On the Ostrogradski instability for higher-order derivative theories and a pseudo-mechanical energy

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Abstract

A two-degree of freedom spring-mass system, described by two coupled second-order ODEs, is considered from a higher-order one-particle (HOOP) viewpoint. A Lagrangian leading to the single fourth-order differential equation now contains time derivatives greater than the first order. The Hamiltonian constructed according to Ostrogradski's method, traditionally regarded as the conserved energy of the system, is negative for one of the modes of vibration, an attribute that would lead to a conclusion in some branches of physics that the mode is unphysical. Reversing this process, a given fourth-order equation, with no apparent underlying second-order structure, is cast into a coupled second-order form allowing one to construct a pseudo-mechanical energy which, unlike the Ostrogradski Hamiltonian, is always positive. The introduction of a viscous damping element leads to velocity and jerk (third derivative)-dependent terms within the HOOP description. In contrast, a physical realisation leading to an isolated velocity-dependent term in the HOOP description shows the instability to be flutter caused by external excitation at the higher natural frequency, rather than an exchange of energy between positive and negative energy modes.

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1. Introduction

Classical mechanics [1] is based upon Newton's laws or, equivalently, the application of Hamilton's principle to a Lagrangian possessing time derivatives no higher than the first order; both approaches lead to second-order differential equations of motion. In turn, the Hamiltonian H, an integral of the motion for conservative systems, is typically equal to the mechanical energy E_m and is positive. In contrast, modern physics has considered the possibility of theories with higher-order time derivatives; examples include corrections to general relativity, string theory, and quantum mechanics [2,3]. The Lagrangian now possesses time derivatives higher than the first order, and the Hamiltonian H is constructed according to a procedure introduced first by Ostrogradski (see Whittaker [4]); being a property of time invariance, H is still regarded as the energy of the system. However, the energy of one of the modes of the higher-derivative theory is very often negative, and there is a widespread view within the literature [5,6] that such modes should be regarded as non-physical or "ghost". This so-called Ostrogradski instability suggests that the amplitude of both positive and

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negative energy modes can simultaneously increase without bound, yet the total energy of the system remains constant. Woodard [5] has suggested that this instability should provide a "no-go" test on the physical validity of a candidate Lagrangian for any particular application, and that "Newton was right to expect that physical laws take the form of second-order differential equations when expressed in terms of fundamental dynamical variables."

The present work was motivated by two papers by Chervyakov and Nesterenko [7] and Nesterenko [8] in which the formulation of Timoshenko beam theory (TBT), with its positive definite mechanical energy, was proposed as a surrogate to avoid this instability. TBT describes the transverse vibration of elastic beams when corrections for shear deformation and rotatory inertia are made to the classic Euler–Bernoulli theory. Newton's laws lead to two coupled second-order time-derivative equations in the transverse displacement and the cross-sectional rotation; one of these variables, usually the rotation, is then eliminated to give a single partial differential equation of the fourth order in both space and time for the transverse displacement. Unfortunately, the second spectrum (TBT2) of natural frequency predictions are largely inaccurate when compared with exact elastodynamic theory [9,10], suggesting that TBT2 is not a fair representation of the actual physics, so the virtue of TBT as a surrogate is somewhat diminished.

At first sight, this instability would appear to be of little importance within classical mechanics, where the fundamental governing equations are of second order. Moreover, the TBT model in Ref. [10] and the two-degree of freedom model studied here, have features which render negative energy physically acceptable. First, the absence of damping implies no interaction with the surroundings, and the negative energy mode would thus be unobservable. Second, factorisation of the frequency equation of TBT into two distinct spectra, or two frequencies as occurs here, implies no interaction between the negative and positive modes; the negative energy mode is then acceptable, as the two modes cannot mutually excite one another. However, mode interaction is a necessity for some TBT end conditions. Below the cut-off frequency $\omega_{\mathrm{co}} = \sqrt{\kappa A G / \rho I}$, one has propagating (TBT1) and evanescent (TBT2) waves associated with trigonometric and hyperbolic functions, respectively; above the cut-off frequency, the hyperbolic functions become trigonometric, implying disturbance propagation. End conditions for the hinged–hinged beam are satisfied by trigonometric functions alone, which means that the frequency equation factorises, and there is no interaction between the positive and negative energy modes. The negative energy second spectrum is thus isolated and can be disregarded; consequently, it was found in Ref. [10] that first spectrum natural frequency predictions some five times greater than the cut-off frequency were in excellent agreement with exact elastodynamic plane stress predictions.

In contrast, for the majority of beam end combinations, such as free–free, the hyperbolic functions are necessary and the frequency equation does not factorise. Above the cut-off frequency, these hyperbolic (evanescent) functions become trigonometric (propagating), which implies the interaction of positive and negative energy modes. In Ref. [11], the concept of a pseudo-second spectrum was introduced to describe these propagating contributions, and it was conjectured that such mode interaction would debase the overall theory. (After all, if the second spectrum predictions are largely inaccurate for the hinged–hinged case, there seems no good reason why the pseudo-second spectrum contributions should be accurate for the free–free.) Comparison with experiment, and finite element and other simulations, for the free–free beam indicated that the cut-off frequency did indeed represent an upper limit for reasonable accuracy of TBT. While stability issues were not considered explicitly in Ref. [11], one might associate a negative energy mode with *defective* physics; in turn, its necessary inclusion for end conditions such as free–free, might be interpreted as leading to *defective* frequency predictions above the cut-off frequency.

