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Improved description of longitudinal strain solitary waves

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Abstract

It is found that simultaneous existence of compressive and tensile localized strain solitary waves in a rod can be described by the model equation containing both quadratic and cubic nonlinear terms. Only propagation of these waves is described by exact travelling wave solutions. However, numerical solution demonstrates that both kinds of these waves may be generated from an arbitrary input and interact each other keeping their shape and velocity. Moreover, one and the same input gives rise to a different number of these kinds of waves, and it is quadratic nonlinearity that determines it.

0. Introduction

The dynamic loading of structural elements (e.g., rods) results in generation of strain waves. The deformation energy carried by these waves may be distributed along a rod or localized in a part of it. Moreover, localized or solitary strain waves may move along the rod keeping their shape and velocity, thereby transmitting considerable energy over long distances. The amplitude of waves may be close to a threshold separating the elastic and plastic strains that is important for estimation of the durability of elastic structures. Furthermore, the parameters of the solitary waves depend upon the elastic features of the material of the rod. Knowledge of these relationships allows to use them for measurement of the high-order elastic moduli of the material by nonlinear acoustics methods.

There are two main types of localized strain waves. First is the envelope solitary wave arising due to a modulation of a harmonic wave. These waves are widely used in nonlinear acoustics, see, e.g., Refs. [1–3]. Another type is a long bell-shaped solitary wave provided by a balance between nonlinearity and dispersion [2–7]. Recently these waves of compression were generated in a rod [4,8,9] and in a plate [4,10]. Nonlinearity in a pure elastic rod is caused by the finite stress values and the elastic material properties while dispersion results from the finite transverse size of the rod. Certainly, the leading nonlinearity for longitudinal waves is quadratic, and usually governing equations were obtained that contain only this nonlinearity. Among the model equations one can list the Korteweg-de Vries equation, the Boussinesq equation [2,3,5], and the double

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dispersion equation (DDE) [4,6,7],

$$v_{tt} - a\, v_{xx} - c_1 (v^2)_{xx} + \alpha_3\, v_{xxtt} - \alpha_4\, v_{xxxx} = 0, \tag{1}$$

where $v(x, t)$ is the longitudinal strain, x is directed along the rod, and the coefficients in Eq. (1) depend upon the radius of the rod and its elastic features.

The sign of the nonlinearity coefficient c_1 defines a possible kind of the localized strain wave (compressive or tensile) according to the exact solitary wave solution of Eq. (1). This prediction is confirmed in experiments. However, some features of the observed waves are not described correctly by the solution of DDE. In particular, the width of the observed wave turns out to be smaller than that of the exact solution. Also varius data for the moduli for plexiglas give rise to the opposite predictions regarding the sign of c_1, thus, to the existence of tensile strain solitary wave instead of the observed compressive one. That is why an improved model for longitudinal strain waves in a rod and in a plate containing both quadratic and cubic nonlinearities was developed in our works [11,12]. An exact solution of the improved equation allowed to explain a narrowing of the solitary wave and achieve a better agreement with experiments on strain solitary wave generation.

In this paper, attention is paid to the possible simultaneous propagation of compressive and tensile solitary waves. This possibility originates from the existence of two exact solitary wave solutions of our improved equation. A numerical study is performed to investigate the simultaneous generation of the solitary waves from an arbitrary input. Then, an interaction between the waves of the same and of the different kinds is considered. Finally, a possible application of the theory to seismic problems is considered.

1. Improved model equation and its solitary wave solutions

Consider an isotropic cylindrical elastic rod with free lateral surface and introduce cylindrical Lagrangian coordinates (x, r, φ) where x is directed along the axis of the rod, $-\infty < x < \infty$, r is the radial coordinate, $0 \leqslant r \leqslant R$, φ is a polar angle, $\varphi \varepsilon [0, 2\pi]$. Assuming that torsion can be neglected the displacement vector can be characterized by two components $\vec{V} = (u, w, 0)$. The so-called nine constants Murnaghan model [13] is used for the density of potential energy,

$$\begin{aligned} \Pi = {} & \frac{\lambda + 2\mu}{2} I_1^2 - 2\mu I_2 + \frac{l + 2m}{3} I_1^3 - 2m I_1 I_2 + n I_3 \\ & + v_1 I_1^4 + v_2 I_1^2 I_2 + v_3 I_1 I_3 + v_4 I_2^2, \end{aligned} \tag{2}$$

