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On the weakly damped vibrations of a vertical beam with a tip-mass

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Abstract

In this paper the wind-induced, horizontal vibrations of a weakly damped vertical Euler–Bernoulli beam with and without a tip-mass will be studied. The damping is assumed to be boundary damping and global Kelvin–Voigt damping. The boundary damping is assumed to be proportional to the velocity of the beam at the top. The horizontal vibrations of the beam can be described by an initial-boundary value problem. In this paper, the multiple-timescales perturbation method will be applied to construct approximations of the solutions of the problem. Also it will be shown that a combination of boundary damping and Kelvin–Voigt damping can be used to damp the wind-induced vibrations of a vertical beam with tip-mass uniformly.

1. Introduction

In many mathematical models oscillations of elastic structures are described by (non)linear wave equations or by (non)linear beam equations. Examples of wave-like or string-like problems are given in Refs. [1,2]. Examples of beam-like problems are given in Refs. [3–7]. In this paper a vertical, cantilevered, uniform Euler–Bernoulli beam with boundary damping and with global Kelvin–Voigt damping (see Fig. 1) as a simple model for a tall building will be considered.

In recent years more and more tall buildings were built. For tall buildings, or high rise buildings, dampers, active or passive, are used to dissipate the energy of the vibrations of the building. Vibrations induced by wind or earthquakes can cause damage to an elastic structure. Vortex-shedding (high-frequency oscillations with small amplitudes) and galloping (the effect of low-frequency vibrations with large amplitudes) can cause material fatigue. Since these small and large amplitudes can cause damage to a building it is important to have damping. To suppress the vibrations of a structure various types of boundary damping can be applied. In this paper, the boundary damping is assumed to be proportional to the velocity of the beam at the top. Some damping mechanisms give rise to a heavy tip-mass, that is why beams with and without such tip-masses will be

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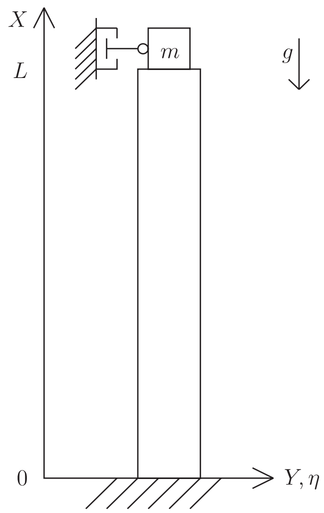

Fig. 1. A simple model for a vertical cantilevered beam with tip-mass and velocity damper.

considered in this paper. Boundary damping for horizontal beams with and without tip-masses has been studied in Refs. [8–11]. In this paper, it is assumed that the beam is made of a viscoelastic material that satisfies the Kelvin–Voigt constitutive equation. Global and local Kelvin–Voigt damping mechanisms for horizontal beams have been studied in Refs. [12,13].

Furthermore, a uniform wind-flow is considered, which causes nonlinear drag and lift forces (F_D, F_L) acting on the structure per unit length. A simple model of a vertical cantilevered Euler–Bernoulli beam equation with Kelvin–Voigt damping subjected to wind-forces is given by

$$EI\eta_{XXXX} + \varsigma EI\eta_{XXXX\tau} + g[(m + \rho A(L - x))\eta_X]_X + \rho A\eta_{\tau\tau} = F_D + F_L, \tag{1}$$

where E is Young's modulus, I is the moment of inertia of the cross-section, ς is the coefficient of the Kelvin–Voigt viscoelastic damping, ρ is the mass density of the beam, A is the cross-sectional area of the beam, L is the length of the beam, η is the deflection of the beam in Y-direction, τ is the time, X is the position along the beam (see Fig. 1), m is the mass of the tip-mass, and g is the acceleration due to gravity. The term $[g(m + \rho A(L - x))\eta_X]_X$ in Eq. (1) is a linearly varying compression force due to the weight of the beam and the tip-mass. In Ref. [14] the Ritz–Galerkin method and perturbation methods have been used to determine closed-form approximate solutions of the vibrations of a vertical beam.

The main goal of this paper is to study the possibility to stabilize vertical cantilevered beams with and without tip-masses at the top in a wind-field. Explicit asymptotic approximations of the solutions for this problem, which are valid on a long timescale, will be given.

A simple model for the damped, vertical, cantilevered Euler–Bernoulli beam subjected to wind-forces is given by Eq. (1) and the boundary conditions $\eta(0,\tau) = \eta_X(0,\tau) = 0$, and

$$EI\eta_{XXX}(L,\tau) + \varsigma EI\eta_{XXX\tau}(L,\tau) = m\eta_{\tau\tau}(L,\tau) - gm\eta_X(L,\tau) + \hat{c}\eta_\tau(L,\tau), \tag{2}$$

$$EI\eta_{XX}(L,\tau) + \varsigma EI\eta_{XX\tau}(L,\tau) = 0, \tag{3}$$

where $\hat{c}$ is a positive constant, the damping parameter. In Ref. [2] it has been shown that $F_D + F_L$ can be approximated by

$$F_D + F_L = \frac{\rho_a d v_\infty a}{2}\left(\eta_\tau + \frac{b}{v_\infty^2}\eta_\tau^3\right), \tag{4}$$

where ρ_a is the density of the air, d is the diameter of the cross-sectional area of the beam, v_∞ is the uniform wind-flow velocity, and a and b depend on certain drag and lift coefficients, which are given explicitly in Ref. [2]. In this paper the linearized partial differential equation (1) will be considered. The nonlinear wind-force $(\rho_a d v_\infty a/2)(\eta_\tau + (b/v_\infty^2)\eta_\tau^3)$ in Eq. (1) will give a coupling between (almost) all oscillation modes. In

Refs. [2,15] also this nonlinear windforce has been considered. It has been shown that the windforce gives a coupling between (almost) all oscillation modes. It is also known that the nonlinear term damps the vibrations. In this paper the linearized initial-boundary value problem will be considered, because the main goal of this paper is to determine the damping. If the damper damps the vibrations due to the linearized wind-force, the damper also damps the vibration due to nonlinear wind-force, because the nonlinear term in the wind-force also damps the vibrations.

To put the model in a non-dimensional form the following substitutions $u(x,t) = (\kappa/v_\infty)(\eta(X,\tau)/L)$, $x = X/L$, and $t = (\kappa/L)\tau$, where $\kappa = (1/L)\sqrt{EI/A\rho}$ will be used. By applying these transformations, the following linearized, dimensionless initial-boundary value problem can be introduced, which describes the horizontal displacement of a damped vertical beam with tip-mass and with a uniform wind-flow acting on it:

$$\mathbb{L}[u] = \varepsilon\alpha u_t, \quad t>0,\ 0<x<1, \tag{5}$$

$$u(0,t) = u_x(0,t) = 0, \quad t\geqslant 0, \tag{6}$$

$$u_{xx}(1,t) + \beta u_{xxt}(1,t) = 0, \quad t\geqslant 0, \tag{7}$$

$$u_{xxx}(1,t) + \beta u_{xxxt}(1,t) = \gamma u_{tt}(1,t) - \varepsilon\gamma u_x(1,t) + \varepsilon c u_t(1,t), \quad t\geqslant 0, \tag{8}$$

$$u(x,0) = f(x), \quad 0<x<1, \tag{9}$$

$$u_t(x,0) = g(x), \quad 0<x<1, \tag{10}$$

where $\varepsilon = g\rho AL^3/EI$ is a small parameter, that is, $0<\varepsilon\ll 1$, $\beta = (\varsigma/L^2)\sqrt{EI/\rho A}$, $\gamma = m/\rho AL$, $\varepsilon\alpha = (\rho_a dL/2A\rho)(v_\infty/\kappa)a$, $\varepsilon c = \hat{c}\sqrt{L^2/EI\rho A}$, and where

$$\mathbb{L}[u] \equiv u_{xxxx} + \beta u_{xxxxt} + \varepsilon[(\gamma + 1 - x)u_x]_x + u_{tt}. \tag{11}$$

The functions $f(x)$ and $g(x)$ represent the initial displacement and the initial velocity of the beam, respectively. It should be observed that α (the parameter due to the wind-force), β (the Kelvin–Voigt damping parameter), c (the boundary damping parameter), γ (the mass of the tip-mass divided by the mass of the beam), and ε are dimensionless parameters. The parameters α and c are ε-independent. The parameters γ and β in general will be small parameters. For the construction of approximations of the solution of Eqs. (5)–(10), however, it will be assumed that β and γ are ε-independent parameters.