While the above concepts are largely concerned with field theories, here they are explored within the context of a discrete, two-degree of freedom spring-mass system, which is perfectly stable in the absence of external excitation. It is shown how the traditional description of coupled 2×2 matrix second-order differential equations (consistent with Newton's second law) can be written either as a coupled 4×4 matrix first-order differential equation (the familiar companion or phase-variable form) or, with equal validity but less familiarity, as a single fourth-order differential equation (the so-called higher-order one-particle, or HOOP, description), which may be determined also from a higher time-derivative Lagrangian. The latter allows the conventional and Ostrogradski Hamiltonians to be related to the mechanical energy. Despite the stability of the system, its HOOP description also displays the Ostrogradski instability. As noted above, the absence of damping and the factorisation of the frequency equation, indicating two non-interacting modes, imply that the

negative energy mode is physically acceptable. On the other hand, it is now difficult to regard the negative energy mode as somehow physically defective, a charge which could be readily levelled at TBT2. This raises questions as to what really can be concluded from a negative energy Ostrogradski Hamiltonian, and the significance of the means by which the higher-derivative theory is constructed—answers to which, unfortunately, are not available here.

The process of generating the higher-derivative formulation is then reversed: a single fourth-order differential equation for which there is no explicit underlying second-order structure is cast into a (non-unique) coupled 2×2 matrix second-order differential form. This allows one to introduce a pseudo-mechanical energy $E_{\text{p-m}}$, an integral of the motion which, unlike the Ostrogradski Hamiltonian, is always positive.

Finally, the effect of damping is studied. Nesterenko [12] has recently considered the effect of a single velocity-dependent term $\gamma \dot{x}$, to represent interaction with the surroundings, on the higher-order differential equation: the amplitude of the lower mode of oscillation decays exponentially with time, while the higher mode increases exponentially. Here, it is shown that the introduction of a damping element within the coupled two-degree of freedom spring-mass system leads to a higher-order equation containing not just an $\dot{x}$ term, but also an $\dddot{x}$ term; thus a single $\dot{x}$ term within the higher-order equation cannot be regarded as a dissipative force. A physical realisation of this single velocity-dependent term is presented in the form of a control system, and the source of the instability becomes clear: it is not the Ostrogradski instability in the sense of an energy exchange between positive and negative energy modes. Rather, the instability can be viewed as flutter under external excitation at the higher natural frequency.

2. Example system

For the two-degree of freedom spring-mass system shown in Fig. 1, the governing equations of motion may be written in the 2×2 matrix form

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}, \tag{1}$$

where dot denotes differentiation with respect to time. Assuming synchronous vibration $x_{1,2} = X_{1,2} \sin \omega t$ leads to natural frequencies $\omega_1 = \sqrt{k/m}$ with mode shape $[\,X_1 \quad X_2\,]^{(1)\text{T}} = [\,1 \quad 1\,]^{\text{T}}$, and $\omega_2 = \sqrt{3k/m}$ with mode shape $[\,X_1 \quad X_2\,]^{(2)\text{T}} = [\,1 \quad -1\,]^{\text{T}}$. (The symmetry of this simple system allows one to identify the first and second mode shapes as symmetric and asymmetric, respectively, from which the natural frequencies follow with little effort.) The general solution of Eq. (1) may be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (A \sin \omega_1 t + B \cos \omega_1 t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (C \sin \omega_2 t + D \cos \omega_2 t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \tag{2}$$

in order to evaluate the constants A, B, C, and D, two initial conditions are required for each mass. Thus, if the system is at rest at time $t = 0$, that is $\dot{x}_1(0) = \dot{x}_2(0) = 0$, one requires $A = C = 0$. Then, if the initial displacements were to be consistent with the first mode of vibration, that is $x_1(0) = x_2(0) = 1$, one has $B = 1$ and $D = 0$, and the solution becomes $[\,x_1 \quad x_2\,]^{\text{T}} = [\,1 \quad 1\,]^{\text{T}} \cos \omega_1 t$. If consistent with the second mode, that is $x_1(0) = -x_2(0) = 1$, one has $B = 0$ and $D = 1$, and the solution becomes $[\,x_1 \quad x_2\,]^{\text{T}} = [\,1 \quad -1\,]^{\text{T}} \cos \omega_2 t$.

The second-order time-derivative 2×2 matrix problem can be re-written in other, equally valid, forms. The more familiar is a state-space formulation, which is the approach adopted within modern theory of dynamical systems: introducing the velocities $v_1 = \dot{x}_1$, and $v_2 = \dot{x}_2$, Eq. (1) can be written as the 4×4 first-order
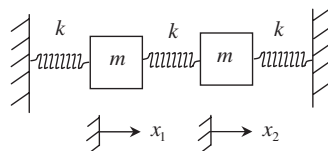

Fig. 1. Two-degree of freedom spring-mass oscillator.

time-derivative equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k/m & k/m & 0 & 0 \\ k/m & -2k/m & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} \tag{3}$$

or more compactly $\dot{\mathbf{s}} = \mathbf{As}$, where the system matrix $\mathbf{A}$ is said to be in *companion* or *phase-variable canonical* form. The underlying second-order origins of this first-order matrix equation are still evident, as the elements of $\mathbf{A}$ are written in algebraic terms. However, if matrix $\mathbf{A}$ was to be obtained by some system identification technique, when only numerical values would be known, extracting the underlying second-order structure is not so straightforward, see Friswell et al. [13] and references therein.

