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Wave propagation in semi-infinite bar with random imperfections of density and elasticity module

J. Náprstek

Institute of Theoretical and Applied Mechanics, CZ-19000 Prague, Czech Republic

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Abstract

Mathematical modeling and properties of a linear longitudinal wave propagating in a slender bar with random imperfections of material density and Young modulus of elasticity is discussed. Fluctuation components of material properties are considered as continuous stochastic functions of the length coordinate. Two types of fluctuation and their influence on response properties have been investigated, in particular the delta correlated and a diffusion-type processes. Investigation itself is based on Markov processes and corresponding Fokker–Planck–Kolmogorov equation. The stochastic moments closure as a solution method has been used. Many effects due to the stochastic nature of the problem have been detected. Along the bar a drop of the mean value of the response with the simultaneous increase of the response variance have been observed. This effect does not represent any conventional damping, but a gradual drop of the deterministic and an increase of the stochastic components of the overall response. The rate of the response indeterminacy increases with the increase of the length coordinate. Increasing values of material imperfection variances and the rising excitation frequency can lead to a critical state when the length of the propagating wave is comparable with the correlation length of imperfections. This state will manifest itself as a radical change of the response character. The problem will pass beyond the boundaries of stochastic mechanics and lose its physical meaning. Similar effects can be observed in the FEM analysis, where there is also a certain permissible upper boundary of the excitation frequency corresponding with the size and type of the element used.

1. Introduction

The problem of propagation of mechanical stress waves in the medium, the physical characteristics of which are burdened by random perturbations, arises in a number of disciplines. By way of example one can mention the propagation of seismic waves, either of natural or technological origin, the wave propagation in materials with microscopic non-homogeneity of a certain degree, etc.

Although the macroscopic mean values of physical parameters (e.g. E, ϱ) are commonly considered as constant, it is impossible to avoid the influence of random imperfections in many cases. They originate from micro polycrystalic structure of metals, from microinclusions in composites with ceramic matrice, etc.

E-mail address: naprstek@itam.cas.cz.

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The random variability of material density and elasticity parameters results in a stochastic component of the response, even if the excitation itself is deterministic. This phenomenon can be observed in experiments if the induced wave motion in the excitation point is compared to the wave motion at various distances from the source. In such a case the dispersion of the results is not determined merely by an unevenness of the experimental equipment, but also by the character of the material itself, see Fig. 1. For the harmonic excitation of the semi-infinite bar in the point x_0 the response frequency curve in the excitation point is a Dirac-like function. With increasing distance $x - x_0$ the response frequency curve is successively dropping.

In principle, the value characterizing the deterministic part of the response is dropping with increasing x, while the stochastic part of the response is increasing in the same time, see Fig. 2 (note: detailed explanation of the $|m_1(x)|, |m_{11}(x)|$ symbols will be given later). The drop of the deterministic part is not accompanied by any mechanical energy loss. Only its form is changed from deterministic into stochastic one so that no thermal energy is produced unless internal viscosity or other source of dissipation would be taken into account. The increasing rate of the stochastic part of the response is not unexpectable. It fully corresponds to the law of the Boltzmann's entropy of probability increase.

As another example can serve the seismic event which approaches a deterministic action in the place of its origin. In the course of propagation through a non-homogeneous medium, the deterministic component of soil movement gradually disappears while its random component is increasing with the distance from the epicenter.

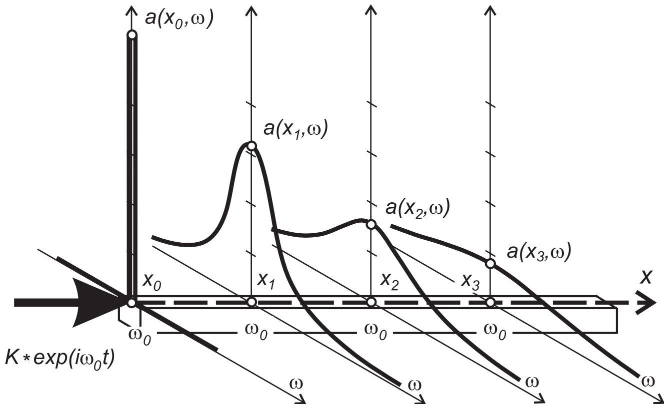

Fig. 1. Successive dropping of the response frequency curve with an increasing distance $|x - x_0|$ from the excitation point x_0; ω_0-excitation frequency; $a(x_0, \omega)$- amplitude of the wave with frequency ω.

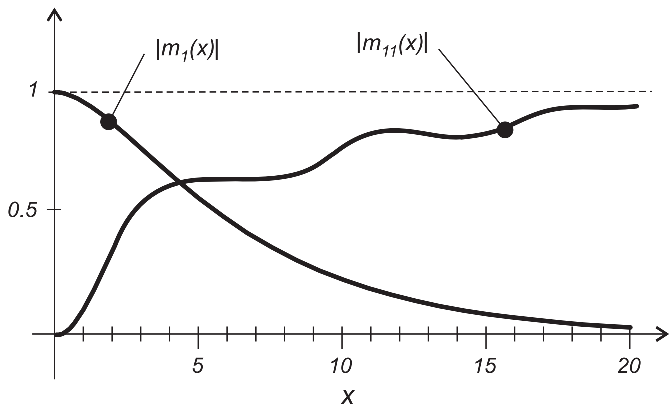

Fig. 2. Decay of the deterministic part $|m_1(x)|$ and an increase of the stochastic part $|m_{11}(x)|$ of the response with increasing distance from the excitation point.

The problem of wave propagation in stochastic medium attracted considerable attention in the past. Generally speaking, however, the works concerned merely certain qualitative estimates of the character of these processes in 2D and 3D media. An extensive survey of these activities with numerous references to further articles can be found in overview papers, for instance [1,2] and many others. Special Issue of the *Probabilistic Engineering Mechanics* has been devoted to problems of materials with random non-homogeneities [3]. Of the other works directly connected with seismicity one can mention e.g. Refs. [4–7], etc. Other authors dealt with special problems arising in connection with various types of internal physical nonlinearity, e.g. Refs. [8,9] or investigated the influence of random roughness of reflection surface on wave scattering [10]. With regard to the philosophy of construction of mathematical models, however, the above papers did not describe a number of effects specific for stochastic media. They were oriented to estimate the global properties of the response rather than to describe detailed properties of the wave propagating in a continuum with random fluctuations in material parameters.

An interesting approach to analysis of wave propagation in 1D continua represents an application of the Lyapunov exponent, see Refs. [11,12]. It is a powerful tool for an assessment of basic properties of waves propagating in a bar of final dimensions. The order of governing differential system remains at two while more detailed models produce the system of the fourth or sixth order. On the other hand, the above stochastic models enable more detailed analysis especially in the neighborhoods of boundaries, where the wave amplitude decrease is more complicated than a simple exponential curve. Boundary effects and main part of the response should be separated when an in-fined bar is to be investigated. Therefore, the analysis envisaged in this study does not make use of the Lyapunov exponent.

To enable a detailed analysis of motion, it is necessary to abandon mathematical models based on the small parameter method. This frequently used procedure does not characterize the motion in detail and, moreover, it gives physically contradictory results regarding the energy equilibrium law. The principal cause of this paradox is the fact that the small parameter method considers the random component of the response to be small and insignificant. This approach is acceptable only in the case of bodies with final dimensions and very sparse spectrum of natural frequencies. In the domain of infinite dimensions, however, the stochastic part of the response becomes entirely dominant at a certain distance from the excitation point and the small parameter method is no longer convergent and provides meaningless results. Also, the usually adopted independence of perturbations in the adjacent points of the region is unacceptable.

Some of these shortcomings could be eliminated by the application of Markov processes theory. It means to introduce on the application level the Ito system. Then, for Gaussian inputs, the corresponding FPK equation for an unknown probability density function (PDF) of the response can be written, e.g. Refs. [13,14] and many others. Another possibility represents an application of spectral decomposition method, which is based on the Wiener-Kchinchin theorem, e.g. Ref. [15]. This conventional method which is widely used in linear dynamics with additive Gaussian excitation, however, does not provide a global insight into the problem and does not allow to sufficiently respect the non-Gaussian character of the response.

Three possible stochastic models are compared in [16]. The first one fully corresponds to author's papers [17,20] where the spectral decomposition approach has been developed and used for a particular analysis. Probably, the first outline concerning application of Markov processes for longitudinal wave propagation is done in this paper as well as the Dyson integral equation remembering some steps of spectral decomposition process.