This paper is organized as follows: in Section 2 the initial-boundary value problem with $c = \alpha = 0$ will be considered. This is the problem of a vertical beam with a tip-mass and with Kelvin–Voigt damping. Also it will be explained why a multiple-timescales perturbation method will be applied. In Section 3, the unperturbed initial-boundary value problem (i.e. $\varepsilon = 0$) will be considered. This is the problem of a beam with tip-mass and Kelvin–Voigt damping. In Section 4 the energy of the initial-boundary value problem without wind-perturbation (i.e. $\alpha = 0$) is considered. The boundedness of the solutions will be shown, assuming the existence of a sufficiently smooth solution. In Section 5, formal approximations for the solutions of the initial-boundary value problem (5)–(10) are constructed by using a two-time-scales perturbation method. Next, in Section 6, the stability of the beam will be discussed. Finally, in Section 7 some conclusions will be drawn and some remarks will be made.

2. The problem (5)–(10) with $c = \alpha = 0$

In this section the wind-forces and the boundary damping acting on the beam are neglected. The horizontal vibrations of a vertical beam with a tip-mass and with Kelvin–Voigt damping are studied. These vibrations can be described by problem (5)–(10), with $c = \alpha = 0$:

$$u_{xxxx} + \beta u_{xxxxt} + \varepsilon[(\gamma + 1 - x)u_x]_x + u_{tt} = 0, \tag{12}$$

$$u(0,t) = u_x(0,t) = u_{xx}(1,t) + \beta u_{xxt}(1,t) = 0, \tag{13}$$

$$\varepsilon\gamma u_x(1,t) + u_{xxx}(1,t) + \beta u_{xxxt}(1,t) - \gamma u_{tt}(1,t) = 0, \tag{14}$$

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x). \tag{15}$$

Now look for non-trivial solutions of the partial differential equation (12) and the boundary conditions (13) and (14) in the form $X(x)T(t)$. By substituting this into Eq. (12) and by dividing the so-obtained equation by $X(x)T(t)$ it follows that

$$\frac{X^{(4)}}{X}\left(1+\beta\frac{T'}{T}\right) + \frac{\varepsilon[(\gamma+1-x)X']'}{X} + \frac{T''}{T} = 0. \tag{16}$$

Now the case $T + \beta T' = 0$ will be considered first. By considering the boundary conditions it can be deduced that, for the case $T + \beta T' = 0$, $X(x)$ has to satisfy

$$\varepsilon\beta^2[(\gamma+1-x)X']' + X = 0, \tag{17}$$

$$X(0) = X'(0) = \varepsilon\beta^2\gamma X'(1) - \gamma X(1) = 0. \tag{18}$$

So, the only solution of Eqs. (17)–(18) is given by the trivial solution. This can be seen in the following way. Multiply Eq. (17) by $(\gamma + 1 - x)X'(x)$, integrate the so-obtained result with respect to x from 0 to 1, and use Eq. (18) to obtain

$$\varepsilon\beta^2\gamma^2(X'(1))^2 + \gamma(X(1))^2 + \int_0^1 X^2(x)\,\mathrm{d}x = 0. \tag{19}$$

From Eq. (19) it follows that $X(x) \equiv 0$. So, the only solution of Eqs. (17)–(18) is given by the trivial solution. Therefore, the case $T + \beta T' = 0$ only leads to trivial solutions. Now to separate the variables in Eq. (16), Eq. (16) can be differentiated with respect to t or to x (see also Refs. [16,17]). Differentiation of Eq. (16) with respect to t, yields

$$\beta\frac{X^{(4)}}{X}\left(\frac{T'}{T}\right)' + \left(\frac{T''}{T}\right)' = 0. \tag{20}$$

Now separate variables to obtain

$$X^{(4)} = \beta_1 X, \tag{21}$$

where $\beta_1 \in \mathbb{C}$ is a separation constant. Then from Eq. (16) it also follows that

$$\beta_1\left(1+\beta\frac{T'}{T}\right) + \frac{T''}{T} + \frac{\varepsilon[(\gamma+1-x)X']'}{X} = 0. \tag{22}$$

Again separate variables to obtain

$$\varepsilon[(\gamma+1-x)X']' = \beta_2 X, \tag{23}$$

where $\beta_2 \in \mathbb{C}$ is also a separation constant. From Eq. (13) it follows that $X(0) = X'(0) = 0$. By substituting $x = 0$ into Eq. (23) it follows that $X''(0) = 0$, and by differentiating Eq. (23) with respect to x and by substituting $x = 0$ into the so-obtained result it follows that $X'''(0) = 0$. Now the differential equation (20) subject to $X(0) = X'(0) = X''(0) = X'''(0) = 0$ only has trivial solutions. So, differentiation of Eq. (16) with respect to t only leads to trivial solutions. Now differentiate Eq. (16) with respect to x to obtain

$$\left(\frac{X^{(4)}}{X}\right)'\left(1+\beta\frac{T'}{T}\right) + \left(\frac{\varepsilon[(\gamma+1-x)X']'}{X}\right)' = 0 \Rightarrow T' = \theta T, \tag{24}$$

where $\theta \in \mathbb{C}$ is a separation constant. Now because $T' = \theta T \Rightarrow T'' = \theta^2 T$ the following eigenvalue problem for $X(x)$ is obtained:

$$(1+\beta\theta)X^{(4)} + \varepsilon[(\gamma+1-x)X']' = -\theta^2 X, \tag{25}$$

$$X(0) = X'(0) = (1+\beta\theta)X''(1) = 0, \tag{26}$$

$$(1+\beta\theta)X'''(1)+\varepsilon\gamma X'(1)-\gamma\theta^2 X(1)=0. \tag{27}$$

This fourth-order differential equation (25) can be solved exactly for $\varepsilon = 0$, but cannot be solved exactly for $\varepsilon \neq 0$.

Now consider the case $\beta = 0$ (this is the case of a vertical beam with a tip-mass but without Kelvin–Voigt damping) and introduce the eigenvalue $\lambda = -\theta^2$. In Ref. [18] it has been shown that the eigenvalues λ of problem (25)–(27) with $\beta = 0$ are real-valued. In addition, it has been shown in Ref. [18] that these eigenvalues are certainly positive for sufficiently small values of ε and γ, that is, if ε and γ satisfy the following inequality:

$$\varepsilon(\gamma + \tfrac{1}{2}) < 1. \tag{28}$$

Moreover, in Ref. [18] it has been proved that the eigenfunctions corresponding to problem (25)–(27) with $\beta = 0$ can be chosen to be real-valued, and it has been shown that these eigenfunctions are orthogonal with respect to the following inner product:

$$\langle u(x), v(x)\rangle = \int_0^1 [1+\gamma\delta(x-1)]u(x)\overline{v(x)}\,\mathrm{d}x, \tag{29}$$

where $\delta(x)$ is the Dirac delta function, with the properties $\int_0^1 \delta(x-1)\,\mathrm{d}x = 1$, and $\delta(x-1)=0$ for $x\neq 1$.

Although some properties of the eigenvalues and the eigenfunctions of problem (25)–(27) with $\beta = 0$ are now known, the fourth-order differential equation (25) for $\beta = 0$ and for $\beta \neq 0$ cannot be solved exactly. To construct an approximation of a solution a perturbation method will be used. It has been assumed that $0<\varepsilon\ll 1$. Then the term $\varepsilon[(\gamma+1-x)X(x)']'$ in Eq. (25) is small compared to the other terms in the equation. In this paper a two-time-scales perturbation method will be used in Section 5 to solve the problem (5)–(10), with $\varepsilon \neq 0$ approximately. The reader is referred to the book of Nayfeh and Mook [19] for a description of this method.