Alternatively, Eq. (1) may be written as a single fourth-order derivative equation; from the first row, one has $x_2 = (m\ddot{x}_1 + 2kx_1)/k$, and substituting into the second gives

$$x_1^{(4)} + 4(k/m)\ddot{x}_1 + 3(k/m)^2 x_1 = 0, \tag{4}$$

where the superscript (4) denotes the fourth differential with respect to time. This is the HOOP viewpoint. According to Kerner [14]: "Physically, the imprint of particle two into the motion of particle one appears equally clear whether one's eye views them both together, or concentrates one's gaze, and analytic machinery, solely on particle one." Eq. (4) may be written as

$$x_1^{(4)} + \left(\omega_1^2 + \omega_2^2\right)\ddot{x}_1 + \omega_1^2\omega_2^2 x_1 = 0, \tag{5}$$

where the natural frequencies ω_1 and ω_2 are defined above; setting $x_1 = X_1 \sin \omega t$ leads to the frequency equation

$$\omega^4 - \left(\omega_1^2 + \omega_2^2\right)\omega^2 + \omega_1^2\omega_2^2 = 0, \tag{6}$$

which factorises as

$$\left(\omega^2 - \omega_2^2\right)\left(\omega^2 - \omega_1^2\right) = 0. \tag{7}$$

Thus, the HOOP formulation leads to precisely the same natural frequencies, as it should.

The general solution to Eq. (5) is

$$x_1 = A \sin \omega_1 t + B \cos \omega_1 t + C \sin \omega_2 t + D \cos \omega_2 t, \tag{8}$$

which is just the first row of Eq. (2), and rather than two initial conditions for each particle, one now requires four initial conditions for the single particle. As before, suppose the system is at rest at time $t = 0$; now one can only set $\dot{x}_1(0) = 0$, to give $A\omega_1 + C\omega_2 = 0$. Suppose, also, that the initial displacement is $x_1(0) = 1$, which gives $B + D = 1$. For this HOOP solution, one requires higher-order initial conditions; thus the acceleration may be expressed as

$$\ddot{x}_1 = -A\omega_1^2 \sin \omega_1 t - B\omega_1^2 \cos \omega_1 t + A\omega_1\omega_2 \sin \omega_2 t + (B - 1)\omega_2^2 \cos \omega_2 t, \tag{9}$$

with initial acceleration

$$\ddot{x}_1(0) = -B\omega_1^2 + (B - 1)\omega_2^2. \tag{10}$$

Then if $B = 1$, the initial acceleration is $\ddot{x}_1(0) = -\omega_1^2$ and one has vibration in the lower frequency mode only; similarly, if $B = 0$, the initial acceleration is $\ddot{x}_1(0) = -\omega_2^2$ and one has vibration in the higher frequency mode only. Last, suppose that the third derivative (the *jerk*) $\dddot{x}_1(0) = 0$; this allows one to determine that $A = C = 0$.

2.1. Lagrangians and Hamiltonians

Governing differential equations are generally found by application of Newton's second law, or by application of Hamilton's principle. The Lagrangian function is (almost always) defined as $L = T - U$, where T is

the kinetic energy and U is the strain energy; for the present example

$$T = m\dot{x}_1^2/2 + m\dot{x}_2^2/2, \quad U = kx_1^2/2 + k(x_2 - x_1)^2/2 + kx_2^2/2, \tag{11a, b}$$

and according to Hamilton's principle the governing equations are generated from $\delta \int_{t_1}^{t_2} L\,dt = 0$. Performing the variation on the action integral, and integrating by parts in the usual way, leads to Eq. (1). The momenta conjugate to the displacement coordinates are $p_i = \partial L/\partial \dot{x}_i$, giving $p_1 = \partial L/\partial \dot{x}_1 = m\dot{x}_1$ and $p_2 = \partial L/\partial \dot{x}_2 = m\dot{x}_2$. The Hamiltonian function $H = \sum_i p_i \dot{x}_i - L$, or $H = p_1\dot{x}_1 + p_2\dot{x}_2 - L = m\dot{x}_1^2 + m\dot{x}_2^2 - T + U = T + U$. Thus, the Hamiltonian H is equal to the sum of the kinetic and strain energies, which is equal to the mechanical energy E_m and is a conserved quantity, an integral of the motion.

