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Chaotic dynamics of simple vibrational systems

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Abstract

This study discusses the development of a technique for analysis of the dynamical regimes of complex mechanical systems consisting of a rotor motor coupled to a system with multi-degrees-of-freedom. To understand the possible qualitatively different dynamical regimes in such systems, a simple mechanical system is considered of the ‘‘rotator-oscillator’’ type with a finite power source. This system has four degrees-of-freedom and is defined in four-dimensional cylindrical phase space with 12 parameters. Near the main resonance the original system is reduced to the Lorenz system with four parameters defined in a three-dimensional Cartesian phase space. This is done with the help of a special change of variables, parameters, and employing an averaging method. Studying the latter system, the existence of one of the chaotic attractors, namely of Lorenz attractor is established. Also established is the Feigenbaum attractor and the alternation. Chaotic limit sets define chaotic behavior of the instantaneous frequency of rotation of the asynchronous motor. The Poincare mappings are presented to show the correspondence of the original 4 dof and averaged 3 dof systems. The qualitative rotational characteristics for different values of the system parameters are obtained. In particular, the system can possess normal Sommerfeld effect, doubled Sommerfeld effect and a so-called scattering of the torque curve. The scattering of the torque curve (which is a known effect in micro-electronics) is likely to be a new effect in mechanics. In contrast to the Sommerfeld effect, when frequency or amplitude jumps occur instantaneously (once the unstable point of the characteristic is reached), the jump to a next stable point may take a certain time, even infinite one. Such chaotic mistuning of the motor frequency would result in random vibrations leading to system wear and damage.

1. Introduction

The studies of dynamics of vibrational mechanisms with the limited power supplies originate in the pioneer papers by Sommerfeld [1] and Timoshenko [2] and continued by Kalischuk [3], Martyshkin [4], Blekhman [5], Kononenko [6,7], etc. Later, studies of different aspects of dynamics of such applied systems have resulted in a huge number of papers. A full review of the publications related to this topic is not presented here but the reader is referred to the classical books by Blekhman [8,9], Alifov and Frolov [10] and Dimentberg [11] and to comprehensive references given there. Note that currently, the models of simplest vibrational

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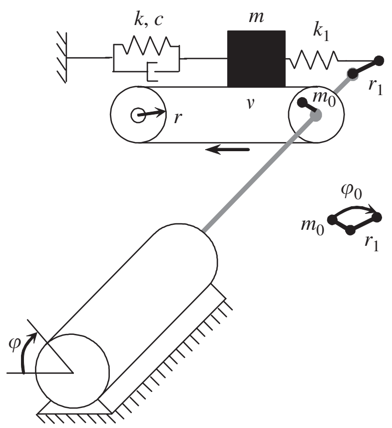

Fig. 1. The model under consideration.

mechanisms became teaching material on vibrotechnics as well as on the theory of the mechanisms and machines [12]. This fact provides extra interest to study such systems for the detailed knowledge of their dynamical properties.

The major studies of the dynamical chaos in nonlinear systems with a small number of degrees of freedom started about three and one-half decades ago. Generally, these studies were done by the mathematicians of the Gorky school of A.A. Andronov, namely by L.P. Shilnikov, Yu.I. Neimark, V.S. Afraimovich, V.V. Bykov et al. [13]. With breakthroughs in bifurcation theory and advances in computer technologies (most of the studies are related to the numerical modeling), the number of publications on the chaotic dynamics of "simple" systems increased enormously. Numerous publications on chaotic dynamics are related to radiophysical systems, superconductive and laser electronics systems, radio-technical systems, etc., but not to the mechanical systems. This paper aims to fill this gap. Although chaotic regimes are not desirable operational regimes of mechanical systems, the necessity to study these regimes is obvious. First, since the simplest vibrational mechanisms have already become teaching material, then information about their dynamics must be thorough and extensive. Second, the breadth and depth of the information is necessary to avoid the unpleasant surprises in the engineering practice when complicated mechanical systems are being designed.

Here, the dynamics of the simplest vibrational mechanism shown in Fig. 1 is considered. Having developed a technique to study such a simple model and having understood its dynamics, one could proceed with more complicated and realistic models. On the basis of transformation techniques for the systems with a cylindrical phase space developed to study dynamics of systems with superconductive junctions [14], the averaging technique is used to show the existence of classical Lorenz and Feigenbaum chaotic attractors in such a simple system and interpret the results.

2. The model

Fig. 1 shows a simple vibrational mechanism: the mass m, which lays on the conveyor band moving by rollers of the radius r, is attached to the immovable wall by the spring–dashpot system with the stiffness k and the viscosity c; the crankshaft of the length r_1 is placed in perpendicular to the motor shaft and attached to the mass by the spring with the stiffness k_1. The crankshaft length is small enough so that the deformation of the elastic coupling might be considered in the horizontal direction only. This model has been taken from the book [10] with slight changes. Firstly, a lubricated conveyer band is considered, so that the horizontal component of the contact force between the mass and the band is proportional to the relative velocity (the friction between the mass and the band has been changed from the dry one into the viscous one with the viscosity v). Secondly, the driving roller has imbalance m_0 placed at the angle φ_0 with respect to the crankshaft in the direction of the shaft rotation. These changes have been done to simplify the study. Note that system under consideration and its variations represent simple models of vibrational mechanisms that are used to enter handbooks on vibrations. Because of that, a necessity to provide a full knowledge on dynamical properties of such systems motivates this study.

