the fundamental characteristic of being whole supports, i.e., the supports' edges are on the perimeter of the membrane. All of the supports cross the geometric centre of the membrane, which leads to the fact that their frequencies belong to those of trapezoidal membranes.

The authors would like to point out that one of the singular novelties of this study is that the proposed internal supports are partial, not necessarily passing through the centroid of the membrane and, besides, their geometry is arbitrary. In fact, the case quoted in Ref. [8] could be considered a particular case within our methodology. A literature research performed by the authors revealed that no solution is available for the case of partial supports. In a later work [9–12], under the assumption that oblique lines "... *vibrate harmonically* ...", a technique is used that has a few points in common with our proposition, but that stays outside the energetic context that our methodology adjudicates here, with a strict justification.

In Section 2, the matrix resolution to find natural frequencies is presented, and it is also indicated how to leave aside the spurious eigenvalues, which do not fit the actual problem. Of course, this technique, based on more-or-less known theorems of matrix calculus, may be applied to any problem, not necessarily variational, where the problem leads to an algorithm formally analogous to the one presented here. Included in an appendix is a demonstration of why in this case, that of membranes with partial intermediate supports, the widely known analogy that natural frequency parameters of polygonal supported plates are the square of those of equally shaped membranes, does not apply [13]. As such, it would be a serious mistake to extend the mentioned analogy to cases such as the one presented here.

A set of selected examples is examined. The numerical values of their natural frequencies and their mode shapes are presented in Section 3. Conclusions and relevant commentaries are included in Section 4.

2. Formulations

The linear problem that we will solve, by means of a generalized solution, is the one ruled by

$$\begin{aligned} &\nabla^2 w + \Omega^{*2} w = 0 \\ &w_{\Gamma_j} = 0 \end{aligned} \qquad (j = 1, 2, \ldots) \tag{1}$$

in the domain of Fig. 1, where $\nabla^2(\bullet)$ is the Laplacian operator in orthogonal cartesian coordinates (XY), $\Omega^{*2} = \omega^2\rho/T$ is the frequency parameter adopted, where ρ and T are, respectively, the uniform density and stress of the membrane, and ω is the natural frequency, since normal ways of vibration are accepted. Also, Γ_k ($k = 1,2,\ldots$) are the linear regions where the mode shape $w = \hat{w}(X, Y)$, ($0 \leqslant X \leqslant a$; $0 \leqslant Y \leqslant b$), is annulled.

Before writing the energetic functional, the problem is non-dimensionalized with respect to edge a. Therefore, if $x = X/a$ and $y = \lambda\ Y/a$, where $\lambda = a/b$, the vibration frequency ω is expressed in terms of the following non-dimensionalized frequency parameter:

$$\Omega^2 = \Omega^{*2} a^2 = \frac{\rho}{T}\omega^2 a^2. \tag{2}$$

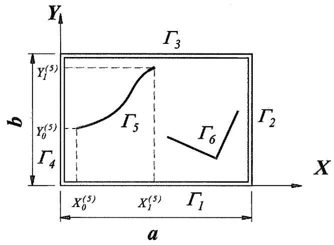

Fig. 1. Simple supported rectangular membrane with partial intermediate linear supports.

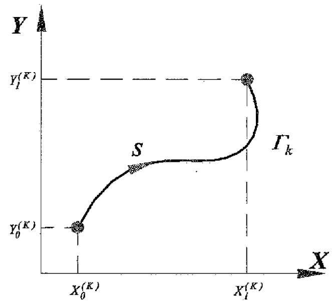

Fig. 2. Local parameters.

Then, for $w = \hat{w}(x, y)$, Eq. (1) turns into

$$\begin{aligned} &w'' + \lambda^2 \bar{\bar{w}} + \Omega^2\, w = 0 \quad (0 \leqslant x, y \leqslant 1), \\ &w_{\Gamma_j} = 0 \qquad\qquad\qquad\quad (j = 1, 2, \ldots) \end{aligned} \tag{3}$$

and the corresponding functional may be written as

$$U = \iint\limits_A \left[(w'^2 + \lambda^2 \bar{w}^2) - \Omega^2 w^2 \right] \mathrm{d}A, \tag{4}$$

where $\mathrm{d}A$ is the element of area, the prime ($\bullet'$) denotes the derivative with respect to x, and ($\bar{\bullet}$) denotes the derivative with respect to y.

The functional equation (4) must be extended with those restrictions that the used sequence does not satisfy identically. In general, if the adopted deflection w is not identically annulled over support Γ_j, the following is proposed. In Fig. 2, consider Γ_k as an internal support of the membrane, and let the function $\mu_k = \mu_k(s)$ is its reaction, where s denotes the arc of the curve. We do not lose generality, if we impose $0 \leqslant s \leqslant l$. Then, the Γ_k curve is defined as

$$(\Gamma_k) \quad \begin{cases} x_k = x_k(s), \\ y_k = y_k(s). \end{cases} \tag{5}$$

We know that the following should also be fulfilled:

$$w(x_k(s), y_k(s)) = w_k(s) = 0 \tag{6}$$

and the virtual work

$$(\mathrm{TV})_k = \int_0^1 \mu_k(s) w_k(s)\, \mathrm{d}s = 0 \tag{7}$$

must be void.

Each and every one of the n regions where w is not identically annulled will generate an integral like Eq. (7), and which will extend the functional equation (4). Then, the extended functional to be used is

$$U_a = U - \sum_{k=0}^{n} (\mathrm{TV})_k = 0. \tag{8}$$

This way of showing the restrictions (restricted edges), leads to a definition of Lagrange multipliers for holonomic continuous problems. We know from the WEM that in a bidimensional space, as in our case, two of the infinite possible series of uniform convergence in a unitary square domain, are

$$w = w(x, y) = \sum_{i=1} A_i(y) \sin(\mathrm{i}\pi x) + xA_0(y) + a(y) \tag{9}$$

and

$$w = w(x, y) = \sum_{i=1} B_i(y) \cos(\mathrm{i}\pi x) + B_0(y). \tag{10}$$

If at the same time, we develop the coefficients of these extended trigonometric series in an analogous way, we find

$$\begin{aligned} w(x, y) = & \sum_{i=1} \sum_{j=1} A_{ij} \sin(\mathrm{i}\pi x) \sin(\mathrm{j}\pi y) \\ & + y\left(\sum_{i=1} A_{i0} \sin(\mathrm{i}\pi x) + b_0 \right) \\ & + x\left(\sum_{j=1} A_{0j} \sin(\mathrm{j}\pi y) + a_0 \right) \\ & + \sum_{i=1} a_{i0} \sin(\mathrm{i}\pi x) \\ & + \sum_{j=1} a_{0j} \sin(\mathrm{j}\pi y) + A_{00}xy + \alpha \end{aligned} \tag{11}$$

and

$$\begin{aligned} w(x, y) = & \sum_{i=1} \sum_{j=1} B_{ij} \cos(\mathrm{i}\pi x) \cos(\mathrm{j}\pi y) \\ & + \sum_{i=1} B_{i0} \cos(\mathrm{i}\pi x) + \sum_{j=1} B_{0j} \cos(\mathrm{j}\pi y) + B_{00}. \end{aligned} \tag{12}$$

Finding the coefficients of Eqs. (11) and (12) in the unitary domain (which is elementary), these series allow any continuous function, to be developed with uniform convergence. Without trying to overextend ourselves over the whole element method, let us just say, that series Eq. (11) preserves its uniform convergence if we take a derivative of it once. On the other hand, the second derivatives lose their uniform convergence, and are only convergent in L_2. With respect to Eq. (12), already its first derivatives are convergent in L_2.