Since the Hamiltonian is a constant of the motion, its time derivative must be zero, from which one finds

$$dH/dt = \dot{x}_1(m\ddot{x}_1 + 2kx_1 - kx_2) + \dot{x}_2(m\ddot{x}_2 + 2kx_2 - kx_1) = 0; \tag{12}$$

this restores the two coupled second-order equations, Eq. (1), under the assumption that the velocities $\dot{x}_1 \neq 0$, $\dot{x}_2 \neq 0$. Note that the kinetic energy is quadratic in the velocities, and strain energy is independent of the velocities, Eqs. (11a, b), and this is the requirement stated in Ref. [1] that the Hamiltonian should be equal to the mechanical energy, that is $H = E_m$.

Now, it is equally valid to write the Langrangian as $L = \alpha(T - U)$ where α is any constant; of particular interest is the case when α is set equal to -1. The governing equations of motion are unchanged, but the generalised momenta become $p_1 = -m\dot{x}_1$ and $p_2 = -m\dot{x}_2$; the Hamiltonian becomes $H = -(T + U) = -E_m$, and is still a conserved quantity. It is mere convention that the Lagrangian is not chosen in this way; the total mechanical energy is a positive quantity, and it is natural that the Hamiltonian should likewise be positive, and α is chosen as equal to $+1$. Quoting Lemos [15] "one cannot refrain from pointing out that there seems to be no *a priori* physical reason to require $H = E_m$ in classical mechanics. The derivation of Hamilton's equations in any modern book on analytical mechanics reveals that the Hamiltonian is constructed directly from the Lagrangian, and its possible connection with the energy is discussed only *a posteriori*."

2.2. Higher-order Lagrangian

The single fourth-order Eq. (4) or (5) can be determined from the higher-order Lagrangian

$$L = \frac{m^2}{2k}\left(\ddot{x}_1^2 - \frac{4k}{m}\dot{x}_1^2 + \frac{3k^2}{m^2}x_1^2\right) \tag{13}$$

or

$$L = \frac{m^2}{2k}\left(\ddot{x}_1^2 - (\omega_1^2 + \omega_2^2)\dot{x}_1^2 + \omega_1^2\omega_2^2 x_1^2\right), \tag{14}$$

and variation of the action integral, leading to the Euler–Lagrange equation (see Whittaker [4])

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) = 0. \tag{15}$$

While this higher-order Lagrangian no longer has the structure $L = T - U$, and now contains a term dependent on the square of the acceleration, the $(\dot{x}_1)^2$ and $(x_1)^2$ terms are negative and positive, respectively, which would be consistent with negative kinetic and strain energies. The Hamiltonian associated with this higher-order Lagrangian can be constructed according to a procedure described first by Ostrogradski, again see Ref. [4]. The canonical coordinates are defined as $q_1 = x_1$, and $q_2 = \dot{x}_1$, and these are conjugate to the generalised momenta $p_1 = (\partial L/\partial \dot{x}_1) - (d/dt)(\partial L/\partial \ddot{x}_1)$ and $p_2 = \partial L/\partial \ddot{x}_1$. One finds $p_2 = (m^2/k)\ddot{x}_1$ and $p_1 = -4m\dot{x}_1 - (m^2/k)\dddot{x}_1$; the latter has no obvious physical meaning. The (Ostrogradski) Hamiltonian is then constructed as $H_O = p_1\dot{q}_1 + p_2\dot{q}_2 - L$, and one finds

$$H_O = \frac{m^2}{2k}\ddot{x}_1^2 - 2m\dot{x}_1^2 - \frac{m^2}{k}\dot{x}_1\dddot{x}_1 - \frac{3}{2}kx_1^2. \tag{16}$$

Since this Hamiltonian is conserved, its time derivative must be zero; from Eq. (16) one finds

$$dH_O/dt = \left(-m^2\dot{x}_1/k\right)\left(x_1^{(4)} + 4(k/m)\ddot{x}_1 + 3(k/m)^2 x_1\right) = 0, \tag{17}$$

which restores the governing fourth-order Eq. (4), under the assumption that the velocity $\dot{x}_1 \neq 0$.

Now write $x_1 = X_1 \sin \omega t$ in Eq. (16) to give

$$H_O = \left(\frac{m^2\omega^4}{2k} - \frac{3k}{2}\right)X_1^2 \sin^2 \omega t + \left(\frac{m^2\omega^4}{k} - 2m\omega^2\right)X_1^2 \cos^2 \omega t; \tag{18}$$

the frequency equation (6) may now be used to show that

$$\left(\frac{m^2\omega^4}{2k} - \frac{3k}{2}\right) = \left(\frac{m^2\omega^4}{k} - 2m\omega^2\right), \tag{19}$$

in which case the Hamiltonian becomes explicitly independent of time t, as it should, and may be expressed as

$$H_O = m\omega^2 X_1^2\left(\frac{m\omega^2}{k} - 2\right). \tag{20}$$

From this HOOP viewpoint, the kinetic and strain energies can be expressed as

$$T = \frac{m\dot{x}_1^2}{2} + \frac{m}{2}\left(\frac{m\ddot{x}_1 + 2k\dot{x}_1}{k}\right)^2, \quad U = \frac{kx_1^2}{2} + \frac{k}{2}\left(\frac{m\ddot{x}_1 + kx_1}{k}\right)^2 + \frac{k}{2}\left(\frac{m\ddot{x}_1 + 2kx_1}{k}\right)^2, \tag{21a, b}$$