3. The problem (5)–(10) with $\varepsilon = 0$

In this section the wind-forces, the effect due to gravity, and the boundary damping are neglected. So, problem (5)–(10), with $\varepsilon = 0$ will be considered:

$$u_{xxxx} + \beta u_{xxxxt} + u_{tt} = 0, \tag{30}$$

$$u(0,t) = u_x(0,t) = u_{xx}(1,t) + \beta u_{xxt}(1,t) = 0, \tag{31}$$

$$u_{xxx}(1,t) + \beta u_{xxxt}(1,t) - \gamma u_{tt}(1,t) = 0, \tag{32}$$

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x). \tag{33}$$

The method of separation of variables will be used to solve the problem (30)–(33). Now look for non-trivial solutions of the partial differential equation (30) and the boundary conditions (31)–(32) in the form $X(x)T(t)$. By substituting this into Eqs. (30)–(32) it follows that

$$\frac{X^{(4)}}{X} = \frac{-T''}{T+\beta T'} = \lambda, \tag{34}$$

where $\lambda \in \mathbb{C}$ is a separation constant. Note that the case $T + \beta T' = 0$ only leads to trivial solutions. By considering the boundary conditions (31)–(32) a boundary value problem for $X(x)$ is obtained:

$$X^{(4)}(x) - \lambda X(x) = 0, \tag{35}$$

$$X(0) = X'(0) = X''(1) = 0, \tag{36}$$

$$X'''(1) + \gamma\lambda X(1) = 0, \tag{37}$$

and the following problem for $T(t)$:

$$T''(t) + \lambda(T(t) + \beta T'(t)) = 0, \tag{38}$$

where $\lambda \in \mathbb{C}$ is a separation constant. The boundary value problem (35)–(37) is the same as problem (25)–(27) with $\varepsilon = \beta = 0$. So the eigenvalues are real-valued and positive; the eigenfunctions can be chosen to be real-valued, and two real-valued eigenfunctions belonging to two different eigenvalues are orthogonal with respect to the inner product (29). Moreover, problem (35)–(37) can be solved analytically. The eigenvalues $\lambda_n = \mu_n^4$ are implicitly given by the roots of

$$h_\gamma(\mu) \equiv 1 + \cosh(\mu)\cos(\mu) + \gamma\mu(\cos(\mu)\sinh(\mu) - \cosh(\mu)\sin(\mu)) = 0, \tag{39}$$

which is equivalent to

$$\tan(\mu) = \frac{(\cos(\mu) + \cosh(\mu) + \gamma\mu\sinh(\mu))}{(\gamma\mu\cosh(\mu) - \sin(\mu))}. \tag{40}$$

The real-valued, positive roots of $h_\gamma(\mu)$ are denoted by μ_n. It can be deduced that $(n-1)\pi < \mu_n < n\pi$, with $n \in \{1, 2, 3, \ldots\}$, the elementary proof will be omitted here. For similar proofs the reader is referred to Ref. [20]. So, there are infinitely many isolated, real-valued, and positive eigenvalues. Definition (39) will have the following approximate form (for large μ) $h_\gamma(\mu) \approx (\gamma/2)\mu e^\mu \cos(\mu)(1 + (1/\gamma\mu) - \tan(\mu))$ and $\mu_n \to (n - \frac{3}{4})\pi$ for $n \to \infty$ and for $\gamma \neq 0$.

The eigenfunctions of the problem (35)–(37) can be determined, and are given by

$$\hat{\phi}_n(x) = \sin(\mu_n x) - \sinh(\mu_n x) + \beta_n(\cosh(\mu_n x) - \cos(\mu_n x)), \tag{41}$$

where

$$\beta_n = \frac{\sin(\mu_n) + \sinh(\mu_n)}{\cos(\mu_n) + \cosh(\mu_n)}.$$

If the tip-mass is zero the eigenvalues and the eigenfunctions are given by Eqs. (39) and (41), respectively, with $\gamma = 0$. These eigenfunctions are also orthogonal with respect to the inner product (29) with $\gamma = 0$, and $\mu_n \approx (n - \frac{1}{2})\pi$ (for large n).

For each eigenvalue the function $T_n(t)$ can be determined from Eq. (38). So infinitely many non-trivial solutions of the initial-boundary problem (30)–(33) have been determined. By using the superposition principle the solution of the initial-boundary value problem is obtained

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\phi_n(x), \tag{42}$$

where

$$T_n(t) = \begin{cases} e^{-(\beta\lambda_n/2)t}(A_n\cos(\sigma_n t) + B_n\sin(\sigma_n t)) & \text{if } \beta^2\lambda_n < 4, \\ (A_n + B_n t)e^{\frac{-2}{\beta}t} & \text{if } \beta^2\lambda_n = 4, \\ A_n e^{\omega_{n_1}t} + B_n e^{\omega_{n_2}t} & \text{if } \beta^2\lambda_n > 4 \end{cases} \tag{43}$$

with

$$\sigma_n = \sqrt{\lambda_n - \left(\frac{\beta\lambda_n}{2}\right)^2}, \tag{44}$$

$$\omega_{n_{1,2}} = -\frac{\beta\lambda_n}{2} \pm \frac{1}{2}\sqrt{\beta^2\lambda_n^2 - 4\lambda_n}, \tag{45}$$

where $\phi_n(x)$ is the normalized eigenfunction

$$\phi_n(x) = \frac{\hat{\phi}_n(x)}{\langle\hat{\phi}_n(x), \hat{\phi}_n(x)\rangle^{1/2}}, \tag{46}$$

where $\hat{\phi}_n(x)$ is given by Eq. (41), and where A_n and B_n are constants. The constants A_n and B_n are determined by the initial displacement $f(x)$ and the initial velocity $g(x)$ in the following way:

$$A_n = \int_0^1 [1 + \gamma\delta(x-1)] f(x)\phi_n(x)\,\mathrm{d}x, \tag{47}$$

$$\sigma_n B_n = \int_0^1 [1 + \gamma\delta(x-1)]\left(g(x) + \frac{\beta\lambda_n}{2} f(x)\right)\phi_n(x)\,\mathrm{d}x \tag{48}$$

if $\beta^2\lambda_n < 4$,

$$a_n = \int_0^1 [1 + \gamma\delta(x-1)] f(x)\phi_n(x)\,\mathrm{d}x, \tag{49}$$

$$B_n = \int_0^1 [1 + \gamma\delta(x-1)]\left(g(x) + \frac{2}{\beta} f(x)\right)\phi_n(x)\,\mathrm{d}x \tag{50}$$

if $\beta^2\lambda_n = 4$, and

$$A_n = \frac{\int_0^1 [1 + \gamma\delta(x-1)](\omega_{n_2} f(x) - g(x))\phi_n(x)\,\mathrm{d}x}{\sqrt{\beta^2\lambda_n^2 - 4\lambda_n}}, \tag{51}$$

$$B_n = \frac{\int_0^1 [1 + \gamma\delta(x-1)](g(x) - \omega_{n_1} f(x))\phi_n(x)\,\mathrm{d}x}{\sqrt{\beta^2\lambda_n^2 - 4\lambda_n}} \tag{52}$$

if $\beta^2\lambda_n > 4$. The eigenfunctions $\phi_n(x)$ form an orthonormal set with respect to the inner product (29). After lengthy but elementary calculations it can be shown that

$$\langle\hat{\phi}_n(x), \hat{\phi}_n(x)\rangle = \left(\frac{\sin(\mu_n) + \sinh(\mu_n)}{\cos(\mu_n) + \cosh(\mu_n)}\right)^2 + \gamma\left(\frac{\sin(\mu_n)\cosh(\mu_n) - \cos(\mu_n)\sinh(\mu_n)}{\mu_n(\cos(\mu_n) + \cosh(\mu_n))}\right)^2, \tag{53}$$

and it can be shown that $\langle\hat{\phi}_n(x), \hat{\phi}_n(x)\rangle \to 1$ if $n \to \infty$. In Section 5 this property will be used to determine the type of damping.