In the current work, we impose that $w(x,y)$ is given by Eq. (11). Fortunately, as the modal shape $w(x, y)$ is annulled over the boundary Γ_1, Γ_2, Γ_3 and Γ_4 (essential or geometric conditions), the series becomes

$$w = w(x, y) = \sum_{i=1} \sum_{j=1} A_{ij} \sin(\mathrm{i}\pi x) \sin(\mathrm{j}\pi y). \tag{13}$$

It is well known that in variational methods of linear problems, it is enough that the extremizing sequences involved in the functional be convergent in L_2, fulfilling only the essential or geometric boundary conditions. As Eq. (11) will be used to derive modal shapes, such will not be the case for membranes, since the first derivatives have uniform convergence, but, for example, they will be of convergence in L_2 for the problem of plates.

In accordance with Eq. (9) or Eq. (10), the reactions $\mu_k(s)$ of each inner partial support could be developed with uniform convergence as

$$\mu_k(s) = \sum_{p_k=1} \gamma_p^{(k)} \sin(p_k \pi s) + \gamma_0^{(k)} s + m^{(k)} \tag{14a}$$

or

$$\mu_k(s) = \sum_{p_k=0} \gamma_p^{(k)} \cos(p_k \pi s). \tag{14b}$$

However, it is easy to demonstrate that it would be enough to impose

$$\mu_k(s) = \sum_{p_k=1} \gamma_p^{(k)} \sin(p_k \pi s) \tag{15}$$

with convergence in L_2 to obtain equal $(\mathrm{TV})_k$ given by Eq. (7).

Then, the extended functional will be

$$\begin{aligned} U_a = \iint \Bigg[& \left(\sum_{i=1}^{M} \sum_{j=1}^{N} \mathrm{i}\pi A_{ij} c_i s_j \right)^2 + \lambda^2 \left(\sum_{i=1}^{M} \sum_{j=1}^{N} \mathrm{j}\pi A_{ij} s_i c_j \right)^2 \\ & - \Omega^2 \left(\sum_{i=1}^{M} \sum_{j=1}^{N} A_{ij} s_i s_j \right)^2 \Bigg] \mathrm{d}A - \sum_{k=1}^{n} \sum_{p_k=0}^{R} \sum_{i=1}^{M} \sum_{j=1}^{N} \gamma_p^{(k)} A_{ij} \\ & \times \int_0^1 \cos(p_k \pi s) \sin(\mathrm{i}\pi x(s)) \sin(\mathrm{j}\pi y(s))\, \mathrm{d}s. \end{aligned} \tag{16}$$

From the stationary condition for U_a, that is

$$\delta U_a = \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{\partial U_a}{\partial A_{ij}} \delta A_{ij} + \sum_{k=1}^{n} \sum_{p_k=0}^{R} \frac{\partial U_a}{\partial \gamma_P^{(k)}} \delta \gamma_p^{(k)} = 0 \tag{17}$$

and if we assume that the variations of the coefficient are independent, we will find out that the following homogeneous system must be fulfilled, which allows us to write

$$\mathbf{Dv} = \mathbf{0}, \tag{18}$$

where $\mathbf{D}$ is a square matrix of order $(MN+n(R+l)$ and $\mathbf{v}$ is the vector of unknows of the same order. M, N and R are the practical limits of the sums of subindexes i, j and p_k, respectively, that is,

$$\begin{aligned} \mathbf{v}^{\mathrm{T}} = [& A_{11} A_{12} \cdots A_{1N} A_{21} A_{22} \cdots A_{2N} \cdots A_{M1} A_{M2} \cdots A_{MN} \\ & | \gamma_0^{(1)} \cdots \gamma_1^{(1)} \cdots \gamma_{R1}^{(1)} \gamma_0^{(2)} \gamma_1^{(2)} \cdots \gamma_{R2}^{(2)} \cdots \gamma_0^{(n)} \cdots \gamma_1^{(n)} \cdots \gamma_{Rn}^{(n)}]. \end{aligned} \tag{19}$$

We will present $\mathbf{D}$ as

$$\mathbf{D} = \begin{bmatrix} \mathbf{\Delta} & \mathbf{K} \\ \mathbf{L} & \mathbf{Q} \end{bmatrix}. \tag{20}$$

For our particular problem, $\mathbf{\Delta}$ is a diagonal square matrix of elements $\mathbf{\Delta}_{IJ}$, where $I, J = 1,2,\ldots(MN)$, $\mathbf{Q}$ is a null square matrix of elements Q_{IJ}, where $I, J = 1, 2, \ldots (n + \sum_{k=1}^{n} R_k)$, $\mathbf{K}$ is a rectangular matrix of elements $K_{I,J}$, where $I = 1,2,\ldots(MN)$, $J = 1,2,\ldots(n + \sum_{k=1}^{n} R_k)$, and $\mathbf{L}$ is the matrix $\mathbf{L} = \mathbf{K}^{\mathrm{T}}$, with elements K_{JI}. Matrix $\mathbf{\Delta}$ is diagonal, because the base that we combined in Eq. (13) is orthogonal, that is

$$\Delta_{IJ} = \begin{cases} 0, & I \neq J, \\ \frac{1}{4}\left[\pi^2(i^2 + \lambda^2 j^2) - \Omega^2\right], & I = J, \end{cases} \tag{21}$$

where

$$I = N(i-1) + j, \quad \begin{matrix} i = 1, 2, \ldots, M, \\ j = 1, 2, \ldots, N, \end{matrix}$$

$$Q_{IJ} = 0\left(I, J = 1, 2, \ldots, n + \sum_{k=1}^{n} R_k\right). \tag{22}$$

Partitioning **K**,

$$\mathbf{K} = \left\lfloor \mathbf{K}^{(1)} | \mathbf{K}^{(2)} | \cdots | \mathbf{K}^{(\mathbf{n})} \right\rfloor \tag{23}$$

we have that

$$K_{IJ_k}^{(k)} = \int_0^1 \cos(p_k \pi s) \sin(\mathrm{i}\pi x(s)) \sin(\mathrm{j}\pi y(s)) \mathrm{d}s \tag{24}$$

with $k = 1,2,\ldots,n$, $I = N(\mathrm{i}-1) + j$ and $J_k = 1 + p_k$.

For the similar problem of plate vibration, the matrixes **K** and **Q** are the same. On the other hand, $\Delta_{II} = \frac{1}{4}\left[\pi^4(i^2 + \lambda^2 j^2)^2 - (\rho h \omega^2 / D) a^4 \Omega^2\right]$ is modified. The other elements of the matrix Δ_{IJ} are null, h is the thickness of the plate, and D is the flexural rigidity of the plate.

The characteristic equation that will allow us to find the frequencies of the rectangular membrane with partial intermediate supports, comes from annulling the determinant of **D**, that is

$$\det \mathbf{D} = 0. \tag{25}$$

Owing to a well-known result of matrix algebra [14], this equals

$$\det \mathbf{D} = \det \boldsymbol{\Delta} \cdot \det(\mathbf{Q} - \mathbf{L}\boldsymbol{\Delta}^{-1}\mathbf{K}) = 0 \tag{26}$$

and in our case it comes down to

$$\det \boldsymbol{\Delta} \cdot \det(\mathbf{K}^{\mathrm{T}} \boldsymbol{\Delta}^{-1} \mathbf{K}) = 0. \tag{27}$$

The frequencies that result from $\det \boldsymbol{\Delta} = 0$, must obviously be discarded, since they belong to the rectangular membrane without intermediate supports, and it would generate spurious eigenvalues. For our case then, the following determinant of order $(n + \sum_{k=1}^{n} R_k)$ must be imposed:

$$\det(\mathbf{K}^{\mathrm{T}} \boldsymbol{\Delta}^{-1} \mathbf{K}) = 0. \tag{28}$$

This is theoretically correct, but due to the practical reasons that we include in the final comments, we preferred to use another characteristic equation, which generates a determinant of order $[M \bullet N + (n + \sum_{k=1}^{n} R_k)]$, which nevertheless leads with greater accuracy to the frequencies that we seek.