The following forces act on the mass: $F_1 = -kx$ is the elastic force acting from the left spring, where x is the displacement of the mass; $F_2 = k_1(r_1 \sin \varphi - x)$ is the elastic force acting from the right spring; $F_3 = v(r\dot{\varphi} - x)$ is the friction force between the mass and the conveyor band; $F_4 = -c\dot{x}$ is the resistance force of the damper. The following moments act on the rotor: $M_1 = \widetilde{M_d}(\dot{\varphi})$ is the motor torque including the moment of the rotor motion resistance forces (load torque); $M_2 = -k_1 r_1(r_1 \sin \varphi - x)\cos \varphi$ is the moment of the elastic force of the right spring; $M_3 = -vr(r\dot{\varphi} - x)$ is the moment of friction force acting from the conveyor band; $M_4 = -M_0 \cos(\varphi + \varphi_0) = -m_0 g\varepsilon \cos(\varphi + \varphi_0)$ is the moment of the gravity force of the imbalance, where ε is the eccentricity. The governing equations have the form:

$$\begin{aligned} &\ddot{x} + \omega_0^2 x = \frac{k_1 r_1}{m} \sin \varphi + \frac{vr}{m}\dot{\varphi} - \frac{v+c}{m}\dot{x}, \\ &I\ddot{\varphi} = \widetilde{M_d}(\dot{\varphi}) - vr(r\dot{\varphi} - \dot{x}) + k_1 r_1 (x - r_1 \sin \varphi)\cos \varphi - M_0 \cos(\varphi + \varphi_0). \end{aligned} \tag{1}$$

The system (1) is defined in the cylindrical phase space $G(\varphi, \dot{\varphi}, x, \dot{x}) = S^1 \times R^3$ and represents an example of a system of the "rotator-oscillator" type [15,16]. Here, $\omega_0^2 = (k + k_1)/m$, I is the normalized moment of inertia of the rotor, $M_0 = m_0 g\varepsilon$.

It is assumed that variables, parameters, and time in the equations of system (1) are reduced to the dimensionless form, and $I^{-1} = \mu \ll 1$, where μ is the small parameter; $(v + c)/m = 2\mu h$ (dissipation in the "oscillatory" part of the system is small enough), $k_1 r_1/m = 2\mu\lambda\omega_0$ and $k_1 r_1 = 2\mu b\omega_0$. The condition of smallness is not applied for other combinations of the parameters.

For the mentioned parameters, the dynamical system (1) is quasi-linear so that asymptotic methods, in particular, the averaging technique, can be applied. This system is studied here just in a zone of main resonance ($\Omega \cong \omega_0$, where Ω is a frequency of rotations of a motor called also a motor speed).

3. Transformation of the dynamical system to the standard form

The technique to transform the equations in system (1) to the system with a fast spinning phase [17], which is used here, is quite a nonstandard technique [14,16]. Due to this reason, this technique is discussed here in detail. This algorithm can be extended for any quasi-linear system with a cylindrical phase space.

For the chosen parameters domain, the equation for the phase has the form

$$\dot{\varphi} = \omega_0 + \mu\Phi(\theta, \eta, \varphi, \xi), \tag{2}$$

here, $\Phi(\theta, \eta, \varphi, \xi)$ is some function, which will be defined later during the transformation of the rotator's equation, and θ,η,ξ is the set of new "slow" variables. Taking into account Eq. (2) and applying the change of variables of the form

$$x = \frac{vr}{m\omega_0} + \theta \sin \varphi + \eta \cos \varphi, \quad \dot{x} = (\theta \cos \varphi - \eta \sin \varphi)\omega_0$$

to the first equation of system (1), the following equations for the new variables θ and η are obtained:

$$\begin{aligned} &\dot{\theta} = \mu F_1(\theta, \eta, \varphi, \xi), \\ &\dot{\eta} = \mu F_2(\theta, \eta, \varphi, \xi), \\ &F_1 = \eta\Phi + \frac{1}{\omega_0} X(.) \cos \varphi, \\ &F_2 = -\theta\Phi - \frac{1}{\omega_0} X(.) \sin \varphi, \\ &X(.) = 2\lambda\omega_0 \sin \varphi + \frac{vr}{m}\Phi - 2h\omega_0(\theta \cos \varphi - \eta \sin \varphi). \end{aligned} \tag{3}$$

It is known that torque of an asynchronous motor is almost a linear function of the form $\widetilde{M_d}(\dot{\varphi}) = M_d - \delta\dot{\varphi}$ with M_d, the constant component (for an AC motor, this parameter is defined by the current in an exciting circuit), and δ, the coefficient that defines a value of the moment of the rotor motion resistance forces [12].

Having used this expression for the motor torque together with Eqs. (3), the equation for the rotator (the second equation in system (1) is transformed). Substituting Eq. (2) into the second equation of the system (1), one obtains

$$\begin{aligned}&\frac{\partial\Phi}{\partial\theta}\mu F_1+\frac{\partial\Phi}{\partial\eta}\mu F_2+\frac{\partial\Phi}{\partial\varphi}(\omega_0+\mu\Phi)+\dot{\xi}\\&\quad=M_d-(\delta+vr^2)\omega_0-\delta\mu\Phi-vr^2\mu\Phi+vr\omega_0(\theta\cos\varphi-\eta\sin\varphi)\\&\qquad+2\mu b\omega_0(\theta\sin\varphi+\eta\cos\varphi-r_1\sin\varphi)\cos\varphi-M_0\cos(\varphi+\varphi_0).\end{aligned}$$

It is assumed that $M_d-(\delta+vr^2)\omega_0=\mu\Delta$ (the zone of main resonance). Having separated terms of the different order with respect to the small parameter μ, one obtains the equation for the variable ξ and the equation for the function $\Phi(\theta,\eta,\varphi,\xi)$ that has the form:

$$\frac{\partial\Phi}{\partial\varphi}\omega_0=vr\omega_0(\theta\cos\varphi-\eta\sin\varphi)-M_0\cos(\varphi+\varphi_0).$$

In particular, this equation has the solution of the form:

$$\hat{O}=vr(\theta\sin\varphi+\eta\cos\varphi)-\frac{M_0\sin(\varphi+\varphi_0)}{\omega_0}+\xi.$$

As a result, the system of equations of the standard form that is equivalent to system (1) is obtained:

$$\begin{aligned}&\dot{\theta}=\mu F_1(\theta,\eta,\varphi,\xi),\\&\dot{\eta}=\mu F_2(\theta,\eta,\varphi,\xi),\\&\dot{\xi}=\mu F_3(\theta,\eta,\varphi,\xi),\\&\dot{\varphi}=\omega_0+\mu\Phi(\theta,\eta,\varphi,\xi).\end{aligned}\tag{4}$$