4. The energy and the boundedness of solutions

The energy of the cantilevered beam with a tip-mass but with no wind force applied to it (i.e. problem (5)–(10), with $\alpha = 0$) is defined to be

$$E(t) \equiv \int_0^1 \frac{1}{2}(u_t^2(x,t) + u_{xx}^2(x,t) - \varepsilon(\gamma + 1 - x)u_x^2(x,t))\,\mathrm{d}x + \frac{1}{2}\gamma u_t^2(1,t). \tag{54}$$

The time derivative of the energy is given by $\mathrm{d}E/\mathrm{d}t = -\varepsilon c u_t^2(1,t) - \beta\int_0^1 u_{xxt}^2(x,t)\,\mathrm{d}x$, where c is the (boundary) damping parameter, and where β is the coefficient of Kelvin–Voigt viscoelastic damping. So, the energy is bounded if the initial energy is bounded and $\varepsilon(\gamma + \frac{1}{2}) < 1$ (see also Eq. (28)). The existence of a solution of $u(x,t)$ is assumed, where $u(x,t)$ is a twice continuously differentiable function with respect to t and a four times continuously differentiable function with respect to x. A proof of this assumption is beyond the scope of this paper. Since $u_x(x,t)$ and $u_{xx}(x,t)$ are continuous it follows that $u(x,t) = \int_0^x u_\xi(\xi,t)\,\mathrm{d}\xi$ and $u_x(x,t) = \int_0^x u_{\xi\xi}(\xi,t)\,\mathrm{d}\xi$. It then can be deduced by using the Cauchy–Schwarz inequality that (see also Ref. [18] for a similar estimate)

$$|u_x(x,t)| \leqslant \int_0^1 |u_{xx}(x,t)|\,\mathrm{d}x \leqslant \sqrt{\int_0^1 u_{xx}^2(x,t)\,\mathrm{d}x} \leqslant \sqrt{\frac{2E(t)}{(1 - \varepsilon(\gamma + \frac{1}{2}))}} \leqslant \sqrt{\frac{2E(0)}{(1 - \varepsilon(\gamma + \frac{1}{2}))}}, \tag{55}$$

where it has been assumed that $\varepsilon(\gamma+\frac{1}{2})<1$. By using $u(x,t)=\int_0^x u_\xi(\xi,t)\,\mathrm{d}\xi$ the following inequality for $|u(x,t)|$ can be derived similarly

$$|u(x,t)|\leqslant\int_0^1|u_x(x,t)|\,\mathrm{d}x\leqslant\int_0^1\sqrt{\frac{2E(0)}{(1-\varepsilon(\gamma+\frac{1}{2}))}}\mathrm{d}x=\sqrt{\frac{2E(0)}{(1-\varepsilon(\gamma+\frac{1}{2}))}}. \tag{56}$$

So, also $u(x,t)$ is bounded if the initial energy is bounded and $\varepsilon(\gamma+\frac{1}{2})<1$.

5. Formal approximations

In this section an approximation of the solution of the initial-boundary value problem (5)–(10) will be constructed. A two-time-scales perturbation method will be used. Conditions like $t>0, t\geqslant 0, 0<x<1$ will be dropped, for abbreviation. Expand the solution in a Taylor series with respect to ε, to obtain

$$u(x,t;\varepsilon)=\hat{u}_0(x,t)+\varepsilon\hat{u}_1(x,t)+\varepsilon^2\hat{u}_2(x,t)+\cdots. \tag{57}$$

It is assumed that the functions $\hat{u}_i(x,t)$ are $\mathcal{O}(1)$. The approximation of the solution will contain secular terms. Since the $\hat{u}_i(x,t)$ are assumed to be $\mathcal{O}(1)$, and because the solutions are bounded on timescales of $\mathcal{O}(\varepsilon^{-1})$, secular terms should be avoided when approximations are constructed on long timescales of $\mathcal{O}(\varepsilon^{-1})$. That is why a two-time-scales perturbation method is applied. By using such a two-time-scales perturbation method the function $u(x,t)$ is supposed to be a function of x, t, and $\tau=\varepsilon t$. So put

$$u(x,t)=w(x,t,\tau;\varepsilon). \tag{58}$$

A result of this is

$$u_t=w_t+\varepsilon w_\tau, \tag{59}$$

$$u_{tt}=w_{tt}+2\varepsilon w_{t\tau}+\varepsilon^2 w_{\tau\tau}. \tag{60}$$

Substitution of Eqs. (58)–(60) into the problem (5)–(10) yields an initial-boundary value problem for $w(x,t,\tau)$. Assuming that

$$w(x,t,\tau;\varepsilon)=u_0(x,t,\tau)+\varepsilon u_1(x,t,\tau)+\varepsilon^2 u_2(x,t,\tau)+\cdots, \tag{61}$$

then by collecting terms of equal powers in ε it follows from the problem for $w(x,t,\tau)$ that the $\mathcal{O}(1)$-problem is

$$u_{0_{xxxx}}+\beta u_{0_{xxxxt}}+u_{0_{tt}}=0, \tag{62}$$

$$u_0(0,t,\tau)=u_{0_x}(0,t,\tau)=0, \tag{63}$$

$$u_{0_{xx}}(1,t,\tau)+\beta u_{0_{xxt}}(1,t,\tau)=0, \tag{64}$$

$$u_{0_{xxx}}(1,t,\tau)+\beta u_{0_{xxxt}}(1,t,\tau)-\gamma u_{0_{tt}}(1,t,\tau)=0, \tag{65}$$

$$u_0(x,0,0)=f(x)\quad\text{and}\quad u_{0_t}(x,0,0)=g(x), \tag{66}$$

and that the $\mathcal{O}(\varepsilon)$-problem is

$$u_{1_{xxxx}}+\beta u_{1_{xxxxt}}+u_{1_{tt}}=\alpha u_{0_t}-[(\gamma+1-x)u_{0_x}]_x-2u_{0_{t\tau}}-\beta u_{0_{xxxx\tau}}, \tag{67}$$

$$u_1(0,t,\tau)=u_{1_x}(0,t,\tau)=0, \tag{68}$$

$$u_{1_{xx}}(1,t,\tau)+\beta u_{1_{xxt}}(1,t,\tau)=-\beta u_{0_{xx\tau}}(1,t,\tau), \tag{69}$$

$$u_{1_{xxx}}(1,t,\tau)+\beta u_{1_{xxxt}}(1,t,\tau)=\gamma u_{1_{tt}}(1,t,\tau)-\gamma u_{0_x}(1,t,\tau)-\beta u_{0_{xxx\tau}}(1,t,\tau)+2\gamma u_{0_{t\tau}}(1,t,\tau)+cu_{0_t}(1,t,\tau), \tag{70}$$

$$u_1(x,0,0)=0, \tag{71}$$

$$u_{1_t}(x,0,0)=-u_{0_\tau}(x,0,0). \tag{72}$$

The solution of the $\mathcal{O}(1)$-problem (62)–(66) has been determined in Section 3 and is given by

$$u_0(x,t,\tau) = \sum_{n=1}^{\infty} T_{0n}(t,\tau)\phi_n(x), \tag{73}$$

where

$$T_{0n}(t,\tau) = \begin{cases} e^{(-\beta\lambda_n/2)t}(A_{0n}(\tau)\cos(\sigma_n t) + B_{0n}(\tau)\sin(\sigma_n t)) & \text{if } \beta^2\lambda_n < 4, \\ (A_{0n}(\tau) + B_{0n}(\tau)t)e^{(-2/\beta)t} & \text{if } \beta^2\lambda_n = 4, \\ A_{0n}(\tau)e^{\omega_{n_1}t} + B_{0n}(\tau)e^{\omega_{n_2}t} & \text{if } \beta^2\lambda_n > 4, \end{cases} \tag{74}$$