where $I_k, k = 1, 2, 3$ are the invariants of the Cauchy–Green deformation tensor $\mathbf{C}$. Both the third-order elastic moduli, or the Murnaghan moduli (l, m, n), and the fourth-order moduli (v_1, v_2, v_3, v_4) can be either positive or negative [14–16]. The governing equation is obtained in the reference configuration using Hamilton's principle,

$$\delta S = \delta \int_{t_0}^{t_1} \mathrm{d}t\, 2\pi \int_{-\infty}^{\infty} \mathrm{d}x \int_0^R r \mathscr{L}\, \mathrm{d}r = 0, \tag{3}$$

where the Lagrangian density per unit volume, $\mathscr{L}$, is defined as the difference of the densities of the kinetic and potential energies, $\mathscr{L} = K - \Pi$.

The absence of the normal and tangential stresses at the lateral surface of the rod helps to define the displacement field as power series in the radius for u and w [11]. Substitution of the series into the boundary conditions allows to define the coefficients in the series and express them through the new unknown function $U(x, t)$ and its derivatives. The series should be truncated due to the validity of the Murnaghan model, Eq. (2), and finally the following expressions can be obtained:

$$u = U(x, t) + a_2\, r^2\, U_{xx}, \tag{4}$$

$$w = b_1\, r\, U_x - b_3 r^3\, U_{xxx} - B_1\, r\, U_x^2 - B_2\, r\, U_x^3, \tag{5}$$

with coefficients b_i, B_i defined in Ref. [11]. The governing equation is obtained from Eq. (3) using Eqs. (2), (4) and (5) for the strain function $v = U_x$,

$$v_{tt} - a\,v_{xx} - c_1\,(v^2)_{xx} - c_2\,(v^3)_{xx} + \alpha_3\,v_{xxtt} - \alpha_4\,v_{xxxx} = 0, \tag{6}$$

where

$$a = \frac{E}{\rho_0}, \quad c_1 = \frac{\beta}{2\rho_0}, \quad c_2 = \frac{\gamma}{3\rho_0}, \quad \alpha_3 = \frac{\nu(1-\nu)R^2}{2}, \quad \alpha_4 = \frac{\nu E R^2}{2\rho_0},$$

and β and γ are the coefficients of quadratic and cubic nonlinearity respectively,

$$\beta = 3E + 2l(1-2\nu)^3 + 4m(1-2\nu)(1+\nu)^2 + 6n\nu^2,$$

$$\begin{aligned}\gamma = \frac{1}{E}[&E^2 - 8l^2(1-2\nu)^5(1+\nu) - 32m^2\nu^2(1-2\nu)(1+\nu)^3\\ &- 8n^2\nu^2(1-2\nu)(1+\nu) + 4l(1-2\nu)^3\{E - 4\nu(1+\nu)(2m(1+\nu)-n)\}\\ &+ 8m(E+4n\nu^2)(1-2\nu)(1+\nu)^2 + 12nE\nu^2 + 8\nu_1 E(1-2\nu)^4\\ &- 8\nu_2 E(1-2\nu)^2(2-\nu)\nu + 8\nu_3 E(1-2\nu)\nu^2 + 8\nu_4 E(2-\nu)^2\nu^2].\end{aligned}$$

The previous model equation, Eq. (1), is obtained assuming $c_2 = 0$ in Eq. (6).

The governing equation, Eq. (6), is non-integrable, and only particular exact solutions may be found. Thus, a travelling wave solution is obtained that depends only upon the phase variable $\xi = x - Vt$. In this case Eq. (6) becomes an ordinary differential equation (ODE) whose order may be reduced by direct integration. Some obvious manipulations give rise to a first-order ODE,

$$v_\xi^2 = \frac{1}{6(\alpha_3 V^2 - \alpha_4)}(6(V^2 - a)v^2 - 4c_1 v^3 - 3c_2 v^4).$$