Here,

$$F_3=-\left(\frac{\partial\Phi}{\partial\theta}F_1+\frac{\partial\Phi}{\partial\eta}F_2+\frac{\partial\Phi}{\partial\varphi}\Phi\right)-(\delta+vr^2)\Phi+2b\omega_0(\theta\sin\varphi+\eta\cos\varphi-r_1\sin\varphi)\cos\varphi.$$

4. Averaged system

Averaging system (4) by the fast phase φ, one obtains

$$\begin{aligned}&\dot{\xi}=\mu(-b_1\xi+b_2\theta+b_3\eta+\Delta),\\&\dot{\theta}=\mu(-b_4\theta+b_5\eta+\eta\xi+b_6),\\&\dot{\eta}=\mu(-b_4\eta-b_5\theta-\theta\xi+b_7),\\&\dot{\varphi}=\omega_0+\mu\xi.\end{aligned}\tag{5}$$

Here,

$$b_1=vr^2+\delta,\quad b_2=-\frac{M_0vr\sin\varphi_0}{2\omega_0},\quad b_3=b\omega_0+\frac{M_0vr\cos\varphi_0}{2\omega_0},\quad b_4=h,\quad b_5=\frac{(vr)^2}{2m\omega_0},$$
$$b_6=-\frac{M_0vr\sin\varphi_0}{2m\omega_0^2}\text{ and }b_7=\frac{M_0vr\cos\varphi_0}{2m\omega_0^2}-\lambda.$$

Introducing new variables and time

$$x=\frac{\xi+b_5}{b_4},\quad y=\frac{b_3\eta+b_2\theta}{b_1b_4}-\Lambda,\quad z=\frac{b_3\theta-b_2\eta}{b_1b_4}+R,\quad \mu b_4\tau=\tau_n,$$

one reduces Eqs. (5) to the system of the form

$$\begin{aligned} \dot{x} &= -\sigma(x-y)+\rho, \\ \dot{y} &= -y+Rx-xz, \\ \dot{z} &= -z+xy+\Lambda x \end{aligned} \tag{6}$$

with

$$\sigma=\frac{b_1}{b_4}=\frac{vr^2+\delta}{h}, \quad \rho=\frac{1}{b_4^2}\left(\Delta+\frac{1}{b_4}(b_1b_4b_5+b_3b_7+b_2b_6)\right),$$
$$R=\frac{b_2b_7-b_3b_6}{b_1b_4^2}=\frac{2A\lambda}{(\delta+vr^2)h^2}\sin\varphi_0, \quad \Lambda=\frac{b_3b_7+b_2b_6}{b_1b_4^2}=\frac{\lambda(A^2-b^2\omega_0^2)}{b\omega_0(\delta+vr^2)h^2},$$
$$A=\frac{M_0qv}{2\omega_0}.$$

Note that after transformations, the averaged system (6) is defined already in the phase space $G^*(x,y,z)=R^3$ but not in the cylindrical phase space, as it normally happens after the introduction of the "amplitude-phase" variables. This significantly simplifies the further study.

Definition 1. The function

$$\Omega=\lim_{T\to\infty}\frac{1}{T}\int_0^T\dot{\varphi}(\tau,t_0)\,\mathrm{d}\tau$$

defined under the parameter space of system (1), and the space of its initial conditions is called the rotation characteristic of rotor (torque–speed curve).

Using the definition and Eq. (2), it is obtained that $\Omega=\omega_0+\mu\overline{\xi^*(t,t_0)}$, where $\overline{\xi^*(t,t_0)}$ is an average value of the variable $\xi(t)$ that corresponds to the limit set of trajectories of system (5) realized for given initial conditions (during the evaluation of a limit, all solutions that correspond to the transient processes will provide the zero average value). Let us review the correspondence of the limit sets of averaged and initial systems [18]: equilibriums of system (5) and system (6) correspond to the limit cycles of system (1), limit cycles of the averaged system correspond to the invariant tori of the system (1) (to the quasi-periodic motions of system (1) if a corresponding torus is ergodic). Generally, if Γ is a limit set of an averaged system with nonzero characteristic measures, lying far enough from the imaginary axis, then it corresponds to the limit set $\Gamma\times S^1$ of the system (1) together with the character of the stability.

Our interest is focused on the qualitatively different torque curves as functions of the constant component of the rotor torque for fixed values of all other system parameters. By definition, to represent a full set of qualitatively different states of the torque curve, a full study of the averaged system (6) is to be carried out, i.e., the classical problem of the decomposition of the phase space into the domains corresponding to the qualitatively different trajectories structures in the phase space is to be solved.

After the application of the new variables and parameters (5) to the system (6), it is obtained that $M_d\sim\Delta\sim\rho$, $\xi(t)\sim x(t)$ (linear dependence). Thus, the curve $\rho=\rho\left(\overline{x^*(t,t_0)}\right)$ has the same qualitative features as a function inversed to the torque curve in a resonance zone. This curve is of further interest as a "rotation characteristic".

5. Properties of the averaged system

(1) System (6) is a dissipative system. This property can be proved with the help of the following quadratic form: $V=\frac{1}{2}(x^2+(y+\Lambda)^2+(z-\sigma-R)^2)$, the derivative of which, taken in accordance to system (6), has the form $\dot{V}=-\sigma x^2-(\rho-\sigma\Lambda)x-y^2-\Lambda y-z^2+(\sigma+R)z$ that is negative outside some sphere $V\leqslant L^2$. It means that all limit sets of the trajectories of system (6) in the phase space $G^*(x,y,z)=R^3$ are limited by a sphere of dissipation.