From the physical viewpoint, it is necessary to admit the stochastic character of both parameters influencing the wave character: (i) material density and (ii) Young modulus of elasticity. In such a case the response has a number of new interesting properties, as the imperfections in the stochastic differential equation influence on the terms with various order of derivatives with respect to x. Generally speaking it is coming to light that the influence of random imperfections on the terms with higher derivatives has weaker local effects, but manifests itself at larger distances and, consequently, is rather of global character.

2. Basic mathematical model

Let us consider the problem of propagation of a linear longitudinal wave in a semi-infinite bar with a constant cross section supposing that the response depends only on the length coordinate $x \in \langle 0; \infty)$ and is

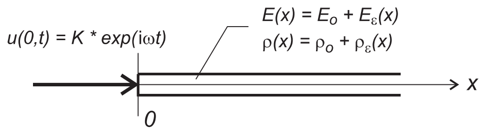

Fig. 3. Prismatic semi-infinite bar with random imperfections of density and elasticity module, kinematically excited in the point $x = 0$.

independent of the lateral coordinates, see Fig. 3. The only point of the external excitation is the origin $x = 0$. The physical characteristics of the bar consist of a constant deterministic part and a small random perturbation along the bar. In such a case the motion of the bar is governed by the following differential equation:

$$(E(x)u'(x,t))' - \varrho(x)\ddot{u}(x,t) = 0, \quad x\varepsilon(0;\infty), \tag{1}$$

$$E(x) = E_0 + E_\varepsilon(x) = E_0(1 + \xi_E(x)),\ \varrho(x) = \varrho_0 + \varrho_\varepsilon(x) = \varrho_0(1 + \xi_\varrho(x)),$$

$$\mathbf{E}\{\xi_E^2(x)\} \ll 1, \mathbf{E}\{\xi_\varrho^2(x)\} \ll 1, \tag{2}$$

where E_0, ϱ_0 are the constant mean values of the Young modulus and the material density, $\xi_E(x), \xi_\varrho(x)$ are continuous centered random Gaussian homogeneous processes describing the fluctuations of Young modulus and material density in the length coordinate x (conditions (2) express that non-dimensional fluctuations $\xi_E(x), \xi_\varrho(x)$ are "small" compared with "one" in the meaning of their variances; a very small but positive probability is admitted that these processes acquire large absolute values in some points x, however respective integrals should conserve their existence), $\mathbf{E}\{\cdot\}$ is the mathematical mean value operator with respect to Gaussian PDF.

Let us introduce the following kinematic excitation of the bar in point $x = 0$:

$$u(x,t)|_{x=0} = K\exp(\mathrm{i}\omega t) \Rightarrow u(x,t) = v(x)\exp(\mathrm{i}\omega t) \tag{3}$$

which leads to the homogeneous stochastic ordinary differential equation with two multiplicative noises:

$$((1 + \xi_E(x))v'(x))' + \Omega^2(1 + \xi_\varrho(x))v(x) = 0, \tag{4}$$

$$\Omega^2 = \omega^2/c^2, \quad c^2 = E_0/\varrho_0, \tag{5}$$

where ω is the excitation frequency, c is the longitudinal wave propagation velocity in the corresponding homogeneous continuum.

Let us transform Eq. (4) into the normal form. To this end the relation $(1 + \xi_E(x))v_1'(x) = v_2(x)$ can be introduced. Taking into account inequalities (2), it holds approximately: $(1 + \xi_E(x))^{-1} \approx (1 - \xi_E(x))$. Then the following stochastic differential system with multiplicative noises can be written:

$$v_1'(x) = (1 - \xi_E(x))\, v_2(x),$$

$$v_2'(x) = -\Omega^2(1 + \xi_\varrho(x))\, v_1(x). \tag{6}$$

Random fluctuations $\xi_E(x), \xi_\varrho(x)$ are considered to admit a large variety of types respecting as much experimental results as possible. For purposes of an analytical investigation they are mostly presented in the form of a correlation function or a spectral density.

The equation of the type (1) or (4) and problems of the fourth-order describing dynamics of a beam on a stochastic subsoil as well as other problems, have been investigated by the author of this study in the past, e.g. Refs. [17–21], using the method of spectral decomposition. Although qualitative properties of the response have been described, many important details and a global insight into the problem remained hidden. For this reason another approach being based on Markov processes or Fokker–Planck–Kolmogorov (FPK) equation is introduced here.

3. Delta correlated material imperfections

Let us try to represent processes $\xi_E(x), \xi_\varrho(x)$ in the most simple way by Gaussian white noises in the coordinate x. These white noises have a multiplicative character. Therefore, Wong-Zakai correction terms (see e.g. Refs. [22,23], or monographs [14,24]) in drift coefficient should be taken into account to express properties of the real physical process of the response. With respect to the form of the stochastic system (6), the following drift and diffusion coefficients can be formulated:

$$\begin{aligned}\kappa_1 &= v_2 + s_{\varrho E}\Omega^2 v_1, \quad \kappa_{11} = s_{\varrho\varrho} v_2^2, \quad \kappa_{12} = s_{\varrho E}\Omega^2 v_1 v_2,\\ \kappa_2 &= -\Omega^2 v_1 + s_{E\varrho}\Omega^2 v_2, \quad \kappa_{21} = s_{E\varrho}\Omega^2 v_1 v_2, \quad \kappa_{22} = s_{EE} v_1^2.\end{aligned} \tag{7}$$

where s_{ij} $(i,j = E, \varrho)$ are the intensities of white noises $\xi_j(x)$ $(s_{E\varrho} = s_{\varrho E} \Rightarrow \kappa_{21} = \kappa_{12})$.

Thus, the respective FPK equation, e.g. Refs. [13,14,24,25], for the response PDF can be written $(p(v_1, v_2, x) = p)$ as:

$$\begin{aligned}\frac{\partial p}{\partial x} = &-\frac{\partial}{\partial v_1}(v_2 + s_{\varrho E}\Omega^2 v_1)p - \frac{\partial}{\partial v_2}(-\Omega^2 v_1 + s_{\varrho E}\Omega^2 v_2)p\\ &+ \frac{1}{2}\left(\frac{\partial^2}{\partial v_1^2} s_{\varrho\varrho} v_2^2 p + 2\frac{\partial^2}{\partial v_1 \partial v_2} s_{\varrho E}\Omega^2 v_1 v_2 p + \frac{\partial^2}{\partial v_2^2} s_{EE}\Omega^4 v_1^2 p\right).\end{aligned} \tag{8}$$

To apply Eq. (8) for an assessment of the basic properties of the propagating wave, the stochastic moment closure procedure will be used, see for instance Ref. [26]. Although this procedure does not provide a convergent series in the general case, it enables to investigate linear systems with ''small'' multiplicative noises. In order to assess the first moment of probability density or the mathematical mean value of the response, let us multiply Eq. (8) successively by factors v_1, v_2 and apply the operator $\mathbf{E}\{\cdot\}$. Using multiple integration by parts on an infinite domain in both coordinates v_1, v_2 and taking into account the fact that the PDF is vanishing together with all derivatives on the boundary of infinite domain, we obtain after a number of adaptations a linear deterministic differential system for the first stochastic moments m_1, m_2 of variables v_1, v_2:

$$\begin{aligned}m_1' &= m_2 + s_{\varrho E}\Omega^2 m_1,\\ m_2' &= -\Omega^2 m_1 + s_{\varrho E}\Omega^2 m_2.\end{aligned} \tag{9}$$

Initial conditions can be introduced, for instance, as: $m_1(0) = K; m_2(0) = 0$, see Eq. (3). Instead of Eq. (9) an equivalent equation of the second order for $m_1(x)$ can be written:

$$m_1'' - 2s_{\varrho E}\Omega^2 m_1' + \Omega^2(1 + s_{\varrho E}^2\Omega^2)m_1 = 0, \quad m_1(0) = K; m_1'(0) = 0. \tag{10}$$

Eq. (10) indicates that the mathematical mean value or ''deterministic part'' of the wave does not depend on the intensities $s_{EE}, s_{\varrho\varrho}$. Only the cross intensity $s_{\varrho E}$ associated with the correction terms in drift coefficients κ_j enters Eq. (10). Therefore, if the noises ξ_E, ξ_ϱ are independent or one of them vanishes, then no influence of material random imperfections on mathematical mean of the response occurs. We may conclude that the mean value of the response of the random bar is equivalent to the response of the deterministic bar with mean values of physical properties. On the other hand if $s_{\varrho E} \neq 0$, solution of Eq. (10) cannot provide a meaningful result unless $s_{\varrho E} \leqslant 0$ (otherwise the m_1 would rise exponentially beyond all limits for increasing x). The above conclusions, however, are controversial and hardly acceptable. First of all, the request for a negative cross intensity $s_{\varrho E}$ is meaningless as this value is primarily a result of an independent measurement.