and where σ_n, ω_{n_1}, ω_{n_2}, the orthonormal eigenfunction $\phi_n(x)$ corresponding to λ_n, $A_{0n}(0)$, and $B_{0n}(0)$ are given by Eqs. (44)–(52). Now the solution of the $\mathcal{O}(\varepsilon)$-problem will be determined. The problem (67)–(72) has an inhomogeneous boundary condition. For classical inhomogeneous boundary conditions the inhomogeneous boundary conditions are made homogeneous. However, for inhomogeneous non-classical boundary conditions such as Eq. (70) a different procedure has to be followed. In fact, a transformation will be used such that the partial differential equation and the inhomogeneous boundary condition, after the transformation, "match"; if a solution which is expanded in eigenfunctions $\phi_n(x)$, defined by Eq. (41), satisfies the transformed partial differential equation it immediately satisfies the transformed inhomogeneous boundary condition. A similar "matching" for a non-self-adjoint string-like problem has been introduced in Ref. [1]. Introduce the following transformation:

$$u_1(x,t,\tau) = v(x,t,\tau) + \left(\frac{-x^2}{2} + \frac{x^3}{6}\right)h(t,\tau). \tag{75}$$

By substituting the latter transformation into Eqs. (67)–(72) it follows that

$$v_{xxxx} + \beta v_{xxxxt} + v_{tt} = \alpha u_{0_t} - [(\gamma + 1 - x)u_{0_x}]_x - 2u_{0_{t\tau}} - \beta u_{0_{xxxx\tau}} - \left(\frac{-x^2}{2} + \frac{x^3}{6}\right)h_{tt}(t,\tau), \tag{76}$$

$$v(0,t,\tau) = v_x(0,t,\tau) = 0, \tag{77}$$

$$v_{xx}(1,t,\tau) + \beta v_{xxt}(1,t,\tau) = 0, \tag{78}$$

$$\begin{aligned} v_{xxx}(1,t,\tau) + \beta v_{xxxt}(1,t,\tau) &= \gamma v_{tt}(1,t,\tau) - \gamma u_{0_x}(1,t,\tau) - \beta u_{0_{xxx\tau}}(1,t,\tau) + 2\gamma u_{0_{t\tau}}(1,t,\tau) + c u_{0_t}(1,t,\tau) \\ &\quad - h(t,\tau) - \beta h_t(t,\tau) - \frac{\gamma}{3}h_{tt}(t,\tau), \end{aligned} \tag{79}$$

$$v(x,0,0) = -\left(\frac{-x^2}{2} + \frac{x^3}{6}\right)h(0,0), \tag{80}$$

$$v_t(x,0,0) = -u_{0_\tau}(x,0,0) - \left(\frac{-x^2}{2} + \frac{x^3}{6}\right)h_t(0,0). \tag{81}$$

Introduce the following infinite sum for $v(x,t,\tau)$:

$$v(x,t,\tau) = \sum_{n=1}^{\infty} v_n(t,\tau)\phi_n(x), \tag{82}$$

and substitute the infinite sum into the partial differential equation (76) and into the boundary condition (79) to obtain

$$\sum_{n=1}^{\infty}(v_{n_{tt}} + \lambda_n(v_n + \beta v_{n_t}))\phi_n(x) = \alpha u_{0_t} - [(\gamma + 1 - x)u_{0_x}]_x - 2u_{0_{t\tau}} - \beta u_{0_{xxxx\tau}} - \left(\frac{-x^2}{2} + \frac{x^3}{6}\right)h_{tt}(t,\tau), \tag{83}$$

and

$$\sum_{n=1}^{\infty}(v_n+\beta v_{n_t})\phi_{n_{xxx}}(1)-\gamma v_{n_{tt}}\phi_n(1)=-\gamma u_{0_x}(1,t,\tau)-\beta u_{0_{xxx\tau}}(1,t,\tau)+2\gamma u_{0_{t\tau}}(1,t,\tau)+cu_{0_t}(1,t,\tau)$$
$$-h-\beta h_t-\frac{\gamma}{3}h_{tt}, \tag{84}$$

respectively. Note that the dependency of $v_n(t,\tau)$, $T_{0n}(t,\tau)$, and $h(t,\tau)$ on t,τ have been dropped for abbreviation. Now the function $h(t,\tau)$ will be determined. By letting x tend to $x=1$ in Eq. (83), by using the first boundary condition in $x=1$ (i.e. $\phi_{n_{xx}}(1)=0$), and by multiplying the so-obtained result by γ, it follows that

$$\gamma\sum_{n=1}^{\infty}(v_{n_{tt}}+\lambda_n(v_n+\beta v_{n_t}))\phi_n(1)=\alpha\gamma u_{0_t}(1,t,\tau)+\gamma u_{0_x}(1,t,\tau)-2\gamma u_{0_{t\tau}}(1,t,\tau)-\beta\gamma u_{0_{xxxx\tau}}(1,t,\tau)+\frac{\gamma}{3}h_{tt}(t,\tau). \tag{85}$$

Now by adding Eqs. (84) and (85), and by using the second boundary condition in $x=1$ (i.e. $\phi_{n_{xxx}}(1)+\gamma\lambda_n\phi_n(1)=0$) and Eq. (35) in $x=1$ (i.e. $\phi_{n_{xxxx}}(1)=\lambda_n\phi_n(1)$) it follows that $h(t,\tau)$ satisfies the following first-order differential equation:

$$h+\beta h_t-(c+\alpha\gamma)u_{0_t}(1,t,\tau)=0. \tag{86}$$

From Eqs. (38), (73), and (86) $h(t,\tau)$ and $h_{tt}(t,\tau)$ can be determined, yielding

$$h(t,\tau)=\tilde{g}(\tau)\mathrm{e}^{-t/\beta}+(c+\alpha\gamma)\sum_{n=1}^{\infty}(\beta\lambda_n T_{0n}+T_{0n_t}))\phi_n(1), \tag{87}$$

$$h_{tt}(t,\tau)=\frac{\tilde{g}(\tau)}{\beta^2}\mathrm{e}^{-t/\beta}-(c+\alpha\gamma)\sum_{n=1}^{\infty}\lambda_n T_{0n_t}\phi_n(1), \tag{88}$$

respectively, and where $\tilde{g}(\tau)$ is an arbitrary function in τ. From now on let $\tilde{g}(\tau)$ be equal to zero, that is, $\tilde{g}(\tau)\equiv 0$. Note that in this way $h(t,\tau)$ is a transformation such that Eqs. (76) and (79) "match". The function $h_{tt}(t,\tau)$ will be used to obtain a differential equation for $v_m(t,\tau)$. Now a differential equation will be obtained for $v_m(t,\tau)$. Eq. (83) can be used to obtain this differential equation for $v_m(t,\tau)$ after expanding $((-x^2/2)+(x^3/6))$ in series of orthonormal eigenfunctions $\phi_n(x)$:

$$\frac{-x^2}{2}+\frac{x^3}{6}=\sum_{n=1}^{\infty}C_n\phi_n(x), \tag{89}$$

where

$$C_n=\int_0^1[1+\gamma\delta(x-1)]\left(\frac{-x^2}{2}+\frac{x^3}{6}\right)\phi_n(x)\,\mathrm{d}x. \tag{90}$$