Effectively, one sees that the matrix $\boldsymbol{\Delta}$ can be expressed as

$$\boldsymbol{\Delta} = \Delta^1 - \Omega^2 \Delta^2, \tag{29}$$

that is

$$\boldsymbol{\Delta}_{IJ} = \Delta_{IJ}^1 - \Omega^2 \Delta_{IJ}^2, \tag{30}$$

where $\Delta_{IJ}^1 = 1/4[\pi^2(i^2 + \lambda^2 j^2)]$, $\Delta_{IJ}^2 = 1/4\delta_{IJ}$, and δ_{IJ} are the second-order deltas of Kronecker.

Therefore, **D** can be written as

$$\mathbf{D} = \mathbf{D}_1 - \Omega^2 D_2 \tag{31}$$

with

$$\mathbf{D}_1 = \begin{bmatrix} \Delta^1 & \mathbf{K} \\ \mathbf{K}^{\mathrm{T}} & \mathbf{0} \end{bmatrix}, \quad \mathbf{D}_2 = \begin{bmatrix} \Delta^2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{32}$$

Now, we find the roots as

$$\det \mathbf{D} = \det(\mathbf{D}_1 - \Omega^2 \mathbf{D}_2) = 0. \quad (33)$$

By the particular feature of Eq. (33), no spurious frequencies are found. This characteristic equation guarantees by itself, that the eigenvalues found are only those belonging to the system. Concluding this item, we indicate a matrix result that can be of interest due to its practical use. In the case that the matrix $\mathbf{Q} = \mathbf{0}$ (see Eq. (20)), as is the case presented here; it must be verified that $MN > (n + \sum_{k=1}^{n} R_k)$ in order to effectively determine the eigenvalues of the problem. The other two possible alternatives do not submit any eigenvalue. The case where $MN < \left(n + \sum_{k=1}^{n} R_k\right)$, verifies that $\det \mathbf{D} = 0$. The particular case where $MN = (n + \sum_{k=1}^{n} R_k)$, since $\mathbf{K}$ and $\mathbf{L}$ are square matrixes, leads to $\det \mathbf{D} = \det \mathbf{K} \cdot \det \mathbf{L}$.

3. Numeric results

In order to illustrate the accuracy and utility of our proposition, we present a series of examples in which we determine the natural frequencies of rectangular membranes with partial intermediate supports of arbitrary geometry. Table 1 shows the first natural oscillation frequencies belonging to square membranes simple supported (SS) on its four edges, with partial linear intermediate supports. The algorithm results are contrasted with ones from the finite element method (FEM). Fig. 3 shows the different models, which are examined. The numeric results for a SS square membrane with curved partial intermediate supports, are shown in Table 2, In this case, the inner support is an arc of circumference of radius $r = 0.25$ and centred on (0.50, 0.50), as it is shown in Fig. 4. Table 3 shows the results belonging to rectangular membranes with multiple intermediate supports (Fig. 5). The frequency parameters are shown with those obtained with the finite element method.

To the authors' knowledge, no values exist that have been obtained with other methodologies. It is important to emphasize that the accuracy of the results depends on the number of terms fixed for the series that reproduces the modal shapes, since our proposition always leads to the exact solution. From this perspective, this implies that the eigenvalues are found to an arbitrary degree of precision, which matches the particular problem.

To ensure the correctness of the present results and to expand the validity of our proposal, comparisons are made with results from one paper available in the open literature [8]. The values obtained are also compared with FEM. Table 4 shows the analysis of natural frequencies, belonging to rectangular membranes with whole linear intermediate supports, i.e., those whose ends extend to the very boundary of the membrane. The different models of rectangular membranes corresponding to this case are presented in Fig. 6. Frequency parameters of the first ten eigenmodes have been presented for all the distinct cases. The modal morphology is

Table 1
Natural frequencies of a square membrane with a partial intermediate support obtained by the proposed method whole element method and finite element method

Natural frequencies	Model (a)		Model (b)		Model (c)		Model (d)	
	WEM	FEM	WEM	FEM	WEM	FEM	WEM	FEM
Ω_1	5.0214	4.9908	5.8759	5.7981	7.0248	7.0348	5.6620	5.6537
Ω_2	7.1705	7.1471	7.0248	7.0297	7.0726	7.0348	7.9649	7.9563
Ω_3	8.1551	8.0953	8.1539	8.1371	9.9346	9.9488	9.3642	9.3598
Ω_4	9.5294	9.5009	9.9346	9.9409	10.0021	9.9488	9.4096	9.3700
Ω_5	10.1806	10.1474	10.4010	10.3007	11.3271	11.3437	10.4984	10.4893
Ω_6	10.8500	10.7844	11.1707	11.0687	11.4042	11.3437	11.6721	11.6565
Ω_7	12.0736	11.9415	11.3271	11.3350	12.9530	12.9721	12.5796	12.5227
Ω_8	12.2980	12.1412	12.2411	12.0656	13.0409	12.9721	12.8829	12.8496
Ω_9	12.9530	12.9562	12.9530	12.9616	14.0496	14.0707	13.2018	13.1855
Ω_{10}	13.2292	13.2088	13.4977	13.4621	14.1451	14.0707	14.1460	14.1316

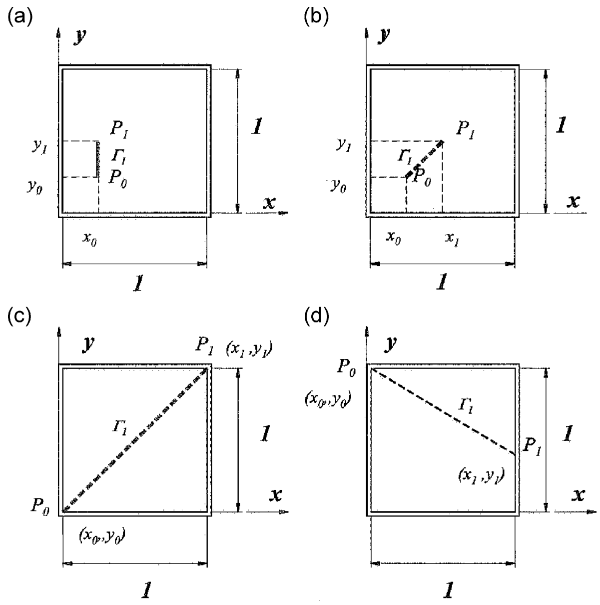

Fig. 3. Square membrane with a partial intermediate support: (a) $M/R = 4$, $P_0 = (0.25, 0.25)$, $P_1 = (0.25, 0.50)$, (b) $M/R = 4$, $P_0 = (0.25, 0.25)$, $P_1 = (50, 0.50)$, (c) $M/R \cong 1$, $P_0 = (0,0)$, $P_1 = (1,1)$, (d) $M/R \cong 1$, $P_0 = (0,l)$, and $P_1 = (l, 0.40)$.

Table 2
Natural frequencies of a square membrane with an intermediate circumferential support obtained by the proposed method whole element method and finite element method

Natural frequencies	Model (a)		Model (b)		Model (c)		Model (d)	
	WEM	FEM	WEM	FEM	WEM	FEM	WEM	FEM
Ω_1	5.3597	5.3151	6.1015	6.0211	7.7940	7.5381	9.6061	9.6196
Ω_2	7.5042	7.4452	8.5102	8.4139	9.8500	9.8306	10.1668	10.1757
Ω_3	8.5622	8.4953	9.8789	9.8670	10.3452	10.2561	10.5928	10.5983
Ω_4	9.9199	9.9206	10.3182	10.2443	10.4168	10.3417	12.7633	12.7772
Ω_5	10.4364	10.3785	10.5269	10.4902	10.6040	10.5524	13.3868	13.4008
Ω_6	10.6524	10.6049	11.1342	11.0464	12.3491	12.0860	14.5612	14.5725
Ω_7	11.8273	11.7433	12.7889	12.5875	13.2337	13.1357	14.8491	14.8521
Ω_8	12.5478	12.3974	13.0376	12.9461	13.3663	13.1648	15.3091	15.3268
Ω_9	13.0984	13.0884	13.3005	13.2943	13.5266	13.5031	17.0136	17.0164
Ω_{10}	13.3204	13.3152	13.5262	13.5017	14.6563	14.5605	18.3144	18.3200

plotted for the first six eigenmodes, except for the rectangular membranes of Table 4, as their frequencies belong to those of trapezoidal membranes.