Therefore, the processes ξ_E, ξ_ϱ should be considered as independent to obtain apparently meaningful result. Introducing such condition, any influence of parameter fluctuations on the response mean value is avoided. It leads, consequently, to equivalence of mathematical mean value of the response and of the response following from a deterministic task for parameter mean values. Although such conclusion can be encountered quite often in literature, see e.g. Refs. [27–29], this equivalence is rather strange. The system is strongly influenced by multiplicative noises. Although they are both Gaussian, the response loses in general the Gaussian character, e.g. Ref. [30]. This fact usually manifests itself as a difference between the above quantities, i.e. between the mean value $m_1(x)$ and the solution of the deterministic task for nominal parameters. The response itself is no more a centered process and some PDF skewness also arises. The other way round, the above equivalence would implicate strong limitations which have been tacitly accepted as to the properties of the material imperfections.

Let us derive from Eq. (8) a system for the second stochastic moments respecting that the cross-correlation $s_{\varrho E}$ vanishes. Using similar procedure as before, one obtains the following differential system:

$$m'_{11} = +2m_{12} + s_{\varrho\varrho} m_{22},$$

$$m'_{12} = -\Omega^2 m_{11} + m_{22},$$

$$m'_{22} = s_{EE}\Omega^4 m_{11} - 2\Omega^2 m_{12}.$$

This system does not display any link with first moments m_1, m_2. In the same time, homogeneous initial conditions should be introduced. In such a case only trivial solution can be obtained and so it is for higher moments. It would mean that the random part of the solution is trivial and parameter fluctuations do not influence the result. Therefore, the energy equilibrium law would be violated, see e.g. Ref. [31].

It follows from these paradoxical results that the mathematical model of material fluctuations in the form of white noises is not satisfactory, because it corresponds with physical reality to a very limited extend of the input parameter properties and leads to hardly applicable results for neglected as well as for non-zero cross-correlation of the processes ξ_E, ξ_ϱ.

Let us refer once again to papers [16,17,20]. Results presented there remain in force. However, the stochastic model in this study is more complex and enables to assess the influence of various levels of cross-correlation of both random coefficients as they are defined in Eq. (2). Therefore, it is obvious that any positive cross-correlation of these input processes leads to non-sensical results that contradict the energy equilibrium law.

4. Material fluctuations of diffuse type

It is obvious that more realistic models of material fluctuations than those represented by the delta-correlated processes should be introduced. They should better correspond with physical reality to avoid paradoxical results of the previous paragraph. Let us suppose that a measurement of the parameters $E(x), \varrho(x)$ enables to adopt a hypothesis that they are centered homogeneous Gaussian processes and, moreover, that their autocorrelations can be characterized by monotonously dropping exponential functions. Such processes

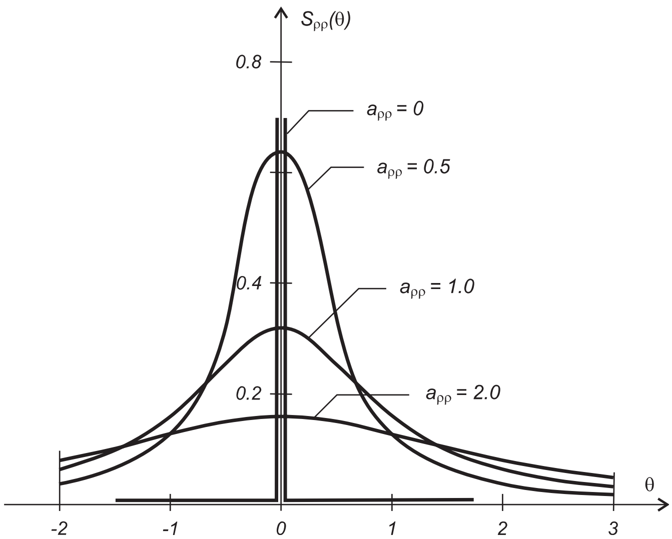

Fig. 4. Spectral density of material imperfections of diffuse type ($\sigma^2_{\varrho\varrho} = 1$).

are described by the following correlation functions and corresponding spectral densities, see Fig. 4:

$$K_{EE}(\hat{x}) = \sigma_{EE}^2 \exp(-a_{EE}|\hat{x}|), \quad S_{EE}(\vartheta) = \frac{\sigma_{EE}^2}{\pi} \frac{a_{EE}}{a_{EE}^2 + \vartheta^2}, \tag{11a}$$

$$K_{\varrho\varrho}(\hat{x}) = \sigma_{\varrho\varrho}^2 \exp(-a_{\varrho\varrho}|\hat{x}|), \quad S_{\varrho\varrho}(\vartheta) = \frac{\sigma_{\varrho\varrho}^2}{\pi} \frac{a_{\varrho\varrho}}{a_{\varrho\varrho}^2 + \vartheta^2}, \tag{11b}$$

where $\sigma_{EE}^2, \sigma_{\varrho\varrho}^2$ are the variances of respective processes, $a_{EE}, a_{\varrho\varrho}$ are the scales of correlation length, $\hat{x}$ is the distance from observation point in the length coordinate: $\hat{x} = x_2 - x_1$, ϑ is the "spacial frequency", i.e. $\vartheta/2\pi$ means the number of waves on a unit length $[1/m]$.

The exponential functions in Eqs. (11) represent a variety wide enough to describe a real situation if the correlation does not descend below zero. This case, however, is not very probable, and that is why Eqs. (11) can be used at least qualitatively. Moreover, the processes characterized by Eqs. (11) can be generated by means of white noises using simple linear filters of the first order:

$$\xi_E'(x) = -a_{EE}\xi_E(x) + \eta_E(x), \tag{12a}$$

$$\xi_\varrho'(x) = -a_{\varrho\varrho}\xi_\varrho(x) + \eta_\varrho(x), \tag{12b}$$

where $\eta_E(x), \eta_\varrho(x)$ are the white noises with intensities:

$$s_{EE} = 2\sigma_{EE}^2 a_E, \quad s_{\varrho\varrho} = 2\sigma_{\varrho\varrho}^2 a_{\varrho\varrho}, \quad s_{E\varrho} = s_{\varrho E} = 0. \tag{13}$$

With respect to the form of Eqs. (12), the processes $\xi_E(x), \xi_\varrho(x)$ can be called diffusion processes and represent the simplest case of spatial correlation of material imperfections acceptable with reference to the above-mentioned criteria.

Let us consider Eqs. (6) and (12) together and remember that processes ξ_E and ξ_ϱ are generated by the input white noise processes $\eta_E(x), \eta_\varrho(x)$. The resulting stochastic differential system contains four components of the response. Completing symbolics introduced in Eq. (6) by $v_3(x) = \xi_E(x), v_4(x) = \xi_\varrho(x)$, one can formulate the following stochastic differential system:

$$v_1'(x) = v_2(x) - v_3(x)v_2(x), \tag{14a}$$

$$v_2'(x) = -\Omega^2 v_1(x) - \Omega^2 v_4(x) v_1(x), \tag{14b}$$

$$v_3'(x) = -a_{EE}v_3(x) + \eta_E(x), \tag{14c}$$

$$v_4'(x) = -a_{\varrho\varrho}v_4(x) + \eta_\varrho(x). \tag{14d}$$

Excitations in Eqs. (14) are additive and consequently drift coefficients do not contain any correction terms. However, Eq. (14) are no more linear. The drift and diffusion coefficients read:

$$\begin{gathered}\kappa_1 = v_2(x) - v_3(x)v_2(x), \\ \kappa_2 = -\Omega^2 v_1(x) - \Omega^2 v_4(x) v_1(x), \quad \kappa_{ij} = 0 \quad \text{with exception:} \\ \kappa_3 = -a_{EE}v_3(x), \quad \kappa_{33} = s_{EE}, \quad \kappa_{34} = 0, \\ \kappa_4 = -a_{\varrho\varrho}v_4(x), \quad \kappa_{43} = 0, \quad \kappa_{44} = s_{\varrho\varrho}.\end{gathered} \tag{15}$$

FPK equation for $p = p(v_1, v_2, v_3, v_4, x)$ has the following form (variable x is omitted):

$$\begin{aligned}\frac{\partial p}{\partial x} = {} & \frac{\partial}{\partial v_1}(-v_2 + v_3 v_2)p + \frac{\partial}{\partial v_2}(\Omega^2 v_1 + \Omega^2 v_4 v_1)p + \frac{\partial}{\partial v_3}(a_{EE}v_3)p + \frac{\partial}{\partial v_4}(a_{\varrho\varrho}v_4)p \\ & + \frac{1}{2}\left(s_{EE}\frac{\partial^2}{\partial v_3^2} + s_{\varrho\varrho}\frac{\partial^2}{\partial v_4^2}\right)p.\end{aligned} \tag{16}$$