(2) Depending on the parameters, system (6) has up to three equilibriums with coordinates

$$x_0=\omega, y_0=(R\omega-\Lambda\omega^2)/(1+\omega^2), \quad z_0=(R\omega^2+\Lambda\omega)/(1+\omega^2),$$

where $\omega_{1,2,3}$ are the solutions of the equation

$$f = \omega + (\Lambda\omega^2 - R\omega)/(1+\omega^2), \quad f = \rho/\sigma. \tag{7}$$

In this case, the parameter ω has actually the sense of the mistuning of a frequency of the rotator's periodical rotations in system (1) and the sense of the oscillator eigenfrequency.

(3) If system (6) has one equilibrium $O(x_0, y_0, z_0)$, then this equilibrium is globally asymptotically stable. This can be proved with the help of the Lyapunov function $V = \frac{1}{2}(mx_1^2 + y_1^2 + z_1^2)$, where $x_1 = x - x_0, y_1 = y - y_0, z_1 = z - z_0$. The derivative of the Lyapunov function, taken in accordance to system (6), has the form $\dot{V} = -(\alpha_1 x_1 + y_1)^2 - (\alpha_2 x_1 + z_1)^2 \leqslant 0, \quad \forall(x_1, y_1, z_1)$,where $\alpha_1 = (\sigma m + R - z_0)/2$, $\alpha_2 = -(y_0 + \Lambda)/2$, and m is a positive root of the equation $\sigma^2 m^2 + 2\sigma(R - z_0 - 2)m + (R - z_0) + (y_0 + \Lambda)^2 = 0$. It may be proven that conditions of existence of a positive root and conditions of unicity of the equilibrium in system (6) (conditions that the function f is one–one function) are the same.

(4) Properties of the equilibriums are as follows. For an arbitrary equilibrium $O(x_0, y_0, z_0)$ of system (6), the characteristic equation has the form $p^3 + a_0 p^2 + a_1 p + a_2 = 0$, where $a_0 = \sigma + 2, \quad a_1 = 2\sigma + 1 + \omega^2 + \sigma(\Lambda\omega - R)/(1+\omega^2)$, $a_2 = \sigma(1+\omega^2)(1 + (R\omega^2 + 2\Lambda\omega - R)/(1+\omega^2)^2)$.
For $\omega = 0 (\rho = 0)$, the criteria of the stability of the equilibrium $O(0,0,0)$ is the inequality $R < 1$. For $\omega \neq 0$, the Gurvitz conditions are equivalent to the following inequalities:

$$a_1 > 0 : f > (<) f_1, \quad a_2 > 0 : f > (<) f_2, \quad a_0 a_1 - a_2 > 0 : f > (<) f_3, \quad \omega > 0 (\omega < 0)$$

with $f_1 = -\omega(\sigma + 1 + \omega^2)/\sigma, \quad f_2 = -\omega(R - 1 + \omega^2)/2, f_3 = -\omega(\sigma^2 + 4\sigma - \sigma R + 2 + 2\omega^2)/\sigma^2$. Thus, the criterion of the stability of any equilibrium of the system is a location of a point of the curve (7), corresponding to this equilibrium, above (below) all of the curves $f_{1,2,3}$ for $\omega > 0 (\omega < 0)$. Equations $f = f_{1,2,3}$, $f = \rho/\sigma$ define, in the parameter space of system (6), bifurcation surfaces corresponding to change of a local structure of its equilibriums. The auxiliary functions have two simple properties.
Function f_2 crosses function f in the positions of extremum and at the origin.
All curves cross each other in three points $\omega_{1,2} = \pm\sqrt{(3\sigma - \sigma R + 2)/(\sigma - 2)}$ and at the origin. The properties of equilibriums will be discussed in more detail during the consideration of the torque curves.

(5) Typical types of chaotic attractors in the averaged system.

The Lorenz attractor: For $\rho = \Lambda = 0$, system (6) represents a well-known Lorenz system [19]. This classical system, for some values of the parameters $\sigma = \sigma^*, \quad R = R^* > R_c, \quad R_c = (\sigma^2 + 4\sigma)/(\sigma - 2)$ has a unique attracting limit set in the phase space—strange attractor (Lorenz attractor). Note that according to the property 4, the expression for R_c is defined by the condition of the coincidence of the zeroes of functions f and f_3. This condition corresponds to the loss of the stability of the equilibriums $O_{2,3}(\pm\sqrt{R-1}, \pm\sqrt{R-1}, R-1)$ since the saddle limit cycles that appear earlier (by the parameter R) from the separatrix loop of the saddle $O_1(0,0,0)$ get stuck in them. In fact, a strange attractor already exists for $R > R^* < R_c$, where R^* is the value of the parameter R that corresponds to the saddle separatrix loop. In this case, a strange attractor and stable equilibriums co-exist having disjointed attraction domains. That is, for $R > R^* < R_c$, depending on the initial conditions, the limit motions of the system will be either equilibriums or those of the strange attractor [20].

"Deformation" and degeneration of the Lorenz attractor for nonzero parameters ρ and Λ have been studied numerically. For $\Lambda = 0$ and increasing ρ from zero, the attractor loses its symmetry because the affix remains mostly in the vicinity of the right saddle focus (for the projection onto the plane (x,z)). Further, this equilibrium becomes stable after the birth of a saddle limit cycle. In this case, depending on the initial conditions, an equilibrium or a chaotic attractor may be realized. With further increase of the parameter ρ, the saddle cycle "gets stuck" in the separatrix loop of a saddle and the equilibrium becomes globally stable. Since system (6) is invariant with respect to the change of the variables $(x, y, z) \mapsto (-x, -y, z)$, $\rho \mapsto -\rho$, the same scenario of the degeneration of the Lorenz dynamical chaos occurs with the change of the parameter ρ towards the negative values. The only difference is that the right equilibrium becomes globally stable. This scenario also takes place for the nonzero Λ, at least for $|\Lambda| < 3$. Asymmetrical Lorenz attractors are depicted in Figs. 2a and c. Note that for $\rho\Lambda > 0$ and small $|\Lambda|$, there exists ρ, for which the attractor has a "visual" symmetry (see Fig. 2b).