4. Conclusions

The tool that we have presented here, is yet another contribution to the study of Helmholtz's equation in rectangular domains with partial intermediate supports. A literature research performed by the authors revealed that no solution is available for this case. Even more, some finite elements codes do not include the natural frequencies for this kind of problems in a direct way. In this current work, we have used FlexPDE to determine the natural vibrations and their modal shapes Figs. 7–16.

Another original aspect that the authors would like to highlight is that it is often found in other works that spurious values have arisen, in the determination of natural frequencies of membranes. In our work, the

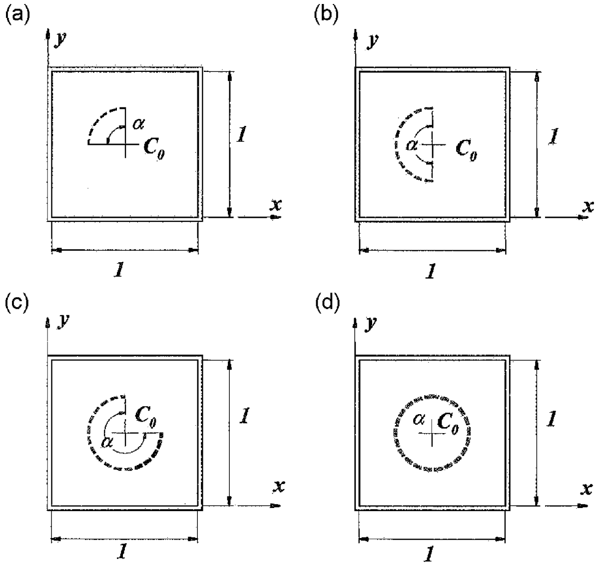

Fig. 4. Square membrane with an intermediate circumferential arc support. Centre $C_0 = (0.50, 0.50)$, $r = 0.25$: (a) $M/R = 2.70$, $\alpha = \pi/2$, (b) $M/R = 2$, $\alpha = \pi$, (c) $M/R = 1.55$, $\alpha = 3\pi/2$, and (d) $M/R = 1.33$, $\alpha = 2\pi$

Table 3
Natural frequencies of a rectangular membrane with multiple intermediate supports obtained by the proposed method whole element method and finite element method. $\lambda = 1.25$

Natural frequencies	Model (a)		Model (b)	
	WEM	FEM	WEM	FEM
Ω_1	6.2865	6.24285	6.9109	6.87680
Ω_2	8.9571	8.90848	8.8684	8.83505
Ω_3	10.2428	10.16590	9.9681	9.93611
Ω_4	11.8417	11.80649	11.6648	11.58715
Ω_5	12.8104	12.75635	12.0644	11.84571
Ω_6	13.5769	13.53110	13.1561	13.12386
Ω_7	14.7017	14.54286	13.4755	13.42944
Ω_8	15.5032	15.18556	14.5274	14.45813
Ω_9	15.7028	15.58244	15.5919	15.44146
Ω_{10}	16.2448	16.22871	16.0256	15.92882

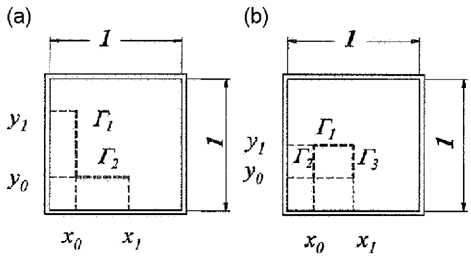

Fig. 5. Rectangular membranes with multiple partial intermediate supports. $\lambda = a/b = 1.25$: (a) $M/R \cong 2$, $x_0 = 0.20$, $y_0 = 0.25$, $x = 0.60$, $y_1 = 0.75$ and (b) $M/R = 2$, $x_0 = 0.20$, $y_0 = 0.25$, $x_1 = 0.50$, $y_1 = 0.50$.

methodology provides a thorough study of the characteristic equation of the problem that allows for the elimination of the spurious values beforehand, therefore guaranteeing that only the solution to the problem is found. This aspect is mentioned in Section 2.

Table 4
Natural frequencies of a rectangular membrane with multiple intermediate supports obtained by the propose method WEM, FEM and the method described in Ref. [8] $\lambda = a/b = 3$

Natural frequencies	Model (a)			Model (b)			Model (c)			Model (d)			Model (e)		
	WEM	FEM	[8]	WEM	FEM	[8]	WEM	FEM	[8]	WEM	FEM	[8]	WEM	FEM	[8]
Ω_1	3.7757	3.7757	3.7760	6.3698	6.3702	6.3714	4.6859	4.6739	4.9301	5.4099	5.3943	5.6510	4.0547	4.0455	4.2604
Ω_2	5.2360	5.2361	–	6.6230	6.6232	–	5.9761	5.9680	–	6.3959	6.3896	–	5.5770	5.5703	–
Ω_3	6.6230	6.6235	–	7.0248	7.0253	–	7.1533	7.1474	–	7.2951	7.2734	–	6.9390	6.9244	–
Ω_4	7.0248	7.0258	–	7.5514	7.5521	–	8.1605	8.1432	–	8.1279	8.1050	–	7.0290	7.0286	–
Ω_5	7.5514	7.5526	–	8.1789	8.1800	–	8.3398	8.3217	–	8.9328	8.9058	–	8.3721	8.3561	–
Ω_6	8.8858	8.8890	–	8.8858	8.8878	–	9.3834	9.3683	–	9.7094	9.6849	–	8.5572	8.5391	–
Ω_7	8.9472	8.9500	–	9.6578	10.9794	–	9.6915	9.6796	–	9.8976	9.8749	–	9.6903	9.6735	–
Ω_8	9.6547	9.6581	–	10.4764	11.5768	–	10.4916	10.4728	–	10.4905	10.4599	–	9.9202	9.9105	–
Ω_9	10.3137	10.3177	–	11.3343	11.6379	–	10.9794	10.9595	–	11.0682	11.0430	–	10.1579	10.1433	–
Ω_{10}	10.4720	10.4770	–	12.2236	12.2221	–	11.5768	11.5597	–	11.2627	11.2386	–	11.0600	11.0393	–

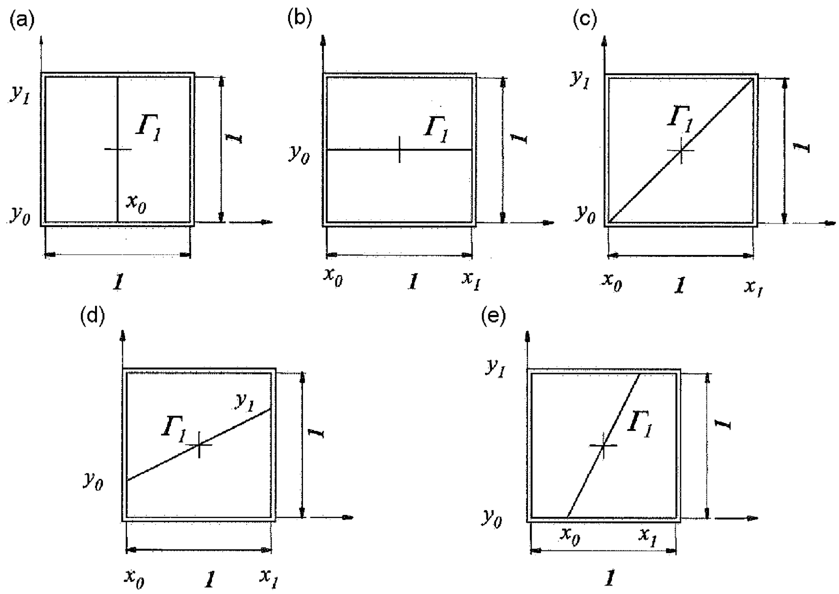

Fig. 6. Rectangular membranes with oblique supports. $X = a/b = 3$: (a) $M/R = 1$, $x_0 = 0.50$, $y_0 = 0$, $y_1 = 1$, (b) $M/R = 1$, $x_0 = 0$, $y_0 = 0.50$, $x_1 = 1$, (c) $M/R = 1$, $x_0 = 0$, $y_0 = 0$, $x_1 = 1$, $y_1 = 1$, (d) $M/R = 1$, $x_0 = 0$, $y_0 = 0.25$, $x_1 = 1$, $y_1 = 0.75$, and (e) $M/R = 1$, $x_0 = 0.25$, $y_0 = 0$, $x_1 = 0.75$,$y_1 = 1.00$.