By using integration by parts and by using that $\phi_n(x)$ is a solution of problem (35)–(37), with $\lambda=\lambda_n$, it follows that

$$C_n=-\frac{\phi_n(1)}{\lambda_n}. \tag{91}$$

Multiply Eq. (83) by $(1+\gamma\delta(x-1))\phi_m(x)$, integrate the so-obtained result with respect to x form 0 to 1, use that the eigenfunctions $\phi_n(x)$ are orthogonal with respect to the inner product (29), and use Eqs. (88) and (91), to obtain

$$v_{m_{tt}}+\lambda_m(v_m+\beta v_{m_t})=-2T_{0m_{t\tau}}-\beta\lambda_m T_{0m_\tau}+2\kappa_m T_{0m_t}+\Theta_{mm}T_{0m}$$
$$+\sum_{n=1,n\neq m}^{\infty}\left(\Theta_{nm}T_{0n}-(c+\alpha\gamma)\phi_n(1)\phi_m(1)\frac{\lambda_n}{\lambda_m}T_{0n_t}\right), \tag{92}$$

where

$$\kappa_m = \frac{\alpha}{2} - \frac{1}{2}(c + \gamma\alpha)\phi_m^2(1), \tag{93}$$

where $T_{0m}(t,\tau)$ is given by Eq. (74), and where $\Theta_{mn} = \int_0^1 (\gamma + 1 - x)\phi_{m_x}(x)\phi_{n_x}(x)\,\mathrm{d}x$. In Ref. [14] explicit expressions for Θ_{nm} have been obtained for the case $\gamma = 0$. From Eq. (74) it follows that $T_{0m}(t,\tau)$ and $T_{0m_t}(t,\tau)$ are solutions of the homogeneous equation corresponding to Eq. (92), and that $T_{0n}(t,\tau)$ and $T_{0n_t}(t,\tau)$ with $n \neq m$ are not solutions of the homogeneous equation corresponding to Eq. (92). Therefore, the right-hand side of Eq. (92) contains terms which are solutions of the homogeneous equation corresponding to Eq. (92). These terms will give rise to unbounded terms, the so-called secular terms, in the solution $v_m(t,\tau)$ of Eq. (92). Since it is assumed in the asymptotic expansions that the functions $u_0(x,t,\tau), u_1(x,t,\tau), u_2(x,t,\tau), \ldots$ are bounded on timescales of $\mathcal{O}(\varepsilon^{-1})$ these secular terms should be avoided. In $T_{0m}(t,\tau)$ the functions $A_{0m}(\tau)$ and $B_{0m}(\tau)$ are still undetermined. These functions will be used to avoid secular terms in the solution of Eq. (92) in the following way. Let the sum of the terms in the right-hand side of Eq. (92) that give rise to secular terms in the solution of Eq. (92) be equal to zero, yielding

$$-2T_{0m_{t\tau}} - \beta\lambda_m T_{0m_\tau} + 2\kappa_m T_{0m_t} + \Theta_{mm} T_{0m} = 0. \tag{94}$$

By substituting $T_{0m}(t,\tau)$, given by Eq. (74), into Eq. (94) (coupled) differential equations for the functions $A_{0m}(\tau)$ and $B_{0m}(\tau)$ can be obtained. From Eq. (74) it follows that $T_{0m}(t,\tau)$ for the case $\beta^2\lambda_m < 4$, $T_{0m}(t,\tau)$ for the case $\beta^2\lambda_m = 4$, and $T_{0m}(t,\tau)$ for the case $\beta^2\lambda_m > 4$ are given in a qualitatively different way. Therefore, from Eq. (94), it follows that qualitatively different differential equations for $A_{0m}(\tau)$ and $B_{0m}(\tau)$ will be obtained for these cases. Now the case $\beta^2\lambda_m < 4$, the case $\beta^2\lambda_m = 4$, and the case $\beta^2\lambda_m > 4$ will be considered.

At first, the case $\beta^2\lambda_m = 4$ will be considered. By substituting $T_{0m}(t,\tau) = (A_{0m}(\tau) + B_{0m}(\tau)t)\mathrm{e}^{(-2/\beta)t}$ into Eq. (94) equations for $A_{0m}(\tau)$ and $B_{0m}(\tau)$ can be obtained. These equations cannot be used to obtain an approximation of the solution of problem (5)–(10). The reason for this is that for the case $\beta^2\lambda_m = 4$ it cannot be expected that the solution of the unperturbed problem (5)–(10) can be expanded in a Taylor series with respect to ε. To show this a so-called auxiliary equation will be introduced. Suppose that the solution of Eq. (38) is given by $T(t) = \mathrm{e}^{rt}$, where r is a parameter to be determined. By substituting $T(t) = \mathrm{e}^{rt}$ into Eq. (38) the auxiliary equation is obtained, given by

$$r^2 + \beta\lambda r + \lambda = 0, \tag{95}$$

where $\lambda > 0$. Now consider the following equation:

$$r^2(\varepsilon) + \beta\lambda(\varepsilon)r(\varepsilon) + \lambda(\varepsilon) = 0, \tag{96}$$

where $\lambda(\varepsilon)$ depends smoothly on ε and where $\lambda(0) = \lambda$. Then Eq. (95) is the corresponding unperturbed equation of Eq. (96). From the implicit function theorem it follows that if

$$2r(0) + \beta\lambda(0) = 0, \tag{97}$$

it cannot be expected that the root $r(\varepsilon)$ of Eq. (96) can be expanded in a Taylor series with respect to ε (see also Ref. [21, Chapter 10]), and that there may be bifurcation solutions. From Eq. (95) it follows that $2r(0) + \beta\lambda(0) = 0$ if $\beta^2\lambda(0) = 4$. From $2r(0) + \beta\lambda(0) = 0$ and $\beta^2\lambda(0) = 4$ it follows that $r(0) = -2/\beta$. Now, it also follows that $r(0) = -2/\beta$ is a bifurcation point. For different values of the parameters β and λ the solution of Eq. (95) will be qualitatively different. Now assume that λ_m is an eigenvalue of the unperturbed problem (i.e. (Eqs. (5)–(10)) with $\varepsilon = 0$) such that $\beta^2\lambda_m = 4$. Then it cannot be expected that the solution of the perturbed problem (i.e. Eqs. (5)–(10)) can be expanded in a Taylor series with respect to ε. To find an approximation of the solution of problem (5)–(10) for the case $\beta^2\lambda_m = 4$ a very different expansion will be needed. Therefore, the case $\beta^2\lambda_m = 4$ will not be considered any further in this paper.

Now the case $\beta^2\lambda_m < 4$ will be considered. By substituting $T_{0m}(t,\tau) = \mathrm{e}^{(-\beta\lambda_m/2)t}(A_{0m}(\tau)\cos(\sigma_m t) + B_{0m}(\tau)\sin(\sigma_m t))$ into Eq. (94), it follows that $A_{0m}(\tau)$ and $B_{0m}(\tau)$ are solutions of the following system of coupled differential equations:

$$\frac{\mathrm{d}A_{0m}}{\mathrm{d}\tau} = \kappa_m A_{0m} - \Omega_m B_{0m}, \tag{98}$$

$$\frac{\mathrm{d}B_{0m}}{\mathrm{d}\tau} = \kappa_m B_{0m} + \Omega_m A_{0m}, \tag{99}$$

where

$$\Omega_m = \left(\frac{\Theta_{mm} - \beta\lambda_m\kappa_m}{2\sigma_m}\right), \tag{100}$$

where κ_m is given by Eq. (93), σ_m by Eq. (44), $\Theta_{mn} = \int_0^1(\gamma + 1 - x)\phi_{m_x}(x)\phi_{n_x}(x)\,\mathrm{d}x$, $\lambda_m = \mu_m^4$, and where μ_m is the mth positive root of Eq. (39). From Eqs. (98) and (99) $A_{0m}(\tau)$ and $B_{0m}(\tau)$ can be determined, yielding

$$A_{0m}(\tau) = \mathrm{e}^{\kappa_m\tau}(A_{0m}(0)\cos(\Omega_m\tau) - B_{0m}(0)\sin(\Omega_m\tau)), \tag{101}$$

$$B_{0m}(\tau) = \mathrm{e}^{\kappa_m\tau}(B_{0m}(0)\cos(\Omega_m\tau) + A_{0m}(0)\sin(\Omega_m\tau)), \tag{102}$$

where $A_{0m}(0)$ and $B_{0m}(0)$ are given by Eqs. (47) and (48), respectively. Hence, for $\beta^2\lambda_m < 4$, $T_{0m}(t,\tau)$ is found to be

$$T_{0m}(t,\tau) = \mathrm{e}^{-(\beta\lambda_m/2)t + \kappa_m\tau}(A_{0m}(0)\cos(\sigma_m t - \Omega_m\tau) + B_{0m}(0)\sin(\sigma_m t - \Omega_m\tau)). \tag{103}$$