Because it can be expected that the response will not differ significantly from Gaussian process, the main characteristics of the deterministic part of the response are given once again by mathematical mean value. Multiplying the FPK equation, Eq. (16), successively by components v_1, v_2, v_3, v_4 and using further the same strategy as in the previous paragraph, the following differential system can be deduced:

$$m_1' = m_2 - m_{23}, \tag{17a}$$

$$m_2' = -\Omega^2 m_1 - \Omega^2 m_{14}, \tag{17b}$$

$$m_3' = -a_{EE} m_3, \tag{17c}$$

$$m_4' = -a_{\varrho\varrho} m_4. \tag{17d}$$

Eqs. (17c,d) are independent and trivial. They do not provide any new information but just confirming an original assumption, that processes of the material fluctuations are centered. Only Eqs. (17a,b) are of interest. Due to the nonlinear character of the original system, Eqs. (14), Eqs. (17) contain the moments higher then of the first order, in particular m_{23}, m_{14}. In order to close the system, Eqs. (17), the previous procedure should be repeated with multipliers $v_1v_3, v_1v_4, v_2v_3, v_2v_4$. After some evaluations, one can obtain the following four equations:

$$m_{13}' = m_{23} - m_{233} - a_{EE} m_{13}, \tag{18a}$$

$$m_{14}' = m_{24} - m_{234} - a_{\varrho\varrho} m_{14}, \tag{18b}$$

$$m_{23}' = -\Omega^2 m_{13} - \Omega^2 m_{134} - a_{EE} m_{23}, \tag{18c}$$

$$m_{24}' = -\Omega^2 m_{14} - \Omega^2 m_{144} - a_{\varrho\varrho} m_{24}. \tag{18d}$$

In Eqs. (18) appear moments of the third order. They have to be approximately transformed using moments of lower order. Note that the processes v_1, v_2 do not differ very much from Gaussian ones, and v_3, v_4 are Gaussian. In this case the higher moments of even degree can be expressed using products of the second-order moments, while the odd moments higher then the first degree vanish. For details and general formula, see e.g. Refs. [32,33] and other monographs.

In particular, processes v_1, v_2 are slightly non-symmetrical but still can be approximately considered to be Gaussian with non-zero mean. Therefore, one can write:

$$v_1(x) \approx m_1(x) + v_{1f}(x), \quad v_2(x) \approx m_2(x) + v_{2f}(x), \tag{19}$$

where v_{1f}, v_{2f} are centered processes. Under the above supposition it holds approximately:

$$m_{233} = \mathbf{E}\{v_2 v_3^2\} \approx m_2 m_{33} = m_2 s_{EE}/2a_{EE} = m_2 \sigma_{EE}^2, \tag{20}$$

and, similarly, using Eq. (13):

$$m_{234} \approx m_2 s_{\varrho E} = 0, \quad m_{134} \approx m_1 s_{E\varrho} = 0, \quad m_{144} \approx m_1 s_{\varrho\varrho}/2a_{\varrho\varrho} = m_1 \sigma_{\varrho\varrho}^2. \tag{21}$$

Moments $m_{233}, m_{234}, m_{134}, m_{144}$ in a decomposed form given by Eqs. (20) and (21) together with cross moments $m_{13}, m_{14}, m_{23}, m_{24}$ express a relation of mathematical means m_1, m_2 with intensities of input processes v_3, v_4 introduced into the system by the white noises η_E, η_ϱ. Using approximative relations Eqs. (20) and (21), the so called closing problem can be considered as finished now, as we are able to write a system of six equations with six unknown moments $m_1, m_2, m_{13}, m_{14}, m_{23}, m_{24}$:

$$m_1' = m_2 - m_{23}, \tag{22a}$$

$$m_2' = -\Omega^2 m_1 - \Omega^2 m_{14}, \tag{22b}$$

$$m_{13}' = -\sigma_{EE}^2 m_2 - a_{EE} m_{13} + m_{23}, \tag{22c}$$

$$m_{14}' = a_{\varrho\varrho} m_{14} m_{24}, \tag{22d}$$

$$m'_{23} = -\Omega^2 m_{13} - a_{EE} m_{23}, \tag{22e}$$

$$m'_{24} = -\sigma^2_{\varrho\varrho}\Omega^2 m_1 - \Omega^2 m_{14} - a_{\varrho\varrho} m_{24}. \tag{22f}$$

A standard procedure can be applied now. It means an exponential form of moments is introduced:

$$m_i(x) = M_i \exp(\lambda x), \quad m_{jk}(x) = M_{jk} \exp(\lambda x), \tag{23}$$

where indices i, j, k should be appointed in correspondence with Eqs. (22). Constants M_i, M_{jk} are components of a column eigenvector of the square matrix (6×6) of coefficients on the right-hand side of the system, Eq. (22). After a cumbersome algebra the following characteristic equation with respect to parameter λ can be obtained:

$$(\lambda^2 + \Omega^2)((\lambda + a_{EE})^2 + \Omega^2)((\lambda + a_{\varrho\varrho})^2 + \Omega^2) - \Omega^4(\sigma^2_{EE}((\lambda + a_{\varrho\varrho})^2 + \Omega^2) + \sigma^2_{\varrho\varrho}((\lambda + a_{EE})^2 + \Omega^2)) + \Omega^6\sigma^2_{EE}\sigma^2_{\varrho\varrho} = 0. \tag{24}$$

Eq. (24) is of the 6th degree and does not enable, in a general case, any closed form solution applicable to further analysis. In order to proceed, numerical procedures should be used. On the other hand, Eq. (24) has a number of properties arising from physical character of the problem it describes. These properties make it possible to solve special cases exactly and the general case approximately. Two of them are discussed in next paragraphs.

5. Fluctuations in material density

An important special case of a general problem seems to be a bar with fluctuations in material density and deterministic Young modulus of elasticity. It means that:

$$\sigma^2_{EE} = 0; \quad a_{EE} = 0. \tag{25}$$

Consequently, also m_{13}, m_{23} vanish. At the same time it should be underlined that $v(x) = v_1(x)$, see Eqs. (6), what implies that the m_1 describes immediately the mean value of the response. Therefore, the system of Eqs. (22) simplifies to the form:

$$m'_1 = m_2, \tag{26a}$$

$$m'_2 = -\Omega^2 m_1 - \Omega^2 m_{14}, \tag{26b}$$

$$m'_{14} = a_{\varrho\varrho} m_{14} + m_{24}, \tag{26c}$$

$$m'_{24} = -\sigma_{\varrho\varrho}\Omega^2 m_1 - \Omega^2 m_{14} - a_{\varrho\varrho} m_{24}. \tag{26d}$$

Analogously with Eq. (23), the general solution of the Eqs. (26) has the form:

$$m_i(x) = M_i \exp(\lambda x); \quad m_{jk}(x) = M_{jk} \exp(\lambda x); \quad i = 1, 2; \; jk = 14, 24 \tag{27}$$

producing the simplified version of the characteristic equation Eq. (24):

$$(\lambda^2 + \Omega^2)((\lambda + a_{\varrho\varrho})^2 + \Omega^2) - \Omega^4\sigma^2_{\varrho\varrho} = 0. \tag{28}$$

Eq. (28) can be converted into a bi-quadratic equation using a transformation $\lambda = \delta - a_{\varrho\varrho}/2$:

$$\delta^4 - 2(a^2_{\varrho\varrho}/4 - \Omega^2)\,\delta^2 + (a^2_{\varrho\varrho}/4 + \Omega^2)^2 - \Omega^4\sigma^2_{\varrho\varrho} = 0. \quad (\delta = \alpha + \mathrm{i}\beta) \tag{29}$$

Hence, one can obtain four roots of Eq. (28):

$$\lambda_1 = -a_{\varrho\varrho}/2 + \alpha + \mathrm{i}\beta, \quad \lambda_2 = -a_{\varrho\varrho}/2 + \alpha - \mathrm{i}\beta, \tag{30a}$$

$$\lambda_3 = -a_{\varrho\varrho}/2 - \alpha + \mathrm{i}\beta, \quad \lambda_4 = -a_{\varrho\varrho}/2 - \alpha - \mathrm{i}\beta, \tag{30b}$$

$$\alpha = \left[\frac{1}{2}(((\Omega^2 + a_{\varrho\varrho}^2/4)^2 - \Omega^4\sigma_{\varrho\varrho}^2)^{1/2} - (\Omega^2 - a_{\varrho\varrho}^2/4))\right]^{1/2}, \tag{30c}$$

$$\beta = \left[\frac{1}{2}(((\Omega^2 + a_{\varrho\varrho}^2/4)^2 - \Omega^4\sigma_{\varrho\varrho}^2)^{1/2} + (\Omega^2 - a_{\varrho\varrho}^2/4))\right]^{1/2} \tag{30d}$$

Roots $\lambda_l, l = 1 - 4$ are eigenvalues of the coefficient matrix on the right-hand side of the system Eqs. (26).