where x_2 has been eliminated from Eqs. (11a, b) by employing $x_2 = (m\ddot{x}_1 + 2kx_1)/k$. Setting $x_1 = X_1 \sin \omega t$ in Eqs. (21a, b) leads to

$$T = \frac{X_1^2 \cos^2 \omega t}{2}\left(5m\omega^2 - \frac{4m^2\omega^4}{k} + \frac{m^3\omega^6}{k^2}\right), \quad U = \frac{X_1^2 \sin^2 \omega t}{2}\left(6k - 6m\omega^2 + \frac{2m^2\omega^4}{k}\right); \tag{22a, b}$$

employing the frequency equation (6), these reduce to

$$T = m\omega^2 X_1^2 \cos^2 \omega t, \quad U = m\omega^2 X_1^2 \sin^2 \omega t, \tag{23a, b}$$

and the mechanical energy $E_m = T + U$ reduces to $E_m = m\omega^2 X_1^2$. The Ostrogradski Hamiltonian, Eq. (20), may now be re-expressed as

$$H_O = m\omega^2 X_1^2\left(\frac{m\omega^2}{k} - 2\right) = E_m\left(\frac{m\omega^2}{k} - 2\right), \tag{24}$$

so the general expression for H_O is no longer equal to the mechanical energy; this might have been anticipated, as the kinetic energy, Eq. (21a), is no longer a quadratic function of the now single velocity.

For the lower natural frequency, $\omega_1 = \sqrt{k/m}$, one finds

$$H_O^{(1)} = -kX_1^2 < 0; \tag{25}$$

in contrast, the mechanical energy associated with this mode can be calculated as $E_m^{(1)} = kX_1^2 > 0$, so one has $H_O^{(1)} = -E_m^{(1)}$. For the higher natural frequency, $\omega_2 = \sqrt{3k/m}$, one finds

$$H_O^{(2)} = 3kX_1^2 > 0, \tag{26}$$

which is equal to the mechanical energy for this mode, that is $H_O^{(2)} = E_m^{(2)}$. This is the Ostrogradski instability: the Hamiltonian associated with one of the natural frequencies—here the lower of the two—is negative. It is simple to confirm that these quantities are indeed conserved during vibration. For example, if one has the initial conditions $x_1(0) = X_1$, $x_2(0) = -X_1$, with $\dot{x}_1(0) = \dot{x}_2(0) = 0$, that is the velocities are initially zero, and the initial displacements are in accordance with the eigenvector for the higher frequency, then vibration will occur only at the higher frequency; the initial strain energy is then equal to $3kX_1^2$ which is the conserved quantity. Similarly, if one has the initial conditions $x_1(0) = X_1$, $x_2(0) = X_1$, with $\dot{x}_1(0) = \dot{x}_2(0) = 0$, then vibration will occur only at the lower frequency; the initial strain energy is then equal to kX_1^2 which is the negative of the conserved quantity according to Eq. (25).

Now, one can equally define the Lagrangian to be the negative of that in Eq. (13), as

$$L = \frac{m^2}{2k}\left(-\ddot{x}_1^2 + \frac{4k}{m}\dot{x}_1^2 - \frac{3k^2}{m^2}x_1^2\right). \tag{27}$$

The canonical coordinates are still defined as $q_1 = x_1$, and $q_2 = \dot{x}_1$ while the conjugate generalised momenta become $p_1 = (\partial L/\partial \dot{x}_1) - (\mathrm{d}/\mathrm{d}t)(\partial L/\partial \ddot{x}_1) = 4m\dot{x}_1 + (m^2/k)\dddot{x}_1$ and $p_2 = (\partial L/\partial \ddot{x}_1) = -(m^2/k)\ddot{x}_1$. The Hamiltonian is again constructed as $H_O = p_1\dot{q}_1 + p_2\dot{q}_2 - L$, and one finds

$$H_O = -\frac{m^2}{2k}\ddot{x}_1^2 + 2m\dot{x}_1^2 + \frac{m^2}{k}\dot{x}_1\dddot{x}_1 + \frac{3}{2}kx_1^2, \tag{28}$$

which is exacty the negative of that given in Eq. (16). In turn, for each of the two modes, one finds expressions for $H_O^{(1)}$ and $H_O^{(2)}$ which are the negative of those in Eqs. (25) and (26); now the higher frequency mode has the negative energy.

We are thus led to the following unsatisfactory state of affairs: a system which is perfectly stable in the absence of external excitation, considered from a HOOP point of view has a negative Ostrogradski Hamiltonian for one mode of vibration; in some branches of modern physics this would be sufficient to conclude that the mode is unphysical. Moreover, since the original sign of the Lagrangian is arbitrary, it seems that one cannot decide which of the two modes is associated with the negative energy. For this example, one has the advantage of the underlying second-order structure and the positive definite mechanical energy; however if one is presented with a given Lagrangian containing time derivatives greater than the first, leading to a fourth-order time-derivative equation, it may not be apparent that the same fourth-order equation can also be derived from two coupled second-order equations, by elimination of one of the variables. Further, one might expect the higher-order Lagrangian/fourth-order equation to have some standing in its own right, irrespective of the means by which it is derived; thus the implication of the negative Ostrogradski Hamiltonian, indeed the existence of the Ostrogradski instability, is called into question. A means of avoiding this issue is now developed.