This equation is integrated giving an implicit dependence of v on ξ. Inversion of this dependence provides two bell-shaped localized (or solitary) wave solutions. The first of them is

$$v = \frac{A}{Q\cosh(k\xi) + 1}, \tag{7}$$

where

$$A = \frac{3(V^2 - a)}{c_1}, \quad Q = \sqrt{1 + \frac{9c_2}{2c_1^2}(V^2 - a)}, \quad k^2 = \frac{V^2 - a}{\alpha_4 - \alpha_3 V^2}. \tag{8}$$

The second bounded solution exists for positive values of c_2 and is given as

$$\eta = -\frac{A}{Q\cosh(k\xi) - 1}. \tag{9}$$

The existence of both solutions requires the parameters of the solutions, Eq. (8), to be real, see about this in Refs. [11,12]. The most important fact is that they always have opposite signs of the amplitudes, and, therefore, both compressive and tensile solitary waves may propagate in the rod made of a material with positive c_2. This prediction is not realized in the framework of the DDE, Eq. (1). Indeed, when $c_2 \to 0$, the first solution (7) transforms into the known solitary wave solution of the DDE,

$$v = A_D \cosh^{-2}(\tfrac{1}{2}k\,\xi), \tag{10}$$

with

$$A_D = \frac{3(V^2 - a)}{2c_1}.$$

On the contrary, another exact solitary wave solution, Eq. (9), tends to the singular exact solution of Eq. (1)

$$v = -A_D \sinh^{-2}(\tfrac{1}{2} k\, \xi). \tag{11}$$

Existence of both kinds of solitary waves independent of the sign of c_1 may help to describe experiments in a plexiglas rod [9]. However, the exact solutions Eqs. (7) and (9) require specific initial conditions while waves of more general form occur in practice. Therefore it is important to know whether Eqs. (7) and (9) are realized in the evolution of an arbitrary initial pulse especially if this pulse contains both compressive and tensile parts.

2. Generation and interaction of solitary waves

Consider only the case $c_2 > 0$ when both localized wave solutions are bounded. According to the exact solutions one can anticipate generation of compressive waves Eq. (7) and tensile waves Eq. (9) for $c_1 < 0$ from negative and positive input correspondingly. Generation of the tensile waves Eq. (7), and compressive waves Eq. (9) occurs for $c_1 > 0$ [11,12]. In the following both generation and an interaction of these waves is studied. To enable such a study, the initial condition, bottom sketch in Fig. 1, is chosen to consist of two equal parts in the form of the Gaussian distribution. However, the initial velocity for the left part located near $x = 500$ is chosen to be zero, while the second part near $x = 750$ is subjected to the initial velocity directed to the left. As a result, the first part is split into two trains of compressive solitary waves which propagate in opposite directions keeping their shape and velocity, while the second one produces a train of four unidirectional solitary waves, see last but one sketch in Fig. 1. Propagation of each wave is described by the exact solution Eq. (7). One can see that the same number of waves is produced by each part of the input, however, the highest amplitude is achieved for the leading wave in the unidirectional train. After generation, the four-waves train interacts with the pair of the waves generated from the left part of the input. It is clearly seen that the waves keep their shapes and velocity after interaction. A similar scenario is seen in Fig. 2 where the input is mirrored