Constants M_i, M_{jk} in Eqs. (27) are the components of lth eigenvector of this matrix. The lth column of modal matrix can be evaluated as ratios of the respective sub-determinants of the matrix being in a singular state due to lth eigenvalue λ_l. They are given by the following formuli:

$$M_1^l = \lambda_l((\lambda_l + a_{\varrho\varrho})^2 + \Omega^2), \tag{31a}$$

$$M_2^l = \lambda_l^2((\lambda_l + a_{\varrho\varrho})^2 + \Omega^2), \tag{31b}$$

$$M_{14}^l = -\lambda_l\Omega\sigma_{\varrho\varrho}^2, \tag{31c}$$

$$M_{24}^l = -\lambda_l\Omega\sigma_{\varrho\varrho}^2(\lambda_l + a_{\varrho\varrho}). \tag{31d}$$

As long as α, β remain real positive numbers, it can be easily shown that $0 \leqslant \alpha \leqslant a_{\varrho\varrho}/2$. Then, the real part of $\lambda_l, l = 1 - 4$ is negative and, consequently, for $x \to \infty$ real parts of the fundamental system are monotonously approaching to zero, as $\lim_{x\to\infty}|\exp(\lambda_l x)| = 0$. At the same time it has to be emphasized that the Sommerfeld condition for wave propagation in a semi-infinite interval $0 \leqslant x < \infty$ is complied with only by solutions for $\lambda_{2,4}$, i.e. containing negative imaginary part $-\mathrm{i}\beta$, while cases for $\lambda_{1,3}$ would describe the solution for negative x. For these reasons, to keep the solution physically meaningful, we shall put adequate integration constants $C_1 = C_3 = 0$.

To determine the integration constants C_2, C_4, we shall apply the initial conditions in the point $x = 0$. The response at this point is fully deterministic being given by harmonic excitation Eq. (3). Taking into account that $v(x) = v_1(x)$, initial value for $m_1(0)$ can be easily introduced. The second condition follows from a hypothesis that cross-moment of the displacement and the material density fluctuation vanishes at this point. The above leads to the simple linear system for integration constants C_2, C_4:

$$m_1(0) = K \Rightarrow \lambda_2((\lambda_2 + a_{\varrho\varrho})^2 + \Omega^2)C_2 + \lambda_4((\lambda_4 + a_{\varrho\varrho})^2 + \Omega^2)C_4 = K,$$

$$m_{14}(0) = 0 \Rightarrow -\lambda_2\Omega^2\sigma_{\varrho\varrho}^2 C_2 - \lambda_4\Omega^2\sigma_{\varrho\varrho}^2 C_4 = 0, \tag{32}$$

which enables to obtain immediately:

$$C_2 = -K\lambda_4/\det, \quad C_4 = K\lambda_2/\det,$$

$$\det = \lambda_2\lambda_4((\lambda_4 + a_{\varrho\varrho})^2 - (\lambda_2 + a_{\varrho\varrho})^2). \tag{33}$$

Performing a back substitution one can write a solution of the system Eqs. (26):

$$m_1(x) = K/D[-((\lambda_2 + a_{\varrho\varrho})^2 + \Omega^2)\mathrm{e}^{\lambda_2 x} + ((\lambda_4 + a_{\varrho\varrho})^2 + \Omega^2)\mathrm{e}^{\lambda_4 x}], \tag{34a}$$

$$m_2(x) = K/D[-\lambda_2((\lambda_2 + a_{\varrho\varrho})^2 + \Omega^2)\mathrm{e}^{\lambda_2 x} + \lambda_4((\lambda_4 + a_{\varrho\varrho})^2 + \Omega^2)\mathrm{e}^{\lambda_4 x}], \tag{34b}$$

$$m_{14}(x) = K\Omega^2\sigma_{\varrho\varrho}^2/D[\mathrm{e}^{\lambda_2 x} - \mathrm{e}^{\lambda_4 x}], \tag{34c}$$

$$m_{24}(x) = K\Omega^2\sigma_{\varrho\varrho}^2/D[(\lambda_2 + a_{\varrho\varrho})\mathrm{e}^{\lambda_2 x} - (\lambda_4 + a_{\varrho\varrho})\mathrm{e}^{\lambda_4 x}], \tag{34d}$$

$$D = (\lambda_4 + a_{\varrho\varrho})^2 - (\lambda_2 + a_{\varrho\varrho})^2. \tag{34e}$$

Let us concentrate now on $m_1(x)$, see Eq. (34a). The mean value of the displacement consists of two periodical damped terms. The period of both terms is the same and corresponds with the parameter β in Eq. (30d).

This parameter increases for $\sigma^2_{\varrho\varrho} \to 0$ monotonously approaching to the value Ω. This value leads to the argument $\beta x = \Omega x = (\omega\sqrt{\varrho_0/E_0})x$ corresponding with the classical problem for homogeneous material. This result coincides with that following from the spectral decomposition procedure, see Refs. [17,20]. But much more information can be gained using Eqs. (34).

The first term in Eq. (34a) is dominant. It is only slightly damped, as for $a_{\varrho\varrho} \to 0$ the parameter α approaches $a_{\varrho\varrho}/2$ from below. For the same reason the second term has the character of a merely local boundary effect with the phase shift $+\pi$. This effect, however, still makes the derivatives of the amplitude $|m_1(x)|$ at point $x = 0$ always low or zero and only for higher x it begins to drop approximately according to $\exp(-a_{\varrho\varrho}/2 + \alpha)x$. It follows that the drop of effective level of the deterministic part of the response is perceptible only at and farther than a certain distance from the point of excitation. For $a_\varrho \to 0$ the expression Eq. (34a) turns into the solution of the classic problem, as the coefficient of the second term turns to zero likewise the damping of the first term.

The effect of a more complicated dropping of the amplitude follows from two cooperating exponential terms in Eq. (34a). It prevents an immediate comparison of the case investigated and the case of wave propagation in continuum with a simple or complicated visco-elastic properties (e.g. respecting the Voigt model). When dealing with a visco-elastic material, the response is described by one exponential only. Therefore a stationary state occurring after a sufficiently long time is characterized by a simple exponential drop of the amplitude, which is the steepest at $x = 0$. In a domain where the second term in Eq. (34a) nearly disappears, the response amplitudes resulting from both stochastic and visco-elastic formulations can be compared, although it should be emphasized, that their physical background is completely different. The drop of the amplitude in the case of a material with random fluctuations does not mean any true damping. The deterministic part of the response is dropping, while the stochastic part is increasing correspondingly.

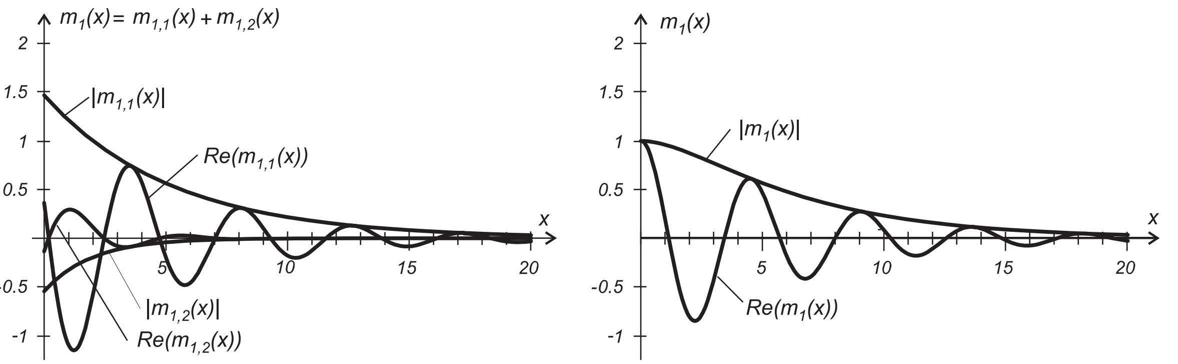

Fig. 5. Mean value of the response consisting of two parts; Symbols $m_{1,1}(x)$ or $m_{1,2}(x)$ mean the first or the second part of the expression (34a) respectively; ($\sigma^2_{\varrho\varrho} = 1.0$, $a_{\varrho\varrho} = 1.0, \Omega^2 = 1.0$).