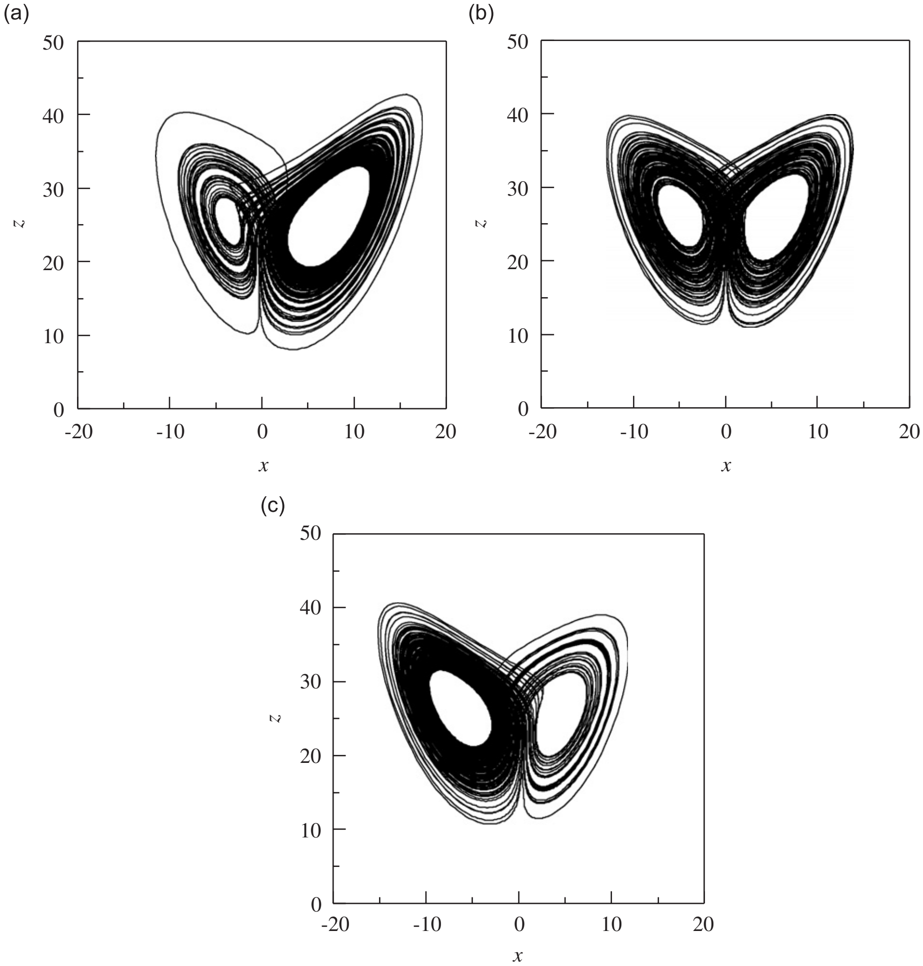

Fig. 2. The Lorenz attractor: (a) $\sigma = 9.7$, $R = 27$, $\Lambda = -1$, $\rho = 28.53$; (b) $\sigma = 9.7$, $R = 27$, $\Lambda = -1$, $\rho = 3.5$; (c) $\sigma = 9.7$, $R = 27$, $\Lambda = -1$, $\rho = 31.67$.

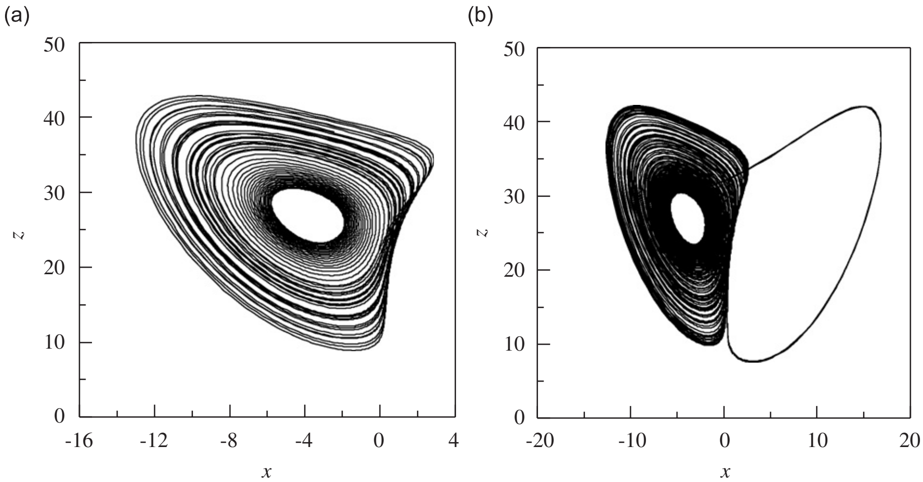

Fig. 3. (a) The Feigenbaum attractor: $\sigma = 9.7$, $R = 27$, $\Lambda = -7$, $\rho = 36.47$; (b) the alternation: $\sigma = 9.7$, $R = 27$, $\Lambda = -7$, $\rho = 36.27$.

The Feigenbaum attractor and the alternation: This type of the chaotic attractors has been found to exist for $\rho\Lambda > 0$ and for high enough absolute values of the parameter Λ. For the numerical calculation, $|\Lambda| \geqslant 7$ has been used. The Feigenbaum attractor [21] is depicted in Fig. 3a. With the increase of the parameter ρ, the left equilibrium of system (6) losses its stability with the birth of a stable limit cycle. Further, this cycle experiences a series of bifurcations of doubling of the period. In some interval of values of ρ, the chaotic attractor has the attraction domain isolated from the attraction domains of other limit sets. With the further increase of ρ, the attraction domain of the attractor intersects with the attraction domain of another chaotic limit set (it has not been studied which one exactly, probably this limit set is an "inheritance" of the asymmetric Lorenz attractor). As a result, a typical alternation [12] occurs (see Fig. 3b). To not overload the figure, just a couple of ejections of the trajectory from the attraction domain of the Feigenbaum attractor have been shown. Since system (6) is invariant with respect to the transformation $(x, y, z) \mapsto (-x, -y, z), \rho \mapsto -\rho, \ \Lambda \mapsto -\Lambda$, the same scenario occurs in the right half-plane (x,z) for the positive Λ and a decrease of ρ.