The utilization of a full set of expanded trigonometric functions of uniform convergence, guarantees beforehand the convergence into exact values. Besides, the fact that the restriction of the partial intermediate supports is considered through the addition of Lagrange multipliers is theoretically exact. The use of trigonometric functions and Lagrange multipliers allow one to obtain values as accurate as necessary, as the convergence of the methodology used here depends on the number of terms adopted in the series. It should be pointed out that in this case, the number of terms has been relatively low, with very little demand of computational time.

It is convenient to briefly indicate the observations derived from the study of convergence. In order to obtain the eigenvalues, it is natural to adopt two of the three parameters that define the limit of the series. Indeed, the study starts by proposing the number M and N, which represent the number of half-waves of a vibration mode in the x and y directions, respectively. Then, a suitable R is adopted to fulfil the condition of null virtual work over each support. It seems logical to assume beforehand, that in order to achieve a better

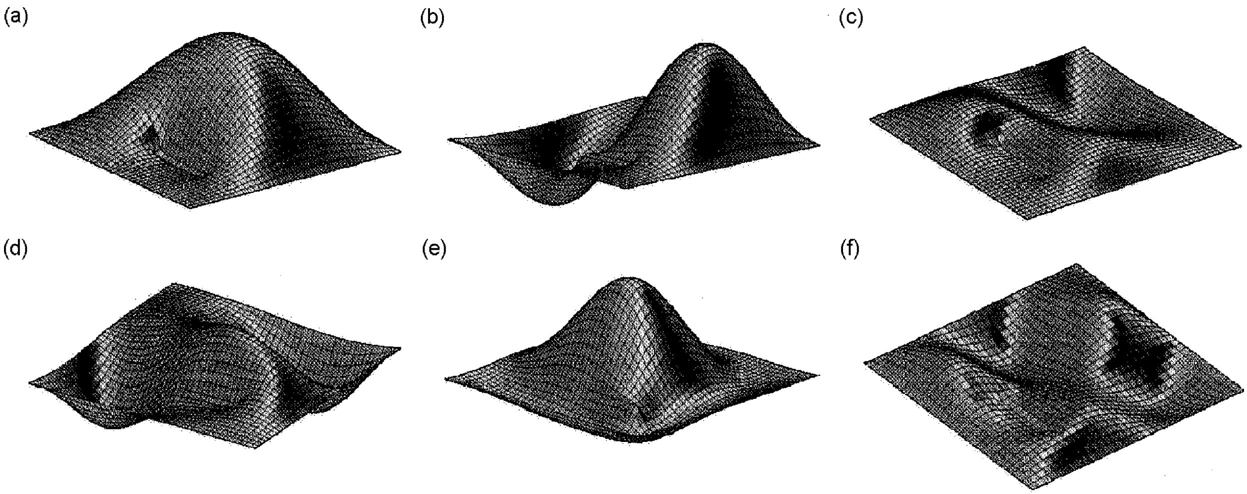

Fig. 7. First six mode shapes of the square membrane Model (a) Table 1 by the proposed method: (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, and (f) 6th mode.

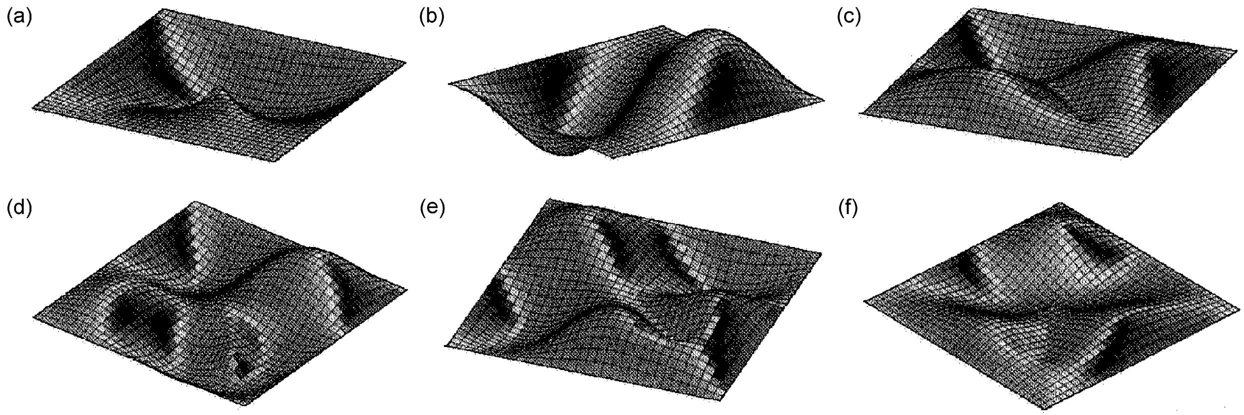

Fig. 8. First six mode shapes of the square membrane Model (b) Table 1 by the proposed method: (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, and (f) 6th mode.

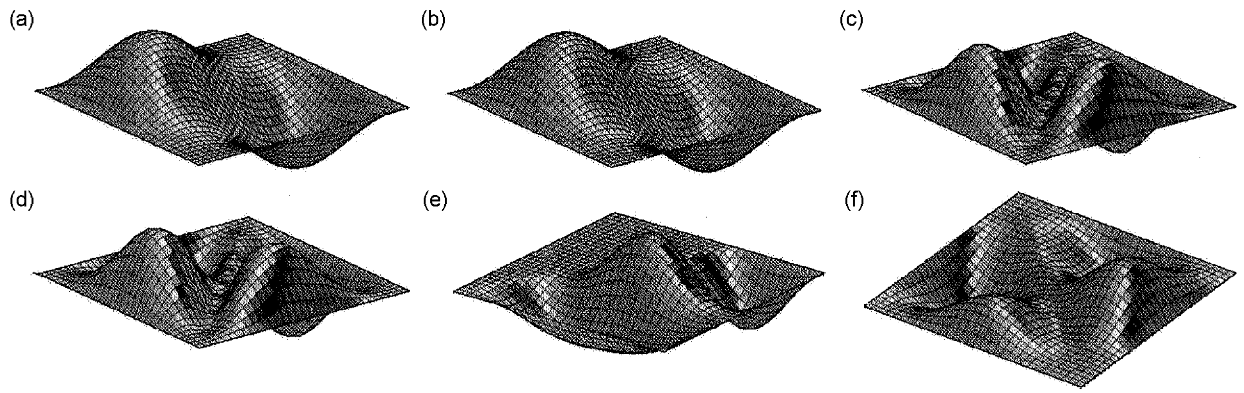

Fig. 9. First six mode shapes of the square membrane Model (c) Table 1 by the proposed method: (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, and (f) 6th mode.