3. Pseudo-mechanical energy

The starting point is now the higher-order time-derivative Eq. (5), also known as the Pais–Uhlenbeck equation [2]. Introducing the expression

$$\ddot{x}_1 = ax_2 + bx_1, \tag{29}$$

and hence $x_1^{(4)} = a\ddot{x}_2 + b\ddot{x}_1$, where a and b are as yet undefined real constants having the units of (radian frequency)^2, into Eq. (5) gives

$$\ddot{x}_2 + \left(b + \omega_1^2 + \omega_2^2\right)x_2 + \left(\frac{b^2 + b(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2}{a}\right)x_1 = 0; \tag{30}$$

the variable x_2 may be thought of as hidden. Eqs. (29) and (30) may be written in the matrix form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} -b & -a \\ \dfrac{b^2 + b(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2}{a} & b + \omega_1^2 + \omega_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}, \tag{31}$$

or more compactly $\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$ where matrix $\mathbf{K}$ and vector $\mathbf{x}$ are defined accordingly. One may regard $\mathbf{K}$ as a pseudo-stiffness matrix for a two-degree of freedom system in which the mass of each particle is unity. Writing $x_{1,2} = X_{1,2}\sin\omega t$ leads to

$$\begin{bmatrix} -b - \omega^2 & -a \\ \dfrac{b^2 + b(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2}{a} & b + \omega_1^2 + \omega_2^2 - \omega^2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \mathbf{0}; \tag{32}$$

for a non-trivial solution the determinant must be zero, and this leads to frequency equation (6), indicating that the choice of the constants a and b has no influence upon the natural frequencies. However, we choose to require that the pseudo-stiffness matrix $\mathbf{K}$ should be symmetric. (This requirement may be unnecessarily restrictive, but it suits the present purpose.) The strain energy associated with the pseudo-stiffness matrix is $U = \mathbf{x}^{\mathrm{T}}\mathbf{K}\mathbf{x}/2$, and we wish this to be positive-definite. It is well known that a positive-definite quadratic form is only defined for a symmetric matrix, and the eigenvalues must be positive. For the present problem, the eigenvalues are the squares of the natural frequencies, that is $\lambda_1 = \omega_1^2$, $\lambda_2 = \omega_2^2$, which are clearly positive. Therefore positive-definiteness is assured if we require that $\mathbf{K}$ be symmetric, that is

$$b^2 + b(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2 = -a^2. \tag{33}$$

This may be expressed as

$$(b + \omega_1^2)(b + \omega_2^2) = -a^2, \tag{34}$$

and since the right-hand side is clearly negative, and ω_2 is taken as the higher of the two frequencies, so one must have $(b + \omega_1^2) < 0$, and $(b + \omega_2^2) > 0$; these results are employed later.

Now define a pseudo-mechanical energy $E_{\text{p-m}}$ as

$$E_{\text{p-m}} = \dot{x}_1^2/2 + \dot{x}_2^2/2 + \mathbf{x}^{\mathrm{T}}\mathbf{K}\mathbf{x}/2, \tag{35}$$

which is clearly positive-definite, leading to

$$2E_{\text{p-m}} = \dot{x}_1^2 + \dot{x}_2^2 - bx_1^2 - 2ax_1x_2 + (b + \omega_1^2 + \omega_2^2)x_2^2. \tag{36}$$

Substitute from Eq. (29), $x_2 = (\ddot{x}_1 - bx_1)/a$, to give

$$2E_{\text{p-m}} = \left\{1 + \frac{b^2}{a^2}\right\}\dot{x}_1^2 - \frac{2b}{a^2}\dot{x}_1\ddot{x}_1 + \frac{1}{a^2}\ddot{x}_1^2 + \left\{b + \frac{b^3 + b^2(\omega_1^2 + \omega_2^2)}{a^2}\right\}x_1^2$$
$$- 2x_1\ddot{x}_1 - \frac{2(b^2 + b(\omega_1^2 + \omega_2^2))}{a^2}x_1\ddot{x}_1 + \left(\frac{b + \omega_1^2 + \omega_2^2}{a^2}\right)\ddot{x}_1^2 \tag{37}$$

and eliminating a in favour of b, using Eq. (34) gives

$$E_{\text{p-m}} = \frac{1}{2(b + \omega_1^2)(b + \omega_2^2)}\Big[(b(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2)\dot{x}_1^2 + 2b\dot{x}_1\ddot{x}_1 - \ddot{x}_1^2$$
$$+ b\omega_1^2\omega_2^2 x_1^2 - 2\omega_1^2\omega_2^2 x_1\ddot{x}_1 - (b + \omega_1^2 + \omega_2^2)\ddot{x}_1^2\Big]; \tag{38}$$

this is the general expression for the pseudo-mechanical energy, and possible alternative to the Ostrogradski Hamiltonian. Since this energy is a constant, its time derivative must be zero; from Eq. (38) one finds

$$\mathrm{d}E_{\text{p-m}}/\mathrm{d}t = (b\dot{x}_1 - \dddot{x}_1)\Big(x_1^{(4)} + (\omega_1^2 + \omega_2^2)\ddot{x}_1 + \omega_1^2\omega_2^2 x_1\Big) = 0, \tag{39}$$

which restores the governing fourth-order Eq. (5), under the condition that $(b\dot{x}_1 - \dddot{x}_1) \neq 0$, or equivalently $\dot{x}_2 \neq 0$.