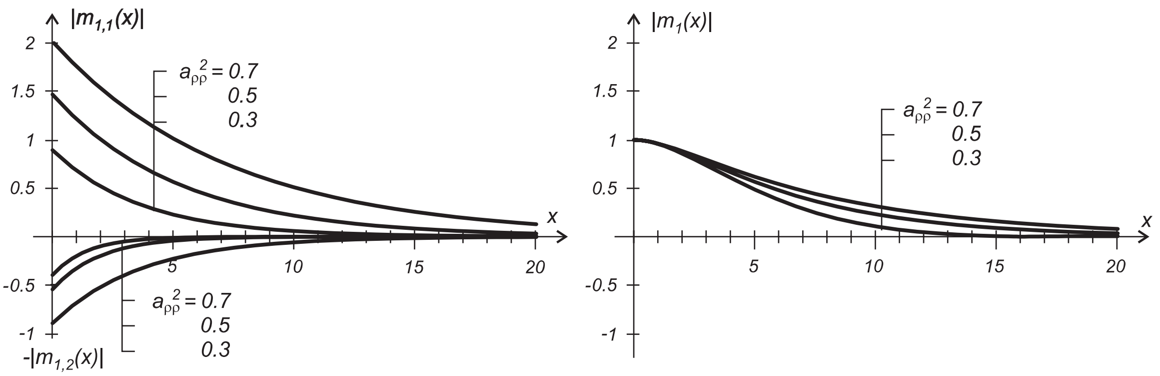

Fig. 6. Mathematical mean value of the response for various levels of imperfections; ($\sigma^2_{\varrho\varrho} = 1.0, \Omega^2 = 1.0$).

The character of this wave consisting of two components is demonstrated in Figs. 5 and 6. The slow decay of the amplitude (absolute value) of the main part and the rapid drop of the boundary effect can be compared in the left-hand parts, while the total result can be seen in the right-hand parts of these figures.

The basic form of the solution, Eq. (34), is based on an assumption that $\sigma_{\varrho\varrho}^2$ is a relatively small value or, in other words, that the condition

$$\alpha^2 > 0 \Rightarrow 0 < \sigma_{\varrho\varrho}^2 < \frac{a_{\varrho\varrho}^2}{\Omega^2} = \frac{a_{\varrho\varrho}^2 E_0}{\omega^2 \varrho_0} \tag{35}$$

is complied with and, consequently, the spectral density Eq. (11b) has no sharp maxima in the point $\vartheta = 0$. The condition Eq. (35) requires that the variance $\sigma_{\varrho\varrho}^2$ of the process $\xi_\varrho(x)$ should be smaller than a certain characteristic quantity $a_{\varrho\varrho}^2 \Omega^{-2}$, which corresponds with a mean correlation length of this process.

It can be expected that the condition Eq. (35) is satisfied for usual materials like metals, composites, etc., if the excitation frequency is not too high. The condition Eq. (35) represents a straight line separating together with horizontal axis $a_{\varrho\varrho}^2/\Omega^2$ domains 1 and 2, see Fig. 7. Some problems can arise for very high excitation frequencies Ω, see Eq. (5). The parameter $a_{\varrho\varrho}/\omega$ represents the inverse value of a certain velocity which is related to the mean correlation length of the process $\xi_\varrho(x)$. It is possible to say that the condition Eq. (35) is complied with, if the length of the propagating wave exceeds the length of this correlation. This limitation reminds of the critical frequencies in discrete or discretized media, e.g. by means of the FEM, except the fact that in our case we do not exceed the natural frequency of the subsystem (or a single element), but do cross the limit of determinism of the material parameter. Below this boundary the problem no longer represents any realistic problem of the stochastic mechanics.

The above results being valid for a continuous model can be compared to a certain extent with those obtained for various types of chains or discretized models of infinite or finite length, e.g. Refs. [34–39], where the basic model of the continuum is either random or periodically deterministic. Although the philosophy of models used in these studies varies from case to case, it can be concluded that the basic effects are nearly identical, namely the decay of determinacy and increase of indeterminacy with the distance from the excitation point. Numerical simulations [40,41] confirm this although being obtained for 3D domains.

Let us return to the cases in which the condition Eq. (35) is not complied with, see Fig. 7, domains 3–6. They can be divided into three areas. The first one is defined predominantly by a condition of positive expression under the internal square root in Eqs. (30c,d):

$$\sigma_{\varrho\varrho}^2 > \frac{a_{\varrho\varrho}^2}{\Omega^2}; \sigma_{\varrho\varrho}^2 < \left(1 + \frac{1}{4}\frac{a_{\varrho\varrho}^2}{\Omega^2}\right)^2 = \left(1 + \frac{1}{4}\frac{a_{\varrho\varrho}^2}{\omega^2}\frac{E_0}{\varrho_0}\right)^2, \tag{36a}$$

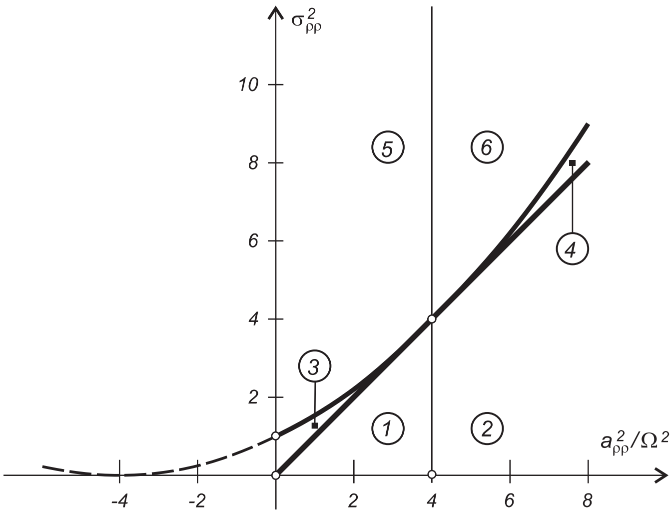

Fig. 7. Domains of the response types 1-6 according parameters $a_{\varrho\varrho}^2/\Omega^2$ and $\sigma_{\varrho\varrho}^2$.

$$a_{\varrho\varrho}^2 < 4\Omega^2 = \frac{4}{\omega^2}\frac{E_0}{\varrho_0}. \tag{36b}$$

The condition Eq. (36b) only puts an upper limit for $a_{\varrho\varrho}^2/\Omega^2$ and separates domains 3 and 4.

In the domain 3, the response is based on the roots

$$\lambda_2 = -\tfrac{1}{2}a_{\varrho\varrho} - i\zeta_1', \lambda_4 = -\tfrac{1}{2}a_{\varrho\varrho} - i\zeta_2',$$

$$\zeta_{1,2}' = \left(\Omega^2 - \frac{1}{4}a_{\varrho\varrho}^2 \pm (\Omega^4\sigma_{\varrho\varrho}^2 - \Omega^2 a_{\varrho\varrho}^2)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \tag{37}$$

where λ_1, λ_3 have been excluded for the same reasons like in Eqs. (30a,b) when domains 1 and 2 have been discussed.

Let us compare the apparent frequency of the wave and its relative damping as a function of the parameter $\Sigma = \sigma_{\varrho\varrho}^2\Omega^2/a_{\varrho\varrho}^2$ characterizing the material density fluctuation variance, correlation length and excitation frequency Ω. Non-dimensional frequency $\beta/a_{\varrho\varrho}$ is identical for both the main part and boundary effect, see Eqs. (30a,b,d). It is mildly decreasing with rising Σ in the domain 1 until the limit given by Eq. (35) is reached. In this point a bifurcation into two branches ζ_1', ζ_2' occurs. They continue throughout domain 3 and finish in the point representing a boundary with domain 5, see the right-hand part of Fig. 8. Concerning the apparent non-dimensional attenuation, in domain 1 two branches representing the main part and boundary effect are visible in the left hand part of Fig. 8. They start from points 0 (main part) or 1 (boundary effect) and continue increasing or decreasing, respectively, as far as the common point 1/2 on the boundary between domains 1 and 3. Within domain 3, both branches remain identical and constant.

Therefore, the response in domain 3 is described once again by the sum of two exponentials, however, is of a different character. Theoretically, a certain beat effect originates. It could manifest itself macroscopically by the presence of an apparent long wave.