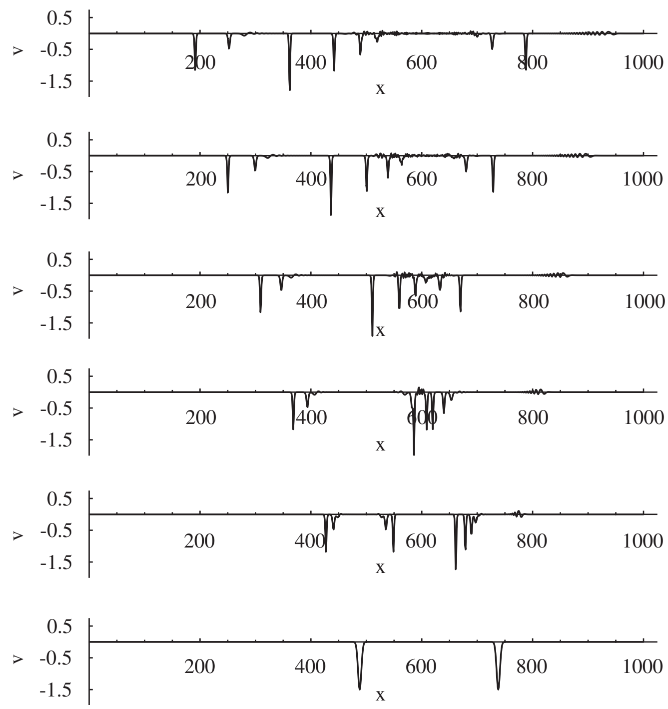

Fig. 1. Formation of strain solitary compression waves (7) from a negative amplitude Gaussian input. Here and in the following time growths from bottom to top, x is measured in arbitrary units of length. Interaction of the wave happens from 3d to 5th stages. The last, 6th stage, demonstrates keeping of the shapes of the waves after interaction, cf. with the waves at the 2nd stage.

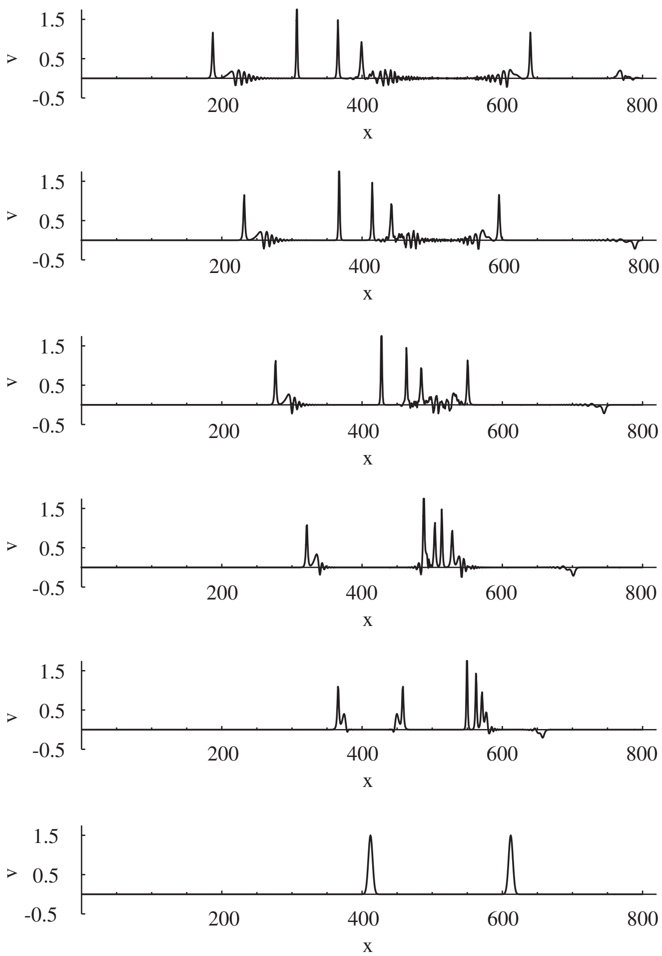

Fig. 2. Formation of strain solitary tensile waves (9) from a positive amplitude Gaussian input. Again the waves generated by the left part of the input are located symmetrically it at the last stage despite interaction with the waves generated by the right part of the input.

relative to that used in Fig. 1. However, now the right part of the input generates only two solitary waves Eq. (9) moving in opposite directions while three solitary waves are generated from the right part of the input. The latter reflects the sensitivity of waves to the value of the initial velocity first found in Ref. [12]. The absolute value of the amplitude of the leading solitary wave in the unidirectional train in Fig. 1 is higher than that of the similar wave in Fig. 2. Like the waves Eq.(7) the waves Eq. (9) interact with each other keeping their shapes and velocity.

A combined input with equal "mass" of positive and negative parts and with zero initial velocity gives rise to simultaneous formation of compressive and tensile solitary waves. One can see in Fig. 3 that two tensile waves Eq. (9) and four compressive waves Eq. (7) do not change their shapes due to the interaction between the waves of different kinds. Here, it is clearly seen that the waves Eq. (7) are faster than the waves Eq. (9). We are able to recognize solitary waves in Fig. 3 thanks to the results shown in Figs. 1 and 2. Even more complicated wave fields may be described separating the solitary waves of each kind.