For negative R and arbitrary values of other parameters, chaotic attractors do not exist in system (6).

6. Direct study of the dynamical chaos in the initial system

The statements about existence of the chaotic attractors of a certain type in the initial system (1) have been made on the basis of the existence of the corresponding attractors in the averaged system (6). To confirm that, the Poincare mapping has been plotted using the secant hyperplane $\varphi = \text{const}$ at the period 2π; i.e., $(\theta, \eta, \xi)_{\varphi=\varphi_0} \to (\bar{\theta}, \bar{\eta}, \bar{\xi})_{\varphi=\varphi_0+2\pi}$. For the rotational motions, this secant is global. For convenience, the mapping has been done for the following system that is equivalent to system (1)

$$\begin{aligned}
\dot{\theta} &= \mu\left(\eta\xi + \left(2\lambda \sin \varphi + \frac{vr}{m\omega_0}\xi - 2h(\theta \cos \varphi - \eta \sin \varphi)\right) \cos \varphi\right), \\
\dot{\eta} &= \mu\left(-\theta\xi - \left(2\lambda \sin \varphi + \frac{vr}{m\omega_0}\xi - 2h(\theta \cos \varphi - \eta \sin \varphi)\right) \sin \varphi\right), \\
\dot{\xi} &= M_d - (\delta + vr^2)\omega_0 - \mu(\delta + vr^2)\xi + vr\omega_0(\theta \cos \varphi - \eta \sin \varphi) \\
&\quad + k_1 r_1 \left(\frac{vr}{m\omega_0} + \theta \sin \varphi + \eta \cos \varphi - r_1 \sin \varphi\right) \cos \varphi - M_0 \cos(\varphi + \varphi_0), \\
\dot{\varphi} &= \omega_0 + \mu\xi.
\end{aligned} \tag{8}$$

This system is obtained from system (1) after the following transformation:

$$x = \frac{vr}{m\omega_0} + \theta \sin \varphi + \eta \cos \varphi, \quad \dot{x} = (\theta \cos \varphi - \eta \sin \varphi)\omega_0, \quad \dot{\varphi} = \omega_0 + \mu\xi$$

with $\mu = I^{-1}$.

Parameters of system (8) have been chosen such that the corresponding parameters of the averaged system would have values close to those that have been used to plot Figs. 2 and 3. Fig. 4a shows the asymmetrical Lorenz attractor in the Poincare domain. The slight difference between this attractor and that shown in Fig. 2a is concerned with the change of the variables done to reduce system (5) to system (6). This change of variables provides parallel displacement and turn of system of coordinates. Thus, attractors depicted in Figs. 4a and 3a represent projections made at different angles. To make them identical, one had to perform extra change of the variables at the Poincare plane. However this is not necessary as qualitative forms of attractors are pretty much recognizable. Relationship between parameters of original system (1) and averaged system has the form

$$\begin{aligned}
\sigma &= (\delta + vr^2)/h, \\
R &= (\lambda M_0 vr \sin(\varphi_0)/(\omega_0(\delta + vr^2)h^2)), \\
\Lambda &= \left(2\mu\lambda\left(\left(\frac{M_0 vr}{2\omega_0}\right)^2 - \left(\frac{k_1 r}{2\mu}\right)^2\right) \Big/ (k_1 r_1(\delta + vr^2)h^2)\right).
\end{aligned}$$

Thus, values of the parameters of system (1): $vr/m = 1, vr^2 = 0.5, vr = 2.17, k_1 r_1 = 0.14, \mu = 0.1, \omega_0 = 1.064, \delta = 0.49, \lambda = 0.2, h = 0.1, M_d = 0.9666, M_0 = 0.671, r_1 = 0.1$, and $\varphi_0 = \pi/2$ correspond to the following

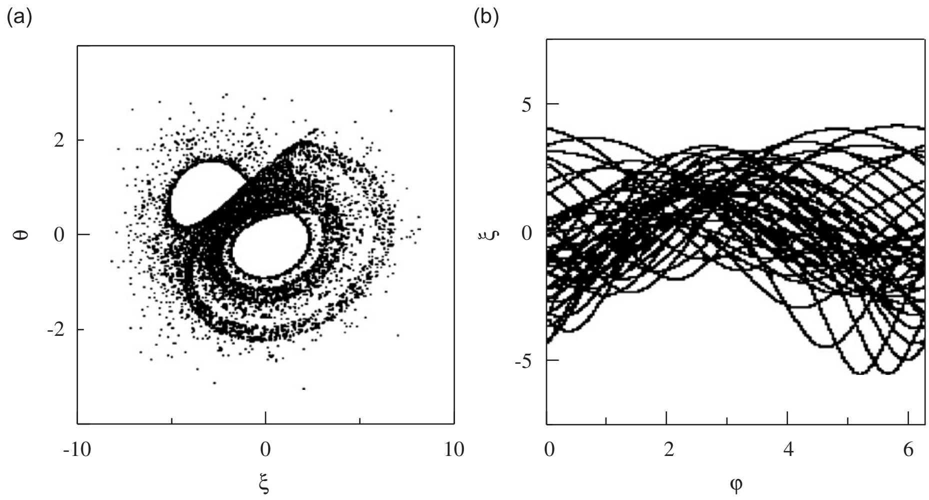

Fig. 4. (a) Lorenz attractor in the Poincare plane and (b) involute of the phase cylinder of the initial system.

