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Vibration of a beam excited by a moving oscillator considering separation and reattachment

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Abstract

The vibration of a beam excited by a moving oscillator is an extensively studied moving-load problem. It has usually been assumed in the previous studies of the moving-oscillator problem that the contact between the moving oscillator and the supporting beam structure is constantly maintained. This restrictive assumption is removed in this paper. Furthermore, a reattachment condition of the moving oscillator to the beam after separation is proposed. Simulated numerical examples indicate that for certain values of system parameters and travelling speeds of the moving oscillator separation can occur frequently and even more than once during its travel, and there is normally a 'jump' in the velocity of the beam at the instant of reattachment.

1. Introduction

The problem of the dynamic effects of a moving system sliding along a beam-like structure has many engineering applications. A number of examples of a simply supported beam excited by different moving loads were presented in Refs. [1,2]. For some simplified solutions, one can also refer to Ref. [3]. Olsson discussed particular computational aspects for the problem of a constant load moving along a beam [4].

The model of the moving mass consists of a point mass sliding along a beam. It brings some improvements to the model of a moving force where there is no inertia effect, but does not allow any consideration of the dynamics of the moving structure. There are many papers that deal with this model using different techniques [5–7]. A general method based on Green's functions was proposed in Ref. [5]. A different but very simple and practical method was put forward by Akin and Mofid in Ref. [6]. Their numerical method based on modal functions can be applied to various beam support conditions. The results of Ref. [5] showed that in some cases the inertia effect induced by the moving mass should be taken into account compared with the moving force model.

A more appropriate model for some applications is the moving oscillator model. In this case, the moving mass is attached to the structure either by a spring or by a mass–spring assembly. The moving oscillator problem has also been described in many papers using different mathematical approaches [1–3,8–10]. At this

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Nomenclature

c	damping coefficient
EI	flexural rigidity of the beam
$f_c(x,t)$	contact force at position x
k	spring stiffness coefficient
m_s	sprung mass
m_u	unsprung mass
q_n	nth natural coordinate of the beam
t_r	the reattachment time
t_s	the separation time
u	the vertical displacement of the unsprung mass
v	moving body speed
w	transverse deflection of the beam
$x(t_r^-) = \lim_{t\to t_r, t<t_r} x(t)$	
$x(t_r^+) = \lim_{t\to t_r, t>t_r} x(t)$	
z	the vertical displacement of the sprung mass
α	ratio of damping coefficient to sprung mass
$\delta(\cdot)$	Dirac delta function
ρA	beam mass per unit length
ω_0^2	ratio of spring stiffness to sprung mass
ω_n	nth natural angular frequency of the beam
ψ_n	nth normal mode of the beam

point, it is worth mentioning that in some cases the moving oscillator problem can be related to the moving mass problem by making the spring very stiff. A short justification of this assertion was given in Ref. [8] and for a more detailed approach Ref. [11] should be consulted.

None of the above-mentioned papers studied the possibility of separation between the moving structure and the supporting structure. Though the possibility of separation was admitted [1,2] or particularly mentioned to appear in some numerical experiments [12,13], Lee is the first researcher who studied separation in a moving-load (actually a moving mass) problem [14,15]. His moving mass model displayed the possibility of separation between the moving structure and the supporting structure. In that case, the contact force was time dependent and under certain conditions it became zero. The transition of the moving contact force from positive to zero was considered to be the onset of separation between the moving and supporting structures. A conclusion of Refs. [14,15] was that the influence of the separation on the beam dynamics is significant. Lee also concluded that the most important parameters to influence the onset of separation were the sliding speed and the ratio between the moving mass and the beam mass.

A more complete approach to the separation problem [16] included a moving oscillator. The paper also presents a novel approach to the reattachment of the moving structure to the supporting structure. The reattachment produces an impact that changes the dynamics of the response.