Now by substituting $\tau = \varepsilon t$ and Eq. (93) into $-(\beta\lambda_m/2)t + \kappa_m\tau$ and by dividing the so-obtained result by t it follows that the damping coefficient ($\theta_{1,m}$), for $\beta^2\lambda_m < 4$, can be approximated by

$$\theta_{1,m} = -\tfrac{1}{2}(\beta\lambda_m - \varepsilon\alpha + \varepsilon(c + \gamma\alpha)\phi_m^2(1)), \tag{104}$$

where

$$\phi_m^2(1) = \frac{4}{1 + \gamma + \gamma^2\mu_m^2\left(\dfrac{2\sin(\mu_m)\sinh(\mu_m)}{1 + \cos(\mu_m)\cosh(\mu_m)}\right)}. \tag{105}$$

From Eq. (44), Eq. (100), and $t = \varepsilon\tau$ it follows that the frequency ($\theta_{2,m}$) can be approximated by

$$\theta_{2,m} = \sqrt{\lambda_m - \left(\frac{\beta\lambda_m}{2}\right)^2} - \varepsilon\left(\frac{\Theta_{mm} - \beta\lambda_m\kappa_m}{2\sigma_m}\right). \tag{106}$$

Now the effect of gravity on the frequency will be considered. By lengthy but elementary calculations it can be shown that the quotient $\Theta_{nn}/2\sigma_n$ is given by (see Ref. [14] for a similar expression)

$$\frac{\Theta_{nn}}{2\sigma_n} = \frac{1}{4\sigma_n}((1 + \mu_n\chi_n)^2 + 3) + \frac{\gamma\mu_n}{2\sigma_n}\left(\gamma\mu_n\left(\frac{s(\mu_n)}{\cos(\mu_n) + \cosh(\mu_n)}\right)^2 + \mu_n\chi_n^2 - 2\chi_n\right), \tag{107}$$

where

$$\chi_n = \frac{\sin(\mu_n) - \sinh(\mu_n)}{\cos(\mu_n) + \cosh(\mu_n)},$$

and where $s(\mu) = \sin(\mu)\cosh(\mu) - \cos(\mu)\sinh(\mu)$. Since $\chi_n \to -1$ and $s(\mu_n) \to 0$ for $n \to \infty$, and since $\sigma_n = \mu_n^2$ if $\beta = 0$ it follows that $\Theta_{nn}/2\sigma_n = \mathcal{O}(1)$ if $\beta = 0$. The compression force due to gravity, the self-weight of the beam, and the mass of the tip-mass is represented by the integral $\varepsilon\Theta_{nm}$. This integral shows up in Eq. (106) and does not show up in Eq. (104). Hence, the compression force does not have a significant effect on the damping rates of the oscillation modes, but only has a significant effect on the frequency of the oscillation modes. Since $\Theta_{nn} > 0$ it follows that the frequency reduces by increasing mass of the tip-mass, that is, by increasing γ and by increasing the mass of the beam itself, that is, by increasing ε.

Lastly, the case $\beta^2\lambda_m > 4$ will be considered. By substituting $T_{0m}(t,\tau) = A_{0n}(\tau)\mathrm{e}^{\omega_{n_1}t} + B_{0n}(\tau)\mathrm{e}^{\omega_{n_2}t}$ into Eq. (94), it follows that $A_{0m}(\tau)$ and $B_{0m}(\tau)$ are solutions of the following differential equations:

$$\frac{\mathrm{d}A_{0m}}{\mathrm{d}\tau} = \frac{2\kappa_m\omega_{m_1} + \Theta_{mm}}{2\omega_{m_1} + \beta\lambda_m}A_{0m}, \tag{108}$$

$$\frac{\mathrm{d}B_{0m}}{\mathrm{d}\tau}=\frac{2\kappa_m\omega_{m_2}+\Theta_{mm}}{2\omega_{m_2}+\beta\lambda_m}B_{0m}, \tag{109}$$

where $\omega_{m_{1,2}}$ and κ_m are given by Eqs. (45) and (93), respectively. From Eqs. (108) and (109) $A_{0m}(\tau)$ and $B_{0m}(\tau)$ can be determined, yielding

$$A_{0m}(\tau)=A_{0m}(0)\exp\left(\frac{(2\kappa_m\omega_{m_1}+\Theta_{mm})\tau}{\mu_m^2\sqrt{\beta^2\lambda_m-4}}\right),$$

$$B_{0m}(\tau)=B_{0m}(0)\exp\left(\frac{-(2\kappa_m\omega_{m_2}+\Theta_{mm})\tau}{\mu_m^2\sqrt{\beta^2\lambda_m-4}}\right),$$

where $A_m(0)$ and $B_m(0)$ are given by Eqs. (51) and (52), respectively. Hence, for $\beta^2\lambda_m>4$, $T_{0m}(t,\tau)$ is found to be

$$T_{0m}(t,\tau)=A_{0m}(0)\exp\left(\omega_{m_1}t+\frac{(2\kappa_m\omega_{m_1}+\Theta_{mm})\tau}{\mu_m^2\sqrt{\beta^2\lambda_m-4}}\right)+B_{0m}(0)\exp\left(\omega_{m_2}t-\frac{(2\kappa_m\omega_{m_2}+\Theta_{mm})\tau}{\mu_m^2\sqrt{\beta^2\lambda_m-4}}\right). \tag{110}$$

The damping properties of $T_{0m}(t,\tau)$ will now be considered. From Eq. (110) and $\tau=\varepsilon t$ it follows that the damping coefficients ($d_{m_{1,2}}$) of $T_{0m}(t,\tau)$ can be approximated by

$$d_{m_{1,2}}=\left(1\pm\frac{2\varepsilon\kappa_m}{\mu_m^2\sqrt{\beta^2\lambda_m-4}}\right)\omega_{m_{1,2}}\pm\left(\frac{\varepsilon\Theta_{mm}}{\mu_m^2\sqrt{\beta^2\lambda_m-4}}\right). \tag{111}$$

Now it will be shown that there exist a constant $\hat{d}<0$ such that $d_{m_{1,2}}<\hat{d}<0$ for all $m\in\mathbb{N}$ with $\beta^2\lambda_m>4$. This property of the damping rates will be used to obtain the type of damping of the problem (5)–(10). From Eq. (45) it follows that there exists an ε-independent constant $\hat{\omega}<0$ such that $\omega_{m_{1,2}}<\hat{\omega}<0$ for all $m\in\mathbb{N}$ with $\beta^2\lambda_m>4$. From Eqs. (93) and (107) it follows that $\kappa_m/\mu_m^2=\mathcal{O}(1)$ and that $\Theta_{mm}/\mu_m^2=\mathcal{O}(1)$. Then there also exists an ε-independent constant $\hat{d}<0$ such that $d_{m_{1,2}}<\hat{d}<0$ for all $m\in\mathbb{N}$ with $\beta^2\lambda_m>4$. Furthermore, it follows from Eq. (111) that the compression force, which is related to Θ_{mm}, has a significant effect on the damping rates.

The functions $A_{0m}(\tau)$ and $B_{0m}(\tau)$ have been determined for the case $\beta^2\lambda_m\neq4$. So, an $\mathcal{O}(\varepsilon)$-approximation, given by Eq. (73), of the initial-boundary value problem (5)–(10) for the case $\beta^2\lambda_m\neq4$, valid on timescales of $\mathcal{O}(\varepsilon^{-1})$, has been determined. It is beyond the scope of this paper to prove that the $\mathcal{O}(\varepsilon)$-approximation are indeed valid on timescales of $\mathcal{O}(\varepsilon^{-1})$.