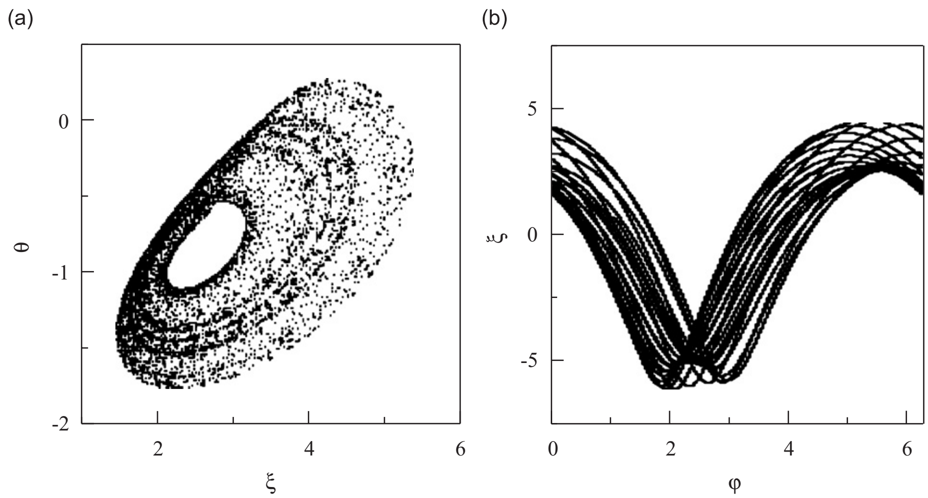

Fig. 5. (a) Feigenbaum attractor in the Poincare plane and (b) involute of the phase cylinder of the initial system.

values of the averaged system: $\sigma = 9.9, R = 27.646$, and $\Lambda = -0.629$. Fig. 4b shows the structure of the chaotic rotational motions at the involute of the phase cylinder.

Fig. 5a shows the Feigenbaum attractor based on the doublings of the invariant torus. The values of the parameters of system (1): $vr/m = 1, vr^2 = 0.5, vr = 2.17, k_1r_1 = 0.1715, \mu = 0.1, \omega_0 = 1.064, \delta = 0.49, \lambda = 0.2$, $h = 0.1, M_d = 0.7466, M_0 = 0.671, r_1 = 0.1$, and $\varphi_0 = \pi/2$ correspond to the following values of the averaged system: $\sigma = 9.9, R = 27.646$, and $\Lambda = -6.293$. Fig. 5b shows the corresponding structure of the chaotic rotational motions at the involute of the phase cylinder.

7. Qualitative forms of the torque curve in the resonance zone

(1) Consider $R \leqslant 0$. In this case, the system dynamics is quite simple: in system (6) there exist only equilibriums (rotational limit cycles in system (1)). Equilibriums experience just one type of bifurcation: confluence of the equilibriums and formation of a saddle-node with its further disappearance. All torque curves have one or two hysteretic loops. The jump in the frequency of the rotor rotations happens at the extremums of the torque curve. Fig. 6 shows torque curve with two loops.

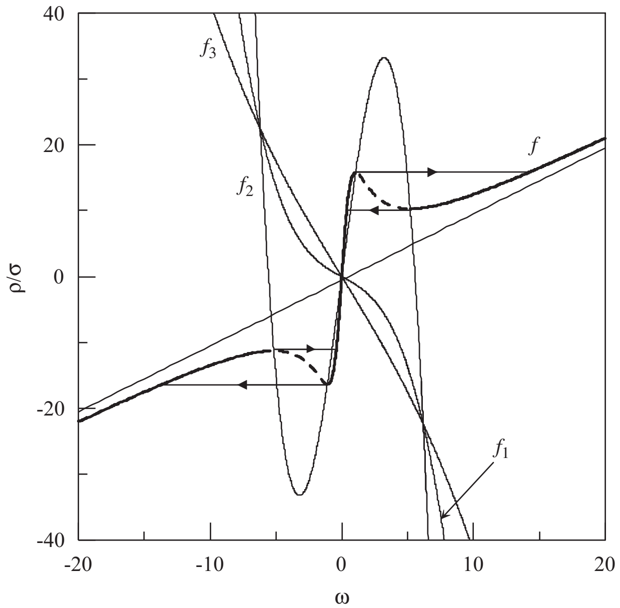

Fig. 6. Torque curve with two loops—doubled Sommerfeld effect, $\sigma = 15$, $R = -30$, $\Lambda = -0.5$.

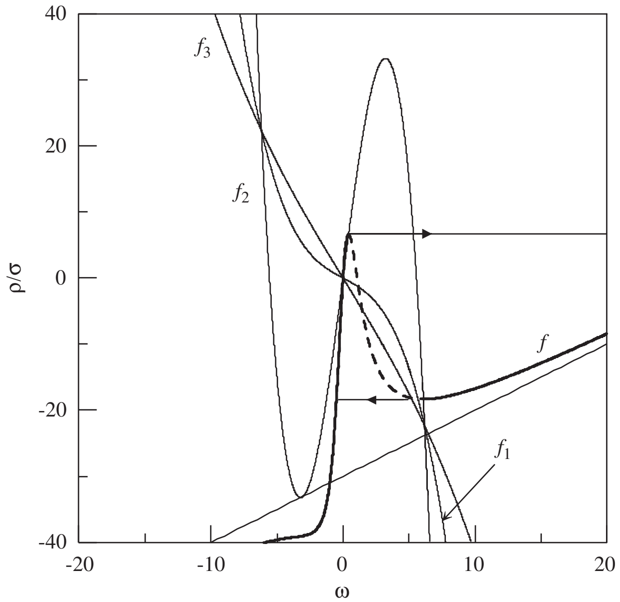

Fig. 7. Torque curve with one loop—the classical Sommerfeld effect, $\sigma = 15$, $R = -30$, $\Lambda = -30$.

The torque curve is shown in Figs. 6–8 by the bold solid line. Its dashed parts correspond to the unstable equilibriums of the type saddle or saddle-focus. In contrast, the solid parts correspond to the stable equilibriums of system (6) (stable limit cycles of system (1)). Thin lines correspond to the auxiliary functions that define the stability and the type [22] of equilibriums. One can say that a so-called doubled Sommerfeld effect occurs in the system [1,2,5].