It is found that the solitary waves Eq. (9) are sensitive to the value of the cubic nonlinearity coefficient in contrast to the waves Eq. (7). The value of c_2 affects the amplitude and the velocity of the wave Eq. (7) but not the number of solitary waves arising from the input [11]. Thus, we found that the larger the positive value of c_2 the higher and faster the arising solitary waves. On the contrary, decreasing this value in comparison with that used in the previous figures, one can achieve no formation of any tensile solitary waves Eq. (9), see Fig. 4,

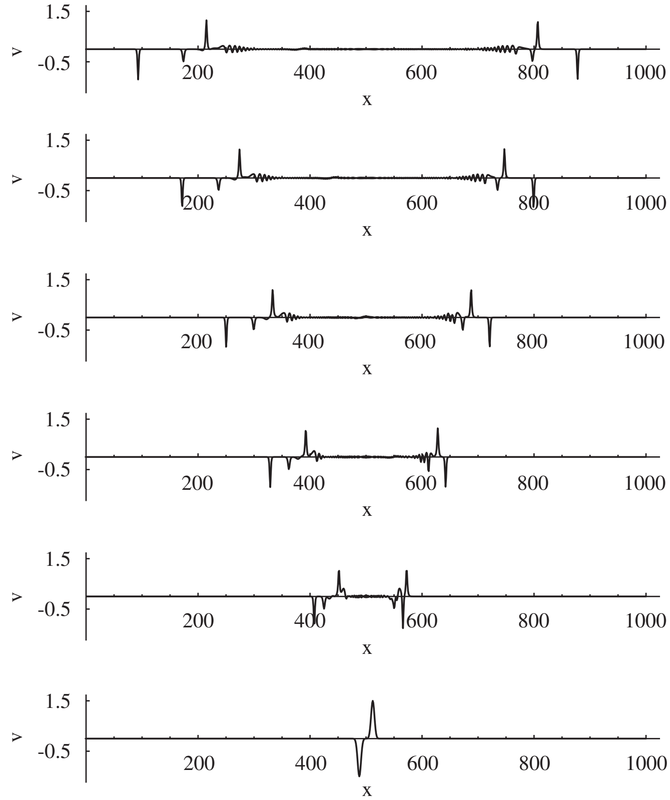

Fig. 3. Simultaneous formation of compression and tensile strain solitary waves and their interaction. Note different number of the waves generated by tensile and compression parts of the input.

while compressive waves are generated as before. Besides the number of the waves Eq. (9), the decrease/increase in c_2 yields the decrease/increase in the waves amplitude.

At the same time, variation of the "mass" of the input affects the number and the amplitudes of both kinds of solitary waves. In particular, an increase in the width of both the positive and negative parts of the input used in Fig. 3 gives rise to an increase in the amplitude and the number of the generated solitary waves.

3. Discussion

The improved description of the strain waves in a rod allows us to explain localization of strain waves due to the simultaneous contribution of quadratic and cubic nonlinearities. The new features of the solitary waves reflect better recent observations. The particular exact solutions describe a more complicated wave field obtained numerically since they reflect balances between nonlinearity and dispersion. Also, the sensitivity of the wave localization to the variation of the values of nonlinear elastic parameters is important for engineering applications.

Isotropic materials were considered here that satisfy the Murnaghan model. However, the results obtained may be used for the study of nonlinear acoustic measurements of the parameters of rocks. Indeed, the one-dimensional model used to model these media [17,18] reads as follows in our notation:

$$P = E(U_x + \Gamma^{(2)} U_x^2 + \Gamma^{(3)} U_x^3), \tag{12}$$

where P is a stress. It corresponds to our model Eq. (3) with Eqs. (4) and (5) being taken into account, and we obtain that $\Gamma^{(2)} = \beta/4$, $\Gamma^{(3)} = \gamma/4$. Of course there is no dispersion in the medium (not in the waveguide) studied in Refs. [17,18]. An important result was found in Refs. [17,18] that nonlinear elastic parameters $\Gamma^{(2)}$,