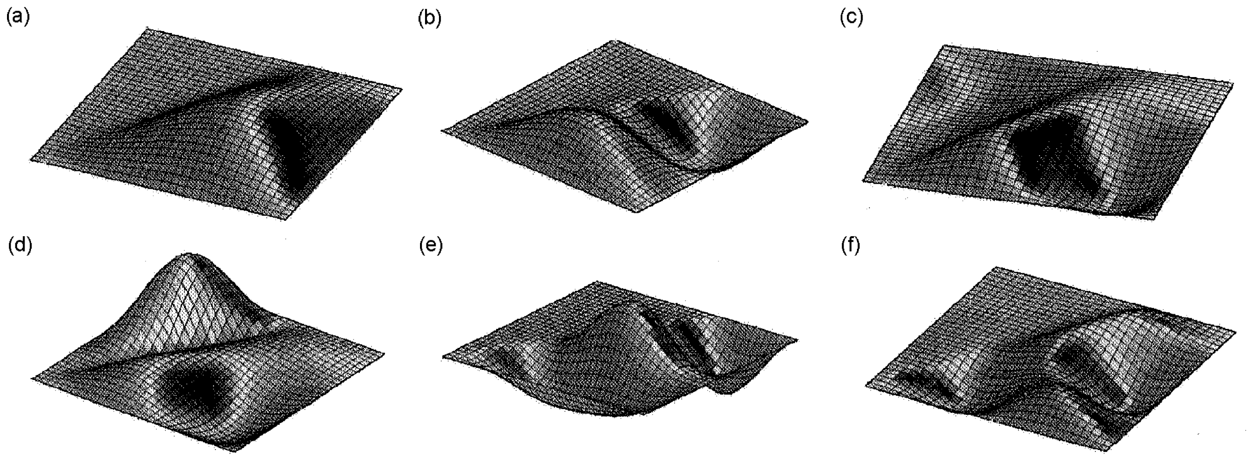

Fig. 10. First six mode shapes of the square membrane Model (d) Table 1 by the proposed method: (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, and (f) 6th mode.

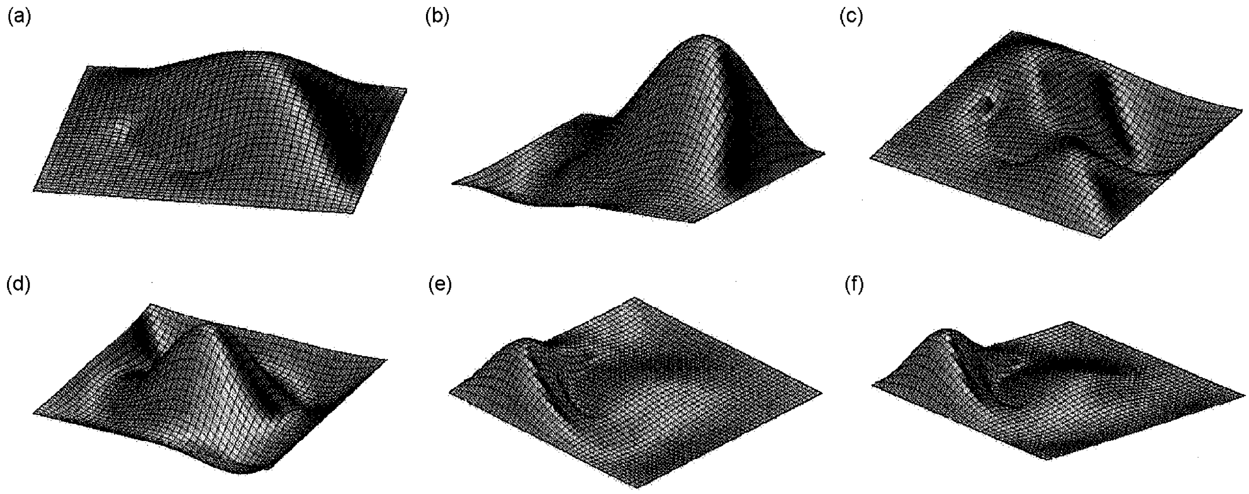

Fig. 11. First six mode shapes of the square membrane Model (a) Table 2 by the proposed method: (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, and (f) 6th mode.

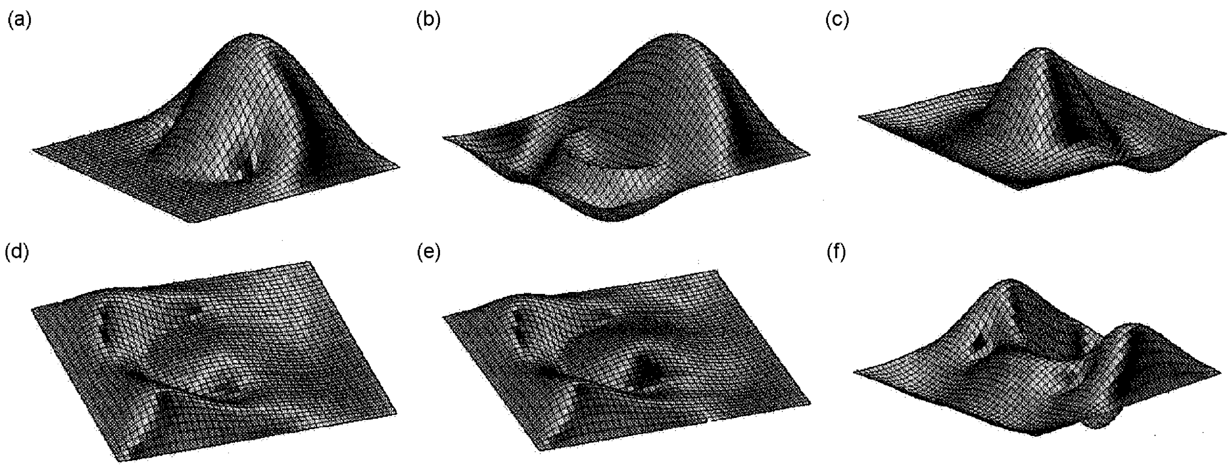

Fig. 12. First six mode shapes of the square membrane Model (b) Table 2 by the proposed method: (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, and (f) 6th mode.

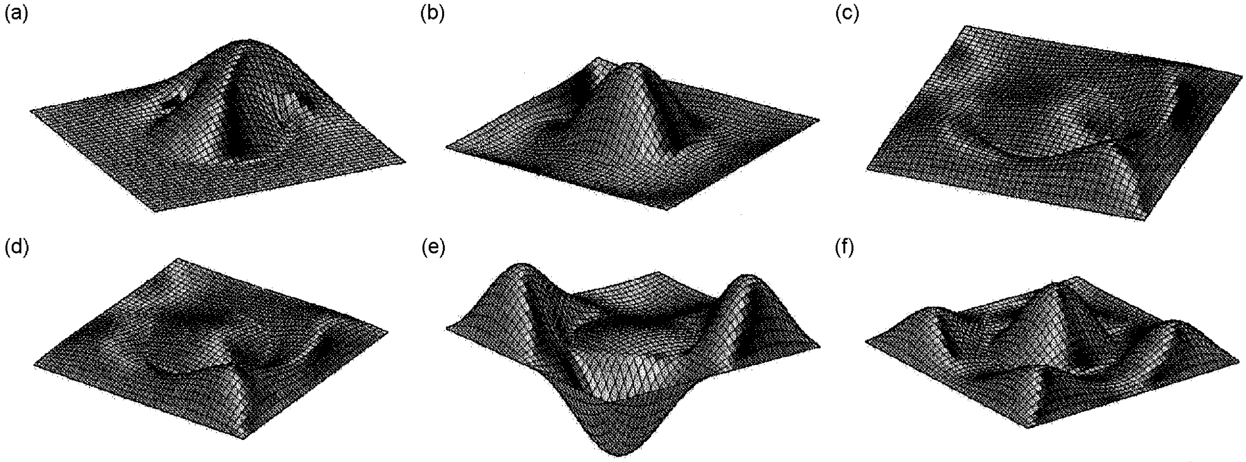

Fig. 13. First six mode shapes of the square membrane Model (c) Table 2 by the proposed method: (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, and (f) 6th mode.