Now demonstrate that $E_{\text{p-m}}$ is independent of time and positive for both modes of vibration: write $x_1 = X_1 \sin \omega t$, to give

$$E_{\text{p-m}} = \frac{X_1^2\cos^2\omega t}{2(b + \omega_1^2)(b + \omega_2^2)}\left\{(b(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2)\omega^2 - 2b\omega^4 - \omega^6\right\}$$
$$+ \frac{X_1^2\sin^2\omega t}{2(b + \omega_1^2)(b + \omega_2^2)}\left\{b\omega_1^2\omega_2^2 + 2\omega_1^2\omega_2^2\omega^2 - (b + \omega_1^2 + \omega_2^2)\omega^4\right\}. \tag{40}$$

For the lower frequency, set $\omega = \omega_1$ to give

$$E_{\text{p-m}}^{(1)} = \frac{\omega_1^2 X_1^2(\omega_2^2 - \omega_1^2)}{2(b + \omega_2^2)}, \tag{41}$$

which is positive, as the denominator $(b + \omega_2^2) > 0$, and explicitly independent of time. For the higher frequency, set $\omega = \omega_2$ to give

$$E_{\text{p-m}}^{(2)} = \frac{-\omega_2^2 X_1^2 (\omega_2^2 - \omega_1^2)}{2(b + \omega_1^2)},$$

(42)

which is also positive, as $(b + \omega_1^2) < 0$, and also independent of time. Thus, in contrast with the Ostrogradski energy, the proposed pseudo-mechanical energy is positive for both natural frequencies.

Now consider possible values for b: the requirements $(b + \omega_2^2) > 0$ and $(b + \omega_1^2) < 0$ imply $\omega_1^2 < -b < \omega_2^2$, and seemingly no more. However, a specific choice for b within the required range has implications on the mode shapes: for example, if one takes $b = -(\omega_1^2 + \omega_2^2)/2$, the pseudo-mechanical energy simplifies to

$$E_{\text{p-m}} = \frac{1}{(\omega_2^2 - \omega_1^2)^2} \left[(\omega_1^4 + \omega_2^4) \dot{x}_1^2 + 2(\omega_1^2 + \omega_2^2) \dot{x}_1 \dddot{x}_1 + 2 \ddot{x}_1^2 \right.$$
$$\left. + \omega_1^2 \omega_2^2 (\omega_1^2 + \omega_2^2) x_1^2 + 4 \omega_1^2 \omega_2^2 x_1 \ddot{x}_1 + (\omega_1^2 + \omega_2^2) \ddot{x}_1^2 \right]$$

(43)

and writing $x_1 = X_1 \sin \omega t$ gives

$$E_{\text{p-m}} = \frac{X_1^2 \cos^2 \omega t}{(\omega_2^2 - \omega_1^2)^2} \left\{ (\omega_1^4 + \omega_2^4) \omega^2 - 2(\omega_1^2 + \omega_2^2) \omega^4 + 2 \omega^6 \right\}$$
$$+ \frac{X_1^2 \sin^2 \omega t}{(\omega_2^2 - \omega_1^2)^2} \left\{ \omega_1^2 \omega_2^2 (\omega_1^2 + \omega_2^2) - 4 \omega_1^2 \omega_2^2 \omega^2 + (\omega_1^2 + \omega_2^2) \omega^4 \right\}.$$

(44)

For the lower frequency, $\omega = \omega_1$, this reduces to

$$E_{\text{p-m}}^{(1)} = \omega_1^2 X_1^2,$$

(45)

while for the higher, $\omega = \omega_2$, one finds

$$E_{\text{p-m}}^{(2)} = \omega_2^2 X_1^2.$$

(46)

The first implication of this particular choice for b is, from Eq. (34), that $a = \pm(\omega_2^2 - \omega_1^2)/2$; further, substituting these values into the first row of matrix Eq. (32) one finds the two possible modes shapes as $X_1 = X_2$ and $X_1 = -X_2$. (The choice of $\pm$ in the expression for a only affects which of the two frequencies—subscript 1 or 2—is associated with which mode shape.) Thus, the specific pseudo-mechanical energies, Eqs. (45,46), are seen to be equal to the mechanical energies, assuming two particles each having a mass of unity for a symmetric system such as that shown in Fig. 1. Alternatively, the pseudo-mechanical energy for both modes is precisely double what one would expect for a *single* particle having unity mass, if governed by the fourth-order time-derivative equation.