Domain 4 differs from domain 3 only in the third condition:

$$\sigma_{\varrho\varrho}^2 > \frac{a_{\varrho\varrho}^2}{\Omega^2}; \sigma_{\varrho\varrho}^2 < \left(1 + \frac{1}{4}\frac{a_{\varrho\varrho}^2}{\Omega^2}\right)^2 = \left(1 + \frac{1}{4}\frac{a_{\varrho\varrho}^2}{\omega^2}\frac{E_0}{\varrho_0}\right)^2, \tag{38a}$$

$$a_{\varrho\varrho}^2 > 4\Omega^2 = \frac{4}{\omega^2}\frac{E_0}{\varrho_0}. \tag{38b}$$

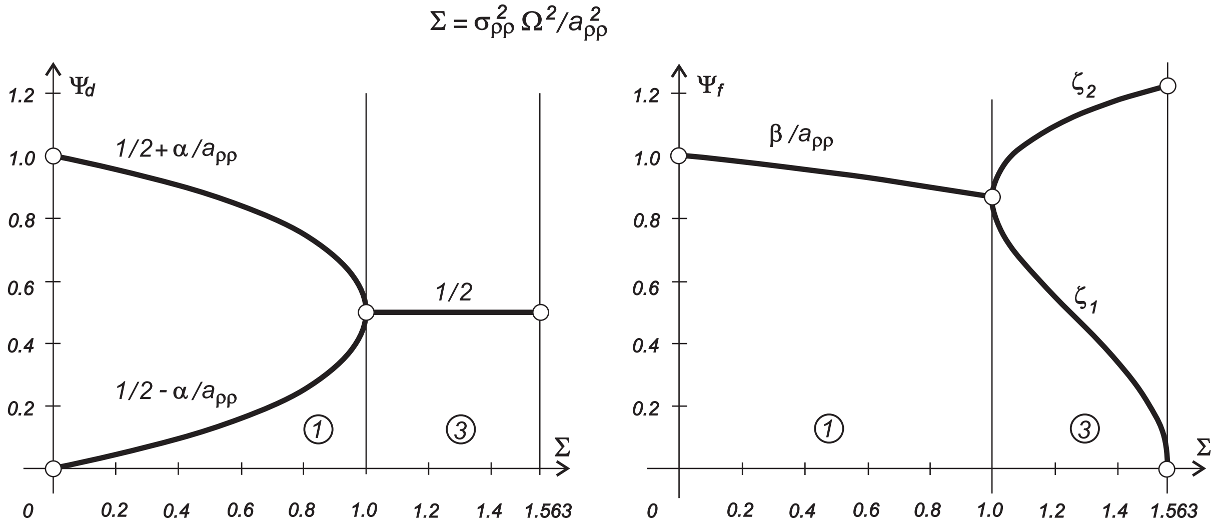

Fig. 8. Apparent damping (left part) and frequency (right part) of the periodic part of the response across domain types 1,3; horizontal axis is scaled by parameter $\Sigma = \sigma_{\varrho\varrho}^2\Omega^2/a_{\varrho\varrho}^2$; vertical axis in the left part $\Psi_d = \mathrm{Re}(\lambda_{2,4})/a_{\varrho\varrho}$; right part $\Psi_f = \beta/a_{\varrho\varrho}$ or $\Psi_f = \zeta_{1,2}$.

This parameter configuration, however, leads to real roots only:

$$\lambda_{1,2} = -\frac{1}{2}\,a_{\varrho\varrho} \pm \zeta_1'', \lambda_{3,4} = -\frac{1}{2}\,a_{\varrho\varrho} \pm \zeta_2'',$$

$$\zeta_{1,2}'' = \left(\frac{1}{4}\,a_{\varrho\varrho}^2 - \Omega^2 \pm (\Omega^4\sigma_{\varrho\varrho}^2 - \Omega^2 a_{\varrho\varrho}^2)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \tag{39}$$

which results in a synthesis of four exponentials with real negative exponents. It would follow that the response in the variable x is not of wave character at all and that, consequently, the excitation (impulse) propagates at an infinitely high velocity. Such material properties would mean too high probability of the state in which the imperfections will overcome completely influence of the parameter nominal value ϱ_0. Even if we admit such a state (due to Gaussian character of imperfections), it must not occur "in a too great extent", which is described by the condition Eq. (35). Otherwise, this case involves big imperfections to which a different mathematical model would have to be applied. It is a question, of course, how realistic is this case and how far it reflects the real nature of the problem. Certainly, it should be eliminated when the assumption of a limited normal distribution of imperfections has been adopted.

Similar approach can be applied also to the third area, domains 5 and 6, lying above the parabola Eq. (38a):

$$\sigma_{\varrho\varrho}^2 > \left(1 + \frac{1}{4}\frac{a_{\varrho\varrho}^2}{\Omega^2}\right)^2 = \left(1 + \frac{1}{4}\frac{a_{\varrho\varrho}^2}{\omega^2}\frac{E_0}{\varrho_0}\right)^2. \tag{40}$$

In this case, characteristic equation (28) has the solutions:

$$\lambda_{1,2} = -\tfrac{1}{2}a_{\varrho\varrho} \pm \zeta_1''', \quad \lambda_{3,4} = -\tfrac{1}{2}a_{\varrho\varrho} \pm \imath\zeta_2'''. \tag{41}$$

It would follow from Eq. (41) that the result would contain, once again, non-periodic components of problematic physical interpretation like in the preceding case.

Consequently, we can state that only the cases satisfying the condition Eq. (35) and being given by the solution Eq. (34) are physically meaningful. Other cases violate basic assumptions of small scale fluctuation of material density, of non-zero correlation in the length coordinate and, in general, of random nature of input/output processes. One can simply conclude that cases not satisfying condition Eq. (35) cannot be investigated using the mathematical model Eq. (14). Transition cases being on limits separating individual domains in Fig. 7 have not been dealt. They are characterized by multiple roots of Eq. (28) and should be subjected to a special analysis. However, it would need a wearisome work and cannot provide any information important from physical point of view.

Results obtained in this part are similar to those obtained using the spectral decomposition method, see Refs. [17,20]. Another special case where stochastic parts of coefficients coincide have been outlined in Ref. [16] for similar diffusion spectral density of coefficients. Results presented in Ref. [16] are very near to those obtained in the last two paragraphs of this study.

6. Fluctuations in both parameters

Let us admit imperfections both in the Young modulus of elasticity and in the density of the material. The full form of the system Eq. (22) and of the characteristic equation (24) must be used. As long as the investigation is oriented into the region of small imperfections (see Eq. (2) and a detailed condition Eq. (35) concerning density fluctuation only), it can be shown using conventional theorems of polynomial algebra, that Eq. (24) has only three pairs of complex conjugate roots. Therefore, the basic formulation of the problem does not admit any real as well as multiple roots, unless a very non-probable case of material parameters match would occur. These cases would lead to various anomalies described in the preceding two chapters and we shall not deal with them.

The first part of Eq. (24) is dominant, while the influence of the second part is determined by small parameters $\sigma_{EE}^2, \sigma_{\varrho\varrho}^2$ and that of the third part by the product $\sigma_{EE}^2\sigma_{\varrho\varrho}^2$. Consequently, on the level of a linear approximation and with regard to Eq. (2) the third part can be neglected. Therefore the position of the roots can be determined approximatively in the form:

$$\lambda_i = \lambda_{0i} + p\sigma_{EE}^2 + q\sigma_{\varrho\varrho}^2, \tag{42}$$

$$\lambda_{01,2} = \pm i\Omega, \quad \lambda_{03,4} = -a_{EE} \pm i\Omega, \quad \lambda_{05,6} = -a_{\varrho\varrho} \pm i\Omega, \tag{43}$$

where λ_{0i} are the roots of the first part of Eq. (24), representing a certain "zero" approximation:

$$(\lambda^2 + \Omega^2)((\lambda + a_{EE})^2 + \Omega^2)((\lambda + a_{\varrho\varrho})^2 + \Omega^2) = 0. \tag{44}$$

If the fluctuations of input parameters disappear, only the first binomial in Eq. (44) will remain meaningful describing the very basic case of a perfect material, while the other two binomials lose their physical significance.