The contact force between the beam and the moving structure depends upon many of the model parameters and the authors' numerical experiments show that in many cases the condition for the onset of separation can easily occur. Many examples indicate that the increase in the speed of the moving structure produces rapid oscillations in itself when it approaches the end of the beam.

To the authors' best knowledge, separation in moving-load problems was studied only in Refs. [14–16] in which either a mass or a moving oscillator traversed an Euler–Bernoulli beam. In this paper the dynamics of an Euler–Bernoulli beam excited by a moving oscillator allowing separation is investigated. It is shown through numerical examples that the separation conditions described in Ref. [14] may occur quite often and the dynamic responses with and without separation may have noticeable differences.

A new and simpler criterion of reattachment after separation is also established in the paper. The reattachment of the mass to the beam may produce an impact and accordingly a jump in the velocity of the beam at the contact point. The approach to this problem differs from the one used in Ref. [16]. This paper presents a simpler method. Only the jump of the velocity is calculated, regardless of the value of the contact force at the instant of impact.

This paper uses the modal functions of the beam for its dynamic response. It also deals with the effect of separation and tries to find a simplified way to include the effect of the impact at the reattachment instant. The internal damping of the beam has been neglected. Actually damping in the beam would not change the technique used to solve the problem.

2. Equations of motion for the moving oscillator

The physical system analysed is an Euler–Bernoulli beam of length L, mass ρAL and flexural rigidity EI, supporting a two-mass oscillator moving at constant speed v, as shown in Fig. 1. The position of the oscillator on the beam is given by $x(t) = vt$. The contact surface between the beam and the oscillator is assumed to be smooth and without friction. When the oscillator slides over the beam, the equation of motion for the beam is described by

$$EI\frac{\partial^4 w}{\partial x^4}(x,t) + \rho A\frac{\partial^2 w}{\partial t^2}(x,t) = -f_c(x,t)\delta(x - vt), \tag{1}$$

where $f_c(x, t)$ is the contact force applied by the moving oscillator and $w(x, t)$ is the transverse deflection of the beam at coordinate x and time t. The beam is considered with different boundary conditions and at rest at $t = 0$.

For the oscillator the vertical displacement $z(t)$ of the sprung mass m_s and the vertical displacement $u(t)$ of the unsprung mass m_u are governed by a set of two equations of motion:

$$\begin{aligned} m_s\ddot{z}(t) &= -k(z(t) - u(t)) - c(\dot{z}(t) - \dot{u}(t)) - m_s g, \\ m_u\ddot{u}(t) &= k(z(t) - u(t)) + c(\dot{z}(t) - \dot{u}(t)) - m_u g + f_c(vt,t), \end{aligned} \tag{2}$$

where the overdot stands for the total derivative $\mathrm{d}/\mathrm{d}t$ and g is acceleration due to gravitation. These equations can be easily modified for arbitrary travelling motion of the moving oscillator along the beam [10,17]. Although in some special cases the initial conditions of the oscillator may incur some changes in the dynamic response of the beam [11], for this case, the oscillator is considered at rest at $t = 0$.

In the system described above, the contact between the beam and the unsprung mass of the oscillator generates a reaction force. This force is assumed to be positive when it is compressive. As the unsprung mass of the oscillator comes into contact with the beam, the vertical deflection of the beam and the displacement of the unsprung mass should be equal. In this respect, the beam and the unsprung mass are considered to be in contact when the following conditions are satisfied [18]:

$$\begin{aligned} u(t) - w(x,t)\big|_{x=vt} &= 0, \\ f_c(vt,t) &> 0. \end{aligned} \tag{3}$$

The beam deflection can be represented as a series expansion in terms of the eigenfunctions of the beam

$$w(x,t) = \sum_{n=1}^{\infty} q_n(t)\psi_n(x), \tag{4}$$

where $q_n(t)$ is the normal coordinate and $\psi_n(x)$ is the mass-normalized modal shape function [19].