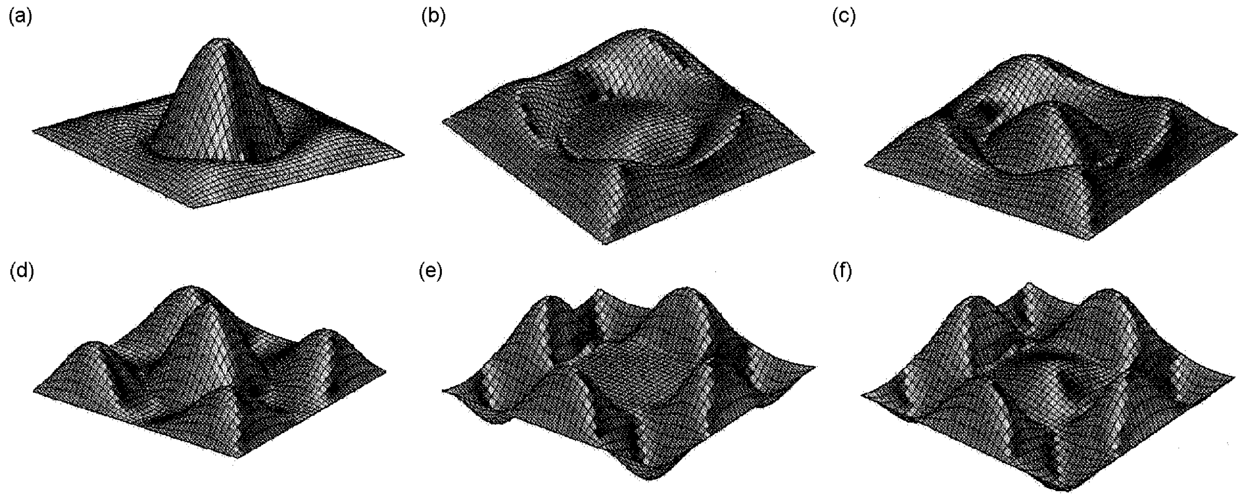

Fig. 14. First six mode shapes of the square membrane Model (d) Table 2 by the proposed method: (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, and (f) 6th mode.

adaptation of the mode shape to the internally restricted geometry of the membrane, a large number (M) of semiwaves should be proposed. Furthermore, in previous work done by this investigation group that has been the norm, resulting in a noticeable demand of computer time. However, from the tests that were done when the algorithm was calibrated, it is concluded that in order to obtain an accurate enough algorithm, there is an M/R ratio for each analysed geometry which offers results that can be considered to have an acceptable accuracy, even for low numbers (M and R) of semiwaves, which were not higher than two digits. Such evidence was manifested at the time when the present algorithm, reproduced the known frequencies of thin plates with the same complexities [7]. With very low values of M and R, that kept a certain ratio, acceptable results were obtained. With an approximately constant M/R ratio, no matter how small the numbers of semiwaves were adopted, eigenvalues with adequate accuracy were found. As it is expected, the accuracy improves as M and R increase. For the sake of brevity, thin plates results are not included.

As an original theorem, the demonstration that the known analogy between membranes and polygonal plates of equal geometry stops working when the domain has partial intermediate supports, is also included in

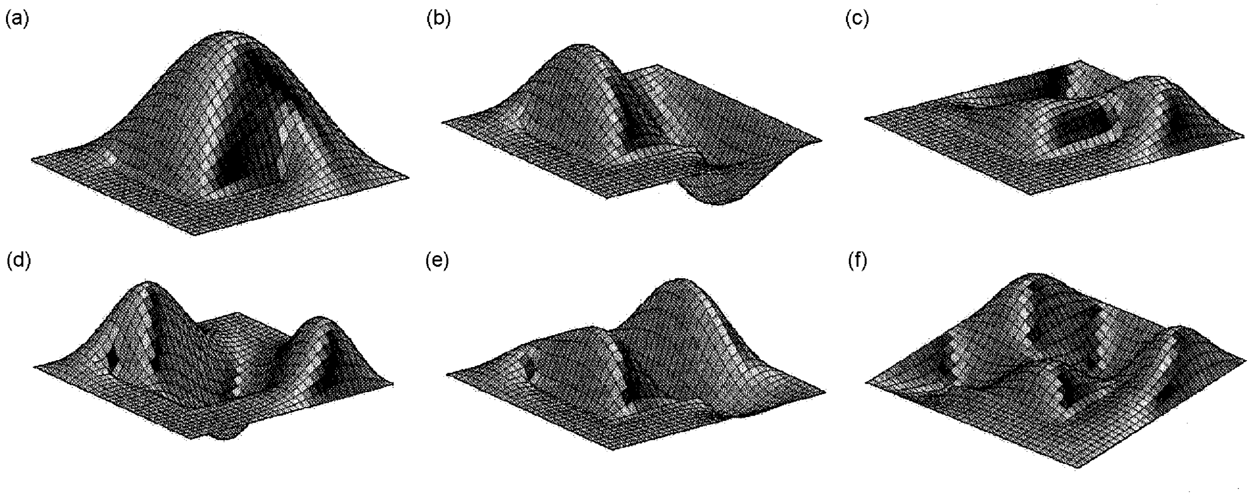

Fig. 15. First six mode shapes of the rectangular membrane Model (a) Table 3 by the proposed method: (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, and (f) 6th mode.

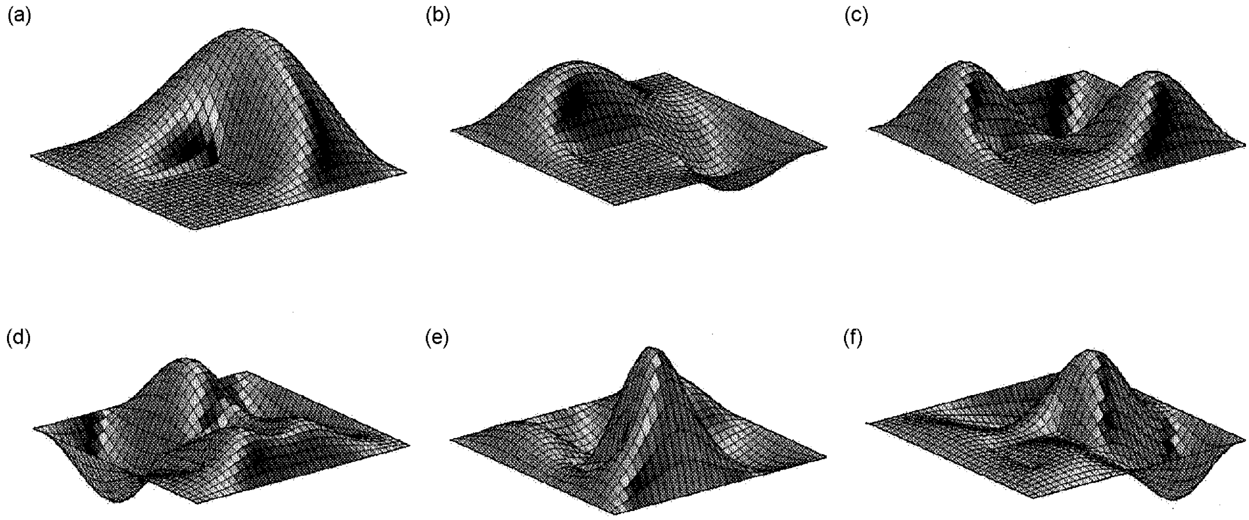

Fig. 16. First six mode shapes of the rectangular membrane Model (b) Table 3 by the proposed method: (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, and (f) 6th mode.

Appendix A. On the other hand, the method of imposing both null displacements and virtual work in several points of the intermediate supports was also used. The difference found between them, "continuous" and "discrete" ones, was irrelevant.

Appendix A

Loss of the analogy between membranes and SS plates when intermediate supports is involved.