With the decrease of $|R|$ and with the increase of $|\Lambda|$, the left loop disappears but not the right one (see Fig. 7). As a result, a classical (one-loop) Sommerfeld effect [5] occurs. Since system (6) is invariant

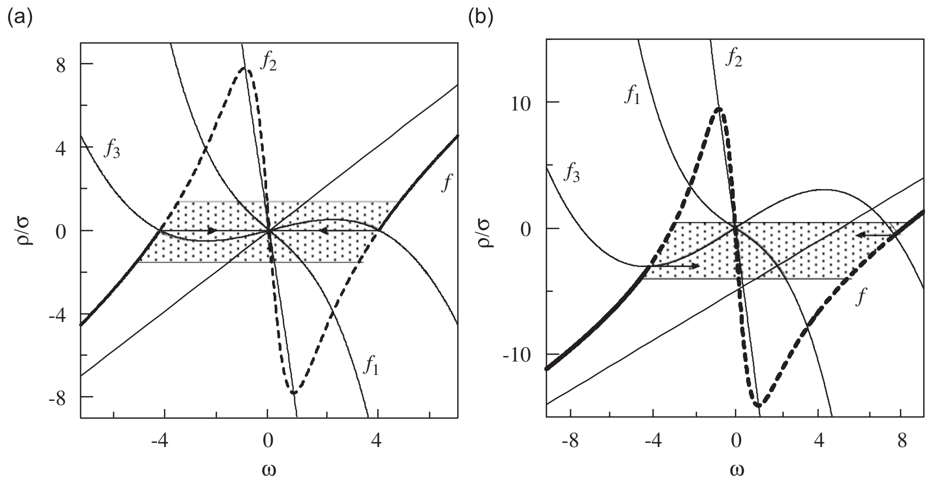

Fig. 8. Effect of scattering of the torque–speed curve: (a) $\sigma = 10$, $R = 17.5$, $\Lambda = 0$; (b) $\sigma = 10$, $R = 25$, $\Lambda = -5$.

with respect to the transformation $(x, y, z) \mapsto (-x, -y, z)$, $\rho \mapsto -\rho$, $\Lambda \mapsto -\Lambda$, for positive and increasing Λ, the right loop disappears and the left one remains.

(2) Consider $R > 0$. In this case, the system dynamics is more diverse during the change of the system parameters. Accordingly, the more diverse is the set of the qualitative forms of the torque curves. Consider only the cases of existence of the chaotic attractors in system (6). Suppose that $R = R_c - o$, where $o > 0$ is small enough (see Fig. 8a). In this case, in some interval of the values of the parameter ρ/σ, there exist two stable equilibriums and the Lorenz attractor. In the figures, this domain is shaded. Thus, depending on the initial conditions, either one of the two equilibriums of system (6) (periodical motions of system (1)) as well as the strange attractor corresponding to the chaotic behavior of the rotor instantaneous frequency may be realized. Outside the zone, for any initial conditions, either pre-resonance or post-resonance regimes of the rotor periodic motions occur (depending on the value of the motor torque).

Fig. 8b shows the torque curve for $R > R_c$. In this case, for any value of the parameter ρ/σ from the shaded domain, there exists a certain chaotic attractor in the phase space of system (6) that is the unique attracting limit set. In other words, in the aforementioned interval, there exists an infinite number of chaotic attractors, every one of which has individual spatio-temporal properties. For each point, bifurcations of the homoclinic trajectories and of the corresponding saddle periodic motions occur. Temporal average value $\langle x(t, t_0) \rangle_t$ for each attractor is different. Moreover, due to the strong dependence of the trajectories on the initial conditions and due to the finiteness of the real averaging interval, this value will strongly depend on the initial time t_0.

As far as a torque curve in the shaded domain is concerned, one can conclude the following:

(a) The torque curve in the shaded domain is irreproducible—during the quasi-stationary increase of the parameter ρ/σ (constant part of the motor torque), one obtains one curve (branch), but for an opposite change (arbitrarily small), one obtains a completely different curve.

(b) The torque curve in the shaded domain has an infinite number of mixed branches that start at the frequency jump points corresponding to the ends of solid bold lines. Due to this reason, torque curve in this zone is not presented.

Such behavior of the torque curve is a so-called effect of the scattering of the torque curve of the rotator. In particular, such effect is known to take place for the synchronization of the superconductive junction by a microwave field [23].

8. Conclusions

Let us explain why the system dynamics changes crucially after the introduction of the rotor imbalance. If the imbalance is absent ($M_0 = 0,\ R = 0$), the system dynamics is simple: there exist periodic motions only, i.e., classical Sommerfeld effect occurs. In this case, the governing equation for the asynchronous motor is, in fact, the first-order equation (with respect to the variable $\dot{\varphi}$). Thus, an asynchronous motor does not represent a "full" rotator, which normally represents a dynamical system, with one degree-of-freedom, that is defined in the cylindrical phase space and has at least one cycle of the second kind. Thus, in the absence of couplings, there is a third-order system (generating case). In the case of the existence of couplings, the corresponding averaged system inherits all properties of the second-order system despite that it is of the third order. Because of this reason, there is no dynamical chaos in the analogous system with the balanced rotor. Of course, existence of the second-order system in a generating case is a necessary condition for existence of the dynamical chaos in vibrational mechanisms. The structure of couplings between rotator and oscillator is also important. These reasonings, of course, are valid for the case, when the normalized moment of inertia is high enough in the corresponding system of equations.

Also note that parameter $R = [2A\lambda/(\delta + vr^2)h^2]\ \sin\ \varphi_0$, where $A = M_0 vr/2\omega_0$ strongly depends on the point of the imbalance placement. If $R<0$ and its absolute value is high enough (see Fig. 6), then the area in between the loops may be quite sharp (for the corresponding choice of the active system parameters). Thus, by choosing the magnitude and the point of the imbalance placement, one could provide a significant stabilization of the rotor motions whereby external influences, including random ones, will not noticeably affect the operational regime of a mechanism.

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