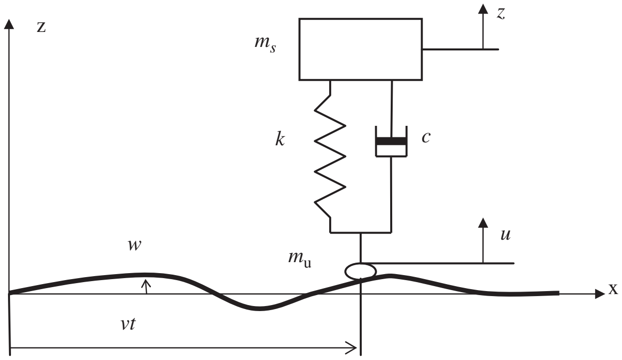

Fig. 1. Beam with a moving oscillator at constant speed v.

Eq. (1) can be converted into a system of differential equations in modal coordinates and then combined with Eqs. (2) and (3) and finally written as

$$\begin{Bmatrix} \ddot{\mathbf{q}} \\ \ddot{z} \end{Bmatrix} = \begin{bmatrix} \mathbf{M}^{-1}C & \frac{c}{\rho AL}\mathbf{M}^{-1}\boldsymbol{\psi}^{\mathrm{T}} \\ \alpha\boldsymbol{\psi} & -\alpha \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}} \\ \dot{z} \end{Bmatrix} + \begin{bmatrix} \mathbf{M}^{-1}\mathbf{K} & \frac{k}{\rho AL}\mathbf{M}^{-1}\boldsymbol{\psi}^{\mathrm{T}} \\ \omega_0^2\boldsymbol{\psi} + v\alpha\boldsymbol{\psi}' & -\omega_0^2 \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ z \end{Bmatrix} + \begin{Bmatrix} \mathbf{M}^{-1}\mathbf{f} \\ -g \end{Bmatrix}, \tag{5}$$

where the time-dependent mass, damping and stiffness matrices and vector **f** are given by

$$\mathbf{M} = \mathbf{I} + \frac{m_u}{\rho AL}\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\psi}, \quad \mathbf{C} = -2\frac{m_u}{\rho AL}v\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\psi} - \frac{c}{\rho AL}\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\psi},$$
$$\mathbf{K} = -\omega_n^2\delta_{kn} - v^2\frac{m_u}{\rho AL}\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\psi} - v\frac{c}{\rho AL}\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\psi} - \frac{k}{\rho AL}\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\psi}, \quad \mathbf{f} = -\frac{m_u}{\rho AL}\boldsymbol{\psi}^{\mathrm{T}}, \tag{6}$$

where

$$\boldsymbol{\psi} = (\psi_1(vt) \quad \psi_2(vt) \quad \dots \quad \psi_m(vt)), \quad \mathbf{q} = (q_1(t) \quad q_2(t) \quad \dots \quad q_m(t))^{\mathbf{T}} \tag{7}$$

denote the modal functions and modal coordinates.

In Eqs. (5)–(7) the superscript T denotes the transpose operation and **I** is an identity matrix of appropriate dimension.

Eq. (5) requires the inverse of matrix **M** which is a time-dependent matrix. Based on the Sherman–Morrison formula [20], the inverse of matrix **M** exists and moreover it has an analytical expression $\mathbf{M}^{-1} = \mathbf{I} - (m_u/\rho AL)\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\psi}/[1 + (m_u/\rho AL)\boldsymbol{\psi}\boldsymbol{\psi}^{\mathrm{T}}]$ [17,20].

3. Separation and reattachment

It is important to check whether or not the moving oscillator separates from the beam during its horizontal travel. This is done by monitoring the values of the contact force. The assumption made in this paper, similar to the one made in Ref. [14] is that separation occurs when the second condition (3) is breached. Subsequently, the contact force remains to be zero during separation.