Let us substitute the approximate expressions Eq. (42) in Eq. (24) and retain only terms up to the first degree of $\sigma_{EE}^2, \sigma_{\varrho\varrho}^2$. Comparing corresponding coefficients, a linear system for unknown parameters p, q can be composed and evaluated. There is to eliminate out three of the six roots Eqs. (43) those laying in the upper half of the Gaussian plane, as they result in solutions not complying with the Sommerfeld condition for $x>0$. The approximate values of $\lambda_2, \lambda_4, \lambda_6$, consequently, are:

$$\lambda_2 = -i\Omega + \sigma_{EE}^2 \frac{-2\Omega^4 + ia_{EE}\Omega^3}{2a_{EE}(a_{EE}^2 + 4\Omega^2)} + \sigma_{\varrho\varrho}^2 \frac{-2\Omega^4 + ia_{\varrho\varrho}\Omega^3}{2a_{\varrho\varrho}(a_{\varrho\varrho}^2 + 4\Omega^2)}, \tag{45a}$$

$$\lambda_4 = -a_{EE} - i\Omega + \sigma_{EE}^2 \frac{2\Omega^4 + ia_{EE}\Omega^3}{2a_{EE}(a_{EE}^2 + 4\Omega^2)}, \tag{45b}$$

$$\lambda_6 = -a_{\varrho\varrho} - i\Omega + \sigma_{\varrho\varrho}^2 \frac{2\Omega^4 + ia_{\varrho\varrho}\Omega^3}{2a_{\varrho\varrho}(a_{\varrho\varrho}^2 + 4\Omega^2)}. \tag{45c}$$

The values given by Eqs. (45) can be taken as approximate eigenvalues of the coefficient matrix of the system Eqs. (22). Evaluating respective eigenvectors, one arrives in this way at an approximate fundamental system and can write the general solution. The three integration constants will result from an application of the initial conditions in $x = 0$. Taking into account symbolics used in the previous paragraph, see Eq. (32), one can formulate the following initial conditions:

$$m_1(0) = K; \quad m_{13}(0) = 0; \quad m_{14}(0) = 0. \tag{46}$$

The approximate roots Eqs. (45) can be used up to about 30% of the critical value of σ_{EE}^2 or $\sigma_{\varrho\varrho}^2$. The dependence of the roots on $\sigma_{EE}^2, \sigma_{\varrho\varrho}^2$ does not differ much from the linear in the rather broad vicinity of zero. The first two terms of Taylor series of the roots Eqs. (30), are identical with approximate roots Eqs. (45), when $\sigma_{EE}^2 = 0, a_{EE} = 0$. The third and higher terms of the expansion yield much lower values in the given domain and, consequently, can be neglected. The approximations Eqs. (45), naturally, do not allow to identify the critical values of σ_{EE}^2 or $\sigma_{\varrho\varrho}^2$ and some further phenomena we have been concerned with in the previous simpler case. However, a qualitative analysis of the character of the solution for small values of σ_{EE}^2 or $\sigma_{\varrho\varrho}^2$ can be made.

Let us study, therefore, the physical meaning and character of the roots of Eq. (24). The first binomial in the dominant part of Eq. (24) results in the solution of $\lambda_{01,2}$ according to Eq. (43). It represents the principal part of the description of the response linking up with classic solutions for the material without fluctuations. Both parts of the correction in Eq. (45) have a real part which means that $|m_1(x)|$ drops due to the influence of both imperfections with the growing x asymptotically to zero. This decrease is the faster, the more significant are the fluctuations. The only source of this pseudo-damping, however, is the correction in Eq. (45), and not the basic value of the root λ_{02}, which is pure imaginary. Due to the imperfection of density $\xi_\varrho(x)$ the spatial frequency of the response decreases, while the fluctuation $\xi_E(x)$ tends to increase this frequency.

The meaning of the remaining two binomials in the dominant part of Eq. (24) depends on the particular values $a_{EE}, a_{\varrho\varrho}$, see Eq. (44). Their influence is the higher, the smaller are the $a_{EE}, a_{\varrho\varrho}$ or the higher are the peaks of the spectral densities Eqs. (11), or, in other words, the more the imperfections are concentrated at small frequencies. However, if a_{EE} and $a_{\varrho\varrho}$ drop, also the significance of the basic root λ_{02} decreases, as the real part of the correction in Eqs. (45) decreases. In any case, however, the roots $\lambda_{04}, \lambda_{06}$ have significant negative real parts and, consequently, the corresponding parts of the response are of the character of boundary effects in the neighborhood of the point $x = 0$. With the growing portion of fluctuations their influence decreases with increasing x more slowly. However, the significance of the roots Eqs. (45b,c) is quantitatively comparable with the root Eq. (45a) only in the domain around the origin.

It is coming to light that on the level of linear approximation Eqs. (45b,c) each of the roots λ_4, λ_6 is always influenced by the fluctuations of one of the parameters only. It is possible to say that λ_4 appertains to the fluctuations of $\xi_E(x)$, and λ_6 to the fluctuations of $\xi_\varrho(x)$. The sum of corrections of the imaginary part of λ_4, λ_6 equals the corrections of the imaginary part of λ_2. Consequently, the principal part of response being described by λ_2 has a periodical part, consisting of two "mutually modulating" harmonic waves, each of which has its counterpart in the harmonic wave coming from λ_4 and λ_6. Consequently, the mathematical mean value of the response has the character of a mixture of two evanescent waves with monotonously decreasing absolute value and lower derivative of their absolute value in the origin than at a certain distance from this point. In other words, the decrement of $m_1(x)$ in the proximity of the origin is small and will manifest itself only later for x outside a neighborhood of the origin. This means that in the proximity of the origin it is possible to admit, on some accuracy level, the equivalence of the classic solution and the mathematical mean value of the response of the continuum with deterministically or randomly varying parameters, e.g. Refs. [42,43]. In such a case the method of small parameter can be used with care on a very short interval of parameter fluctuation. However, the admissible distance from the origin cannot be very large, because the conventional model cannot be accepted particularly for bodies with infinite dimensions, as it results in physical paradoxes.

7. Conclusions

The solution of harmonic wave motion in the continuum the Young modulus of elasticity and the density of which are random variables of the longitudinal coordinate gives rise to a number of special effects. If a harmonic, fully deterministic kinematic excitation is applied to one end of a prismatic semi-infinite bar with the above-mentioned properties, the very longitudinal force at the origin effecting this motion is of stochastic character, if burdened by imperfections of the Young modulus of elasticity. In the bar it is possible to observe a drop of the response mean value with the simultaneous increase of the response variance. Typical is the relatively small drop of the mathematical mean value up to a certain distance from the point of excitation, which is followed by a steep drop and final approach to the horizontal asymptote $|m_1(x)| \to 0$. This is not the damping in the proper meaning of the term, but a gradual drop of the deterministic and an increase of the stochastic components of the overall response. In other words, the rate of indeterminacy of the response increases with the increase of x. With growing x the process of the response approaches a homogeneous process. Its slightly non-Gaussian character can be represented, particularly for major distances from the point of excitation, by a non-centered character of the respective Gaussian curve.

The principal character of the response is the same in case of imperfections of both parameters separately as well as with their simultaneous application. The boundaries between individual response types (rather boundaries of physical applicability of a mathematical model used), in the case of perturbations $E_\varepsilon(x)$ and $\varrho_\varepsilon(x)$, however, are entirely different. For low imperfection levels they do not differ much.

In case of increasing values of dispersion of imperfections $(\sigma_{EE}^2, \sigma_{\varrho\varrho}^2)$, the rising excitation frequency ω and rising sharpness (given by $a_{EE}, a_{\varrho\varrho}$) of the imperfections spectral densities, it is possible to attain a critical boundary when the length of the propagating wave is comparable with the "correlation length" of imperfections. This state will manifest itself by a radical change of the response character. This transitory stage will terminate soon, the problem will pass beyond the boundaries of stochastic mechanics and will lose its physical meaning in this formulation. Similar effects can be observed in the FEM, where there is also a certain permissible upper boundary of the excitation frequency corresponding with the size and type of the element used.

The mathematical model describing the imperfections as white noise in the longitudinal coordinate is problematic, as it yields the results which either cannot respect cross-correlation of material parameter fluctuations or they are at discrepancy with the energy equilibrium law. For similar reasons also the small parameter method is hardly applicable. It is necessary to use as basis at least the diffusion model of imperfections which is characterized by exponential correlation in space. From this viewpoint it is necessary to impose stricter requirements on the internal structure of imperfections of the Young modulus of elasticity than on that of imperfections of material density.

Concerning the methods, a mathematical model following from the theory of Markov processes and respective FPK equations seems to be the most flexible and capable of avoiding several ambiguous steps which are necessary when using integral decomposition procedure. Nevertheless, the main character of results obtained when using any of these methods coincide in principle even if some details are slightly different. Also conclusions obtained when investigating discrete either finite or infinite systems of periodical or random character are comparable with those obtained in this study.

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