6. Damping results

In this section the damping properties of the wind-induced vibrations of a weakly damped vertical beam with a tip-mass will be discussed. These vibrations are described by Eqs. (5)–(10). In the previous section an approximation of the solution of problem (5)–(10) for the case $\beta^2\lambda_m\neq4$ has been found and is given by Eq. (73), where $T_{0m}(t,\tau)$, for the case $\beta^2\lambda_m<4$, is given by Eq. (103), and where $T_{0m}(t,\tau)$, for the case $\beta^2\lambda_m>4$, is given by Eq. (110). The damping rates of the modes such that $\beta^2\lambda_m<4$ are given by Eq. (104) and the damping rates of the modes such that $\beta^2\lambda_m>4$ are given by Eq. (111). Now the modes of $u_0(x,t,\tau)$, given by Eq. (73), will be damped uniformly (i.e. exponentially) if there exist constants $\hat{\theta}$ and $\hat{d}$ such that $\theta_{1,m}<\hat{\theta}<0$ for all $m\in\mathbb{N}$ with $\beta^2\lambda_m<4$, and such that $d_{m_{1,2}}<\hat{d}<0$ for all $m\in\mathbb{N}$ with $\beta^2\lambda_m>4$. If such constants $\hat{\theta}$ or $\hat{d}$ do not exist, but $\theta_{1,m}<0$ for all $m\in\mathbb{N}$ with $\beta^2\lambda_m<4$, and $d_{m_{1,2}}<0$ for all $m\in\mathbb{N}$ with $\beta^2\lambda_m>4$, the modes will be

Table 1
Numerical approximations of $\phi_n^2(1)$ and of the damping coefficient $\theta_{1,n}$ for $\beta=0$ and $\gamma=1$

n	$\phi_n^2(1)$	$\theta_{1,n}$
1	0.80753	$\varepsilon\alpha/2-0.40376(c+\alpha)\varepsilon$
2	0.08998	$\varepsilon\alpha/2-0.04499(c+\alpha)\varepsilon$
3	0.03395	$\varepsilon\alpha/2-0.01698(c+\alpha)\varepsilon$
4	0.01717	$\varepsilon\alpha/2-0.00859(c+\alpha)\varepsilon$
5	0.01033	$\varepsilon\alpha/2-0.00516(c+\alpha)\varepsilon$
6	0.00688	$\varepsilon\alpha/2-0.00344(c+\alpha)\varepsilon$
7	0.00491	$\varepsilon\alpha/2-0.00246(c+\alpha)\varepsilon$
8	0.00368	$\varepsilon\alpha/2-0.00184(c+\alpha)\varepsilon$
9	0.00286	$\varepsilon\alpha/2-0.00143(c+\alpha)\varepsilon$
10	0.00228	$\varepsilon\alpha/2-0.00114(c+\alpha)\varepsilon$

damped strongly (i.e. asymptotically). In the last paragraph of the previous section is has been shown that there exist a constant $\hat{d}$ such that $d_{m_{1,2}}<\hat{d}<0$ for all m with $\beta^2\lambda_m>4$. So the modes of $u_0(x,t,\tau)$ with $\beta^2\lambda_m>4$ will be damped uniformly. Now the value of the damping coefficients $(\theta_{1,m})$ of the modes of $u_0(x,t,\tau)$ with $\beta^2\lambda_m<4$ will be considered for several values of the parameters β, c, γ, and α.

First, consider the case that the Kelvin–Voigt damping is not included (i.e. $\beta=0$). Hence $\beta^2\lambda_m<4$ and therefore Eq. (104) is the damping coefficient for all modes. Now if a beam without a tip-mass (i.e. $\gamma=0$) is considered, it follows that $\theta_{1,m}=(\alpha/2)-2c$. So, the oscillation modes of a vertical beam subjected to wind-forces will be damped uniformly if $c>\alpha/4$. And a vertical beam not subjected to wind-forces will be damped uniformly for every positive value of the damping parameter c.

Now the damping rates of a vertical beam with a tip-mass but not subjected to Kelvin–Voigt damping (i.e. $\gamma>0, \beta=0$) will be considered. Since $\mu_m\to(m-\frac{3}{4})\pi$ for $m\to\infty$ and for $\gamma>0$ it follows that

$$\left(\frac{\sin(\mu_m)\sinh(\mu_m)}{1+\cos(\mu_m)\cosh(\mu_m)}\right)\to 1 \quad \text{for } m\to\infty \text{ and for } \gamma>0.$$

Hence it follows from Eq. (105) that $\phi_m^2(1)\to 0$ for $m\to\infty$ and for $\gamma>0$. Now consider Eq. (104) where the parameter $\varepsilon\alpha$ is the negative damping due to the wind. If this wind-force is not included (i.e. $\alpha=0$) it can similarly be deduced that the damping rates $\theta_{1,m}$ tend to zero for $m\to\infty$. Hence, for this case, the modes will be damped strongly, but not uniformly, because c is a positive parameter and because $\theta_{1,m}\to 0$ for $m\to\infty$. The first ten damping coefficients for this case with $\gamma=1$ are listed in Table 1. If the wind-force is included (i.e. $\alpha>0$) not all modes of the wind-induced vibrations of the vertical beam will be damped by the boundary velocity damper, with damping parameter $c>0$. If γ (the ratio of the mass of the tip-mass and the mass of the beam) is a small parameter also $\gamma\mu_m$ will be small. Then the damping coefficients of the lower-order modes can be approximated by $\theta_m\approx(\alpha/2)-2c$. Hence the velocity damper will damp the lower modes if $c>\alpha/4$. However, a velocity damper is not sufficient to suppress the wind-induced modes of vibrations of a vertical beam with a tip-mass. In particular, the higher-order modes will hardly be damped.

Since low- and high-frequency vibrations can cause damage to a building it is important to have damping for all of the oscillation modes. Now the damping coefficients $\theta_{1,m}$ of a vertical beam with boundary damping, with Kelvin–Voigt damping, and with a tip-mass in a wind-field will be considered. It follows in this case that the modes will be damped uniformly if $\alpha<(\beta\mu_m^4/\varepsilon)+(c+\alpha\gamma)\phi_m^2(1)$ for all $m\in\mathbb{N}$, where $\mu_m\to(m-\frac{3}{4})\pi$ for $m\to\infty$ and where $(m-1)\pi<\mu_m<m\pi$ (see Section 3). So, if $\beta\mu_m^4>\varepsilon\alpha$ for $m=1$ the velocity damper is not necessary to obtain uniform damping. But if there exists an integer $M\geqslant 1$ such that $\beta\mu_m^4\leqslant\varepsilon\alpha$ for all $m\leqslant M$ and $\beta\mu_m^4>\varepsilon\alpha$ for all $m>M$ the velocity damper is necessary to obtain damping for the first M oscillation modes. These M modes will be damped uniformly if the damping parameter c is such that $(\beta\lambda_m/\varepsilon)+(c+\alpha\gamma)\phi_m^2(1)>\alpha$ for all $m\leqslant M$.

7. Conclusions

In this paper a weakly damped vertical beam with and without a tip-mass in a wind-field has been considered. Boundary damping and global Kelvin–Voigt damping have been considered. The boundary damping is assumed to be proportional to the velocity of the beam at the top. By using the energy integral it has been shown that the solutions (assuming the existence of a sufficiently smooth solution) are bounded in absence of a windforce. Explicit asymptotic approximations of the solutions have been derived. The damping rates for several cases have been considered. It has been shown that if the damping parameter is large enough (i.e. $c > \alpha/4$) that the wind-induced vibrations of a vertical beam without tip-mass and without Kelvin–Voigt damping will be damped uniformly. The vibrations of a vertical beam with a tip-mass but without Kelvin–Voigt damping and not subjected to wind-forces will be damped strongly. Finally it has been shown that a combination of boundary damping and Kelvin–Voigt damping can be used to damp the wind-induced vibrations of a vertical beam with tip-mass uniformly. It also has been shown that the compression force due to the mass of the tip-mass and due to the mass of the beam itself has a significant effect on the frequency.

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