We will present a simple demonstration of the above. The equation for free vibrations for thin plates (Germain–Lagrange) is

$$\nabla^2\nabla^2 v - \lambda^2 v = 0 \tag{A.1}$$

in which

$$\nabla^2(.) = (.)_{xx} + (.)_{yy},$$

$$(.)_x = \frac{\partial(.)}{\partial x},\ (.)_y = \frac{\partial(.)}{\partial y},$$

$$\lambda^2 = \frac{\rho h}{D}\omega^2, \quad v = v(x, y), \tag{A.2}$$

where h is the thickness and D the flexural rigidity of the plate. If the plate is SS on the boundary (Γ), it is seen that

$$v^{(a)}_{(\Gamma)} = 0, \tag{A.3a}$$

$$M^{(b)}_{n(\Gamma)} = 0, \tag{A.3b}$$

where M_n indicates the bending moment in the plane (zn), being M_x, M_y and M_{xy} the bending moments and the twisting moment, respectively. Owing to the tensorial nature of the stress, M_n fulfills the following:

$$M_n = M_x n_1^2 + 2M_{xy} n_1 n_2 + M_y n_2^2, \tag{A.4}$$

where n_i ($i = 1,2$) are the cosine directors.

From the theory of plates, we know that

$$\begin{aligned} M_x &= -D(v_{xx} + \nu v_{yy}), \\ M_y &= -D(v_{yy} + \nu v_{xx}), \\ M_{xy} &= -D(1-\nu)v_{xy}, \end{aligned} \tag{A.5}$$

where ν is the Poisson's coefficient.

The directional derivatives with regard to orthogonal directions $\hat{s}$ and $\hat{n}$ are (Fig. A1)

$$(.)_s \equiv (.)_t = \text{grad}(.)\hat{t}, \tag{A.6a}$$

$$(.)_n = \text{grad}(.)\hat{n} \tag{A.6b}$$

being

$$\text{grad}(.) = (.)_x\hat{i} + (.)_y\hat{j}, \tag{A.7}$$

$$\hat{t} = -n_2\hat{i} + n_1\hat{j}, \tag{A.8a}$$

$$\hat{n} = n_1\hat{i} + n_2\hat{j}. \tag{A.8b}$$

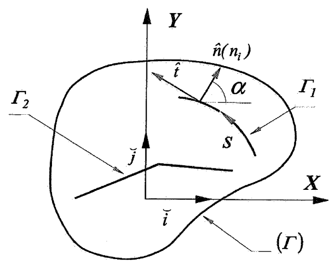

Fig. A1. Directional derivatives with regard to orthogonal directions $\bar{s}$ and $\bar{n}$.

We calculate the second directional derivatives; with $\alpha = \alpha(s)$ we find that

$$(.)_{ss} = (.)_{xx}n_2^2 - 2(.)_{xy}n_1n_2 + (.)_{yy}n_1^2 - \alpha_s(.)_n, \tag{A.9a}$$

$$(.)_{nn} = (.)_{xx}n_1^2 - 2(.)_{xy}n_1n_2 + (.)_{yy}n_2^2. \tag{A.9b}$$

We notice from Eqs. (A.4), (A.5) and (A.9) that

$$M_n = -D[v_{nn} + \nu(v_{ss} + \alpha_s v_n)]. \tag{A.10}$$

We also need the Laplacian in coordinates n and s; so, we operate with the sum of Eqs. (A.9a) and (A.9b), and we find

$$\nabla^2(.) = (.)_{xx} + (.)_{yy} = (.)_{ss} + \alpha_s(.)_n + (.)_{nn}. \tag{A.11}$$

Now, moving over to the perimeter where $\hat{t}$ and $\bar{n}$ are the unit tangent and normal vectors, respectively, from condition equation (A.3a)

$$v_{(\Gamma)} = 0 \Rightarrow v_{s(\Gamma)} = v_{ss(\Gamma)} = 0. \tag{A.12}$$

Therefore, condition equation (A.3b) can be written, considering Eq. (A.12), as

$$M_{n_{(\Gamma)}} = 0 \Rightarrow (v_{nn} + \nu\alpha_s v_n)_{(\Gamma)} = 0, \tag{A.13}$$

since $-D\neq 0$. Also, from Eqs. (A.11) and (A.12)

$$(\nabla^2 v)_{(\Gamma)} = (v_{nn} + \alpha_s v_n)_{(\Gamma)} = \left(\frac{\nu - 1}{\nu}\right)(v_{nn})_{(\Gamma)}. \tag{A.14}$$

Now, let us remember Helmholtz's equation for the vibrating membrane

$$\nabla^2 w + \Omega^2 w = 0 \tag{A.15}$$

added to the boundary condition

$$w_{(\Gamma)} = 0. \tag{A.16}$$

Seeking an analogy between both problems, we rewrite Eq. (A.1) adding and subtracting $(\lambda^2\nabla^2 v)$, that is

$$\nabla^2\nabla^2 v - \lambda^2 v + (\lambda^2\nabla^2 v - \lambda^2\nabla^2 v) = 0 \tag{A.17}$$

or

$$\nabla^2(\nabla^2 v - \lambda v) + \lambda(\nabla^2 v - \lambda v) = 0. \tag{A.18}$$

If we denote

$$w^* = \nabla^2 v - \lambda v, \tag{A.19}$$

Eq. (A. 19) for vibrating plates can be written as

$$\nabla^2 w^* + \lambda w^* = 0. \tag{A.20}$$

Then, comparing Eq. (A.15) with Eq. (A.20), we see that both equations will be the same if $\lambda = \Omega^2$, as long as

$$w^*_{(\Gamma)} \equiv (\nabla^2 v - \lambda v)_{(\Gamma)} = 0. \tag{A.21}$$

Now, let us analyze under which conditions Eq. (A.21) will be verified. Owing to Eq. (A.3) $v_{(\Gamma)} = 0$; we need to find when $(\nabla^2 v)_{(\Gamma)} = 0$. We deduct from Eq. (A.14) that it must occur with $(\nu-1)/\nu\neq 0$, that is

$$(v_{nn})_{(\Gamma)} = 0. \tag{A.22}$$

So, considering that Eq. (A.22) is satisfied, the boundary condition Eq. (A.3b) for SS plates, due to Eq. (A.14), would imply that

$$(\alpha_s v_n)_{(\Gamma)} = 0. \tag{A.23}$$

The normal derivative of SS plate is generally not null; therefore, in order to fulfil the analogy

$$(\alpha_s)_{(\Gamma)} = 0. \tag{A.24}$$

Therefore, and this is a result known for the last century, it is enough that the plate's shape be polygonal (like the membrane's), that is, a domain formed by straight SS edges, with which Eq. (A.24) is evidently fulfilled on each length. In this way, the boundary conditions of the polygonal SS plate become

$$\begin{aligned}(v)_{(\Gamma)} &= 0,\\ (v_{nn})_{(\Gamma)} &= 0,\end{aligned} \tag{A.25}$$

so $(\nabla^2 v)_{(\Gamma)} = 0$ is verified, and therefore, also Eq. (A.21). Then, solving the λ frequencies of these plates, we also find that

$$\Omega = \sqrt{\lambda}, \tag{A.26}$$

which are the frequency parameters of the membranes of the same geometry.

When the membrane has partial intermediate supports (Fig. A1), besides Eq. (A.16) condition, it must be satisfied

$$(w)_{(\Gamma k)} = 0, \quad k = 1, 2, \ldots, n, \tag{A.27}$$

where n is the number of inner supports.

Now, in general, a plate with the same shape and same supports will not fulfil Eq. (A.21), despite the inner supports are polygonal, because the condition $(M_n)_{(\Gamma k)} = 0$ will not be true. That is, even accepting $(\alpha_s)_{(\Gamma k)} = 0$ (straight inner supports) $(v_{nn})_{(\Gamma k)} \neq 0$. This shows that the known and useful analogy is lost.

Directional derivatives with regard to orthogonal directions $\bar{s}$ and $\bar{n}$ are shown in Fig. Al.

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