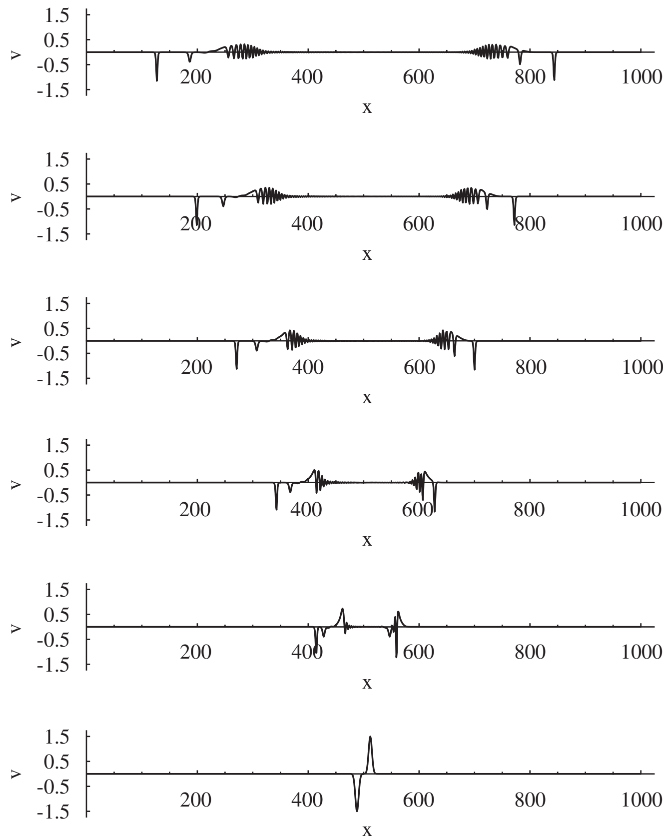

Fig. 4. Formation of strain solitary compression waves (7) and delocalization of tensile waves (9) due to the alteration of the value of c_2.

$\Gamma^{(3)}$ change in a wider interval than the velocity of sound during the static loading of a rock or when alterations in its structure occur. This points to a more important role of nonlinearity (both quadratic and cubic!) in acoustic testing than supposed before. Note that the values of cubic nonlinearity coefficient found in Refs. [17,18] for some media are positive.

Also a phenomenon of abnormal nonlinearity for some complex media was studied consisting of several interacting components, e.g., fluidized granular media [17,18]. Indeed, the cubic nonlinearity is usually weaker than the quadratic one for classic elastic materials. Using the data for aluminium [16]: $l = -2.9\times10^{11}\,\mathrm{N/m^2}$, $m = -3.1\times10^{11}\,\mathrm{N/m^2}$, $n = -2.3\times10^{11}\,\mathrm{N/m^2}$, $v_1 = -1.4\times10^{12}\,\mathrm{N/m^2}$, $v_2 = -5.3\times10^{12}\,\mathrm{N/m^2}, v_3 = 1.7\times10^{12}\,\mathrm{N/m^2}, v_4 = -2.9\times10^{12}\,\mathrm{N/m^2}$, we first obtain $\beta<0, \gamma>0$. Therefore, both kinds of solitary waves may exist in an aluminium rod. However, the ratio between the third and the second terms in brackets in Eq. (12) is about 10^{-2} for typical strains of order 10^{-3}. At the same time, the data for a medium with tuff grains [17], $\Gamma^{(2)} = 130$, $\Gamma^{(3)} = 4\times10^4$, gives rise to almost equal contributions of nonlinearities for the observed strains of the order 10^{-4}. Similar estimations follow using the data for a loam soil where $\Gamma^{(2)} \cong 10^3$, $\Gamma^{(3)} \cong 10^7$ [17]. Therefore, our model might be employed for acoustic study of the rocks and seismic soils. Besides, a significant role of nonlinearity, the sensitivity of the solution to the variation of $\Gamma^{(3)}$ looks promising.

Finally, one can make an important observation regarding application of the Gardner equation whose structure is similar to our model equation Eq. (6). The Gardner equation is used to account for internal waves in a two-layer fluid, but quadratic and cubic nonlinearities are found to be of the same order only when the widths of the layers and the density ratio lie in a confined interval. However, the features of the solitary wave solutions of the Gardner equation are efficiently used to describe observed waves in the ocean even when the widths of the layers considerably differ from each other [19], i.e., outside the range of their formal mathematical applicability.

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