4. Damping

Nesterenko [12] considered recently the inclusion of a single viscous damping term into the HOOP description, when Eq. (5) becomes

$$x_1^{(4)} + (\omega_1^2 + \omega_2^2) \ddot{x}_1 + \omega_1^2 \omega_2^2 x_1 + \gamma \dot{x}_1 = 0$$

(47)

and γ is assumed small and positive; setting $x_1 = X_1 \text{e}^{\text{i}\omega t}$ gives

$$\omega^4 - (\omega_1^2 + \omega_2^2) \omega^2 + \omega_1^2 \omega_2^2 + \text{i}\gamma\omega = 0.$$

(48)

Writing

$$\omega = \omega_{1,2} + \text{i}\Delta\omega_{1,2},$$
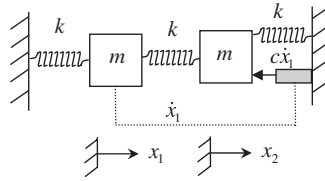
(49)

Fig. 2. Two-degree of freedom spring-mass oscillator, with servo.

where the $\Delta\omega_{1,2}$ are assumed small, ignoring terms in $(\Delta\omega_{1,2})^2$ and higher, and also $\gamma\Delta\omega_{1,2}$, leads to

$$\Delta\omega_1 = \frac{\gamma}{2(\omega_2^2 - \omega_1^2)} > 0 \tag{50}$$

for the lower frequency, and

$$\Delta\omega_2 = \frac{\gamma}{2(\omega_1^2 - \omega_2^2)} < 0 \tag{51}$$

for the higher. In turn, the response at the lower frequency is

$$x_1 = X_1 e^{i\omega_1 t} e^{-\Delta\omega_1 t}, \tag{52}$$

which represents a stable decaying oscillation as $\Delta\omega_1$ is positive; on the other hand, for the higher frequency

$$x_1 = X_1 e^{i\omega_2 t} e^{-\Delta\omega_2 t}, \tag{53}$$

which is unstable, as $\Delta\omega_2$ is negative.

Now consider how such a term might arise. The inclusion of a viscous damper into the system, Fig. 1, will in general lead to terms in both $\dot{x}_1$ and $\dddot{x}_1$ in the HOOP description; for example, if a single damper (coefficient c) is placed to the left of the left-hand mass (mass 1), or to the right of the right-hand mass (mass 2), the resulting higher-derivative equation is

$$x_1^{(4)} + 4(k/m)\ddot{x}_1 + 3(k/m)^2 x_1 + (2kc/m^2)\dot{x}_1 + (c/m)\dddot{x}_1 = 0. \tag{54}$$

Thus the single $\gamma\dot{x}$ term in Eq. (47) cannot be regarded as a simple dissipative force within the context of a second-order formalism.

A physical realisation of Eq. (47) is now presented: consider the coupled equations

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}. \tag{55}$$

Again the first row is employed to eliminate the fundamental variable x_2, to give

$$x_1^{(4)} + 4(k/m)\ddot{x}_1 + 3(k/m)^2 x_1 + (kc/m^2)\dot{x}_1 = 0; \tag{56}$$

this is identical in form to Eq. (47) when one sets $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$ as in Section 2, and $\gamma = kc/m^2$. The first row of matrix Eq. (55) indicates that the left-hand mass (mass 1) of Fig. 1 experiences no damping force, while the second row indicates that the right-hand mass (mass 2) experiences a force dependent solely on the velocity of mass 1. This can be realised by the addition of a control system, as depicted in Fig. 2. The dotted line represents the signal from a transducer measuring the velocity of mass 1, $\dot{x}_1$, which feeds into a high-input impedance amplifier; for an ideal instrumentation system, this will have no effect on mass 1. The amplifier then provides a force $c\dot{x}_1$ precisely proportional to the velocity of mass 1, with no phase lag. The effect of this ideal servo is clear: in the lower mode of vibration, the two masses move in-phase and the force $c\dot{x}_1$ will oppose the motion of mass 2, and thereby damp oscillation of the system. On the other hand, for the higher mode the two masses move in anti-phase; suppose mass 1 is moving to the right (that is $\dot{x}_1$ is positive), then mass 2 will be moving to the left and the force $c\dot{x}_1$ will excite its motion. The instability can thus be seen as a flutter caused by external excitation at the higher natural frequency.

5. Conclusions

A two-degree of freedom spring-mass system has been considered from the higher-order one-particle (HOOP) point of view. The single governing fourth-order differential equation may be derived by elimination of one of the fundamental variables from the coupled second-order equations, or from a Lagrangian containing derivatives higher than the first order. The Ostrogradski Hamiltonian associated with the HOOP description, regarded as the conserved energy of the system, is negative for one of the modes; in some branches of physics, this would be sufficient to conclude that the mode is unphysical. However, given the obvious stability of the system in the absence of external excitation, and the apparently arbitrary sign of the Lagrangian, and in turn the Ostrogradski Hamiltonian, the existence of the Ostrogradski instability has been called in to question.

To avoid these issues, a given fourth-order equation with no explicit underlying second-order structure, is cast into a non-unique second-order matrix form, allowing the construction of a pseudo-mechanical energy, a conserved quantity which is always positive.

The effect of damping has also been considered; a single velocity-dependent term within the HOOP description, previously known to result in the instability of the higher-mode of vibration, cannot be regarded as a simple dissipative force within the second-order formalism. A physical realisation consists of an ideal control system, and the source of the instability becomes clear: it is not the exchange of energy between positive and negative energy modes, rather a flutter caused by external excitation at the higher natural frequency.

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