The separation conditions are:

$$u(t) - w(x,t)\big|_{x=vt} > 0,$$
$$f_c(vt,t) = 0. \tag{8}$$

With these conditions Eqs. (1) and (2) are now explicitly written as

$$EI\frac{\partial^4 w}{\partial x^4}(x,t) + \rho A\frac{\partial^2 w}{\partial t^2}(x,t) = 0, \tag{9}$$

$$m_s\ddot{z}(t) = -k(z(t) - u(t)) - c(\dot{z}(t) - \dot{u}(t)) - m_s g,$$
$$m_u\ddot{u}(t) = k(z(t) - u(t)) + c(\dot{z}(t) - \dot{u}(t)) - m_u g. \tag{10}$$

These equations constitute the equations of motion during separation. The separation is considered to happen with no discontinuity in the velocity or the deflection of the beam. This way, the initial conditions for Eqs. (9) and (10) are given by the solutions of Eqs. (1) and (2) at t_s (see Fig. 2).

After separation, the first condition in Eq. (8) is monitored for reattachment. The assumption made for reattachment is that the beam displacement $w(x,t)$ equals the vertical coordinate of the unsprung mass $u(t)$ as in Eq. (3). A further assumption that seems plausible is that the unsprung mass just after the instant of reattachment (that is at t_r^+) takes the values of the displacement and the velocity of the beam at t_r^+. This is the assumption made in Ref. [14]. Another approach is proposed in Ref. [16] where a very short duration impact force is considered to act on the beam within (t_r^-, t_r^+). The initial velocity at t_r^+ is then determined as a constraint optimization problem. This is a more natural condition and allows for a jump in the beam velocity at the reattachment.

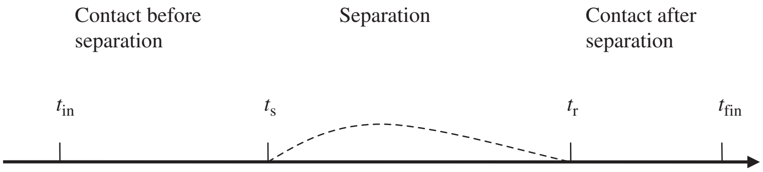

Fig. 2. Time history for a succession of contact–separation–reattachment.

A simpler approach is proposed in this paper whereby the velocity of the unsprung mass at t_r^+ is determined by the momentum theory and hence a jump in the velocity of the beam normally occurs. A further simplifying assumption for the reattachment is that the impact is perfectly plastic and from t_r^+ onwards the unsprung mass sticks to the beam, until a new separation takes place. This means the velocity at t_r^+ is usually different from that in the previous study [14], depending upon the characteristics of the impact. Any effect of bouncing and skimming of the moving structure on the beam is neglected as that will incur more difficulties and require a much more complicated analysis [21].

During the time period (t_s, t_r) the contact force is zero. The equation of motion of the beam due to impact is given by

$$EI\frac{\partial^4 w}{\partial x^4}(x,t) + \rho A\frac{\partial^2 w}{\partial t^2}(x,t) = -p\delta(x - vt)\delta(t - t_r), \tag{11}$$

where p is the impulse due to the impact. The equations of motion for the oscillator are:

$$\begin{aligned} m_s\ddot{z}(t) &= -k(z(t) - u(t)) - c(\dot{z}(t) - \dot{u}(t)) - m_s g, \\ m_u\ddot{u}(t) &= k(z(t) - u(t)) + c(\dot{z}(t) - \dot{u}(t)) - m_u g + p\delta(t - t_r). \end{aligned} \tag{12}$$

After instant t_r, when the unsprung mass reattaches to the beam, the equations of motion will be Eqs. (1) and (2) again with the initial conditions derived from Eqs. (11) and (12) under the assumption that after the impact, both the unsprung mass velocity and the beam velocity are equal.

By multiplying Eq. (11) with $\psi_n(x)$ and integrating over the beam length a new equation is obtained as

$$\ddot{q}_n(t) + \omega_n^2 q_n(t) = -\frac{p}{\rho AL}\psi_n(vt)\delta(t - t_r). \tag{13}$$

Eq. (11) can be solved using the distribution theory [22]. A similar approach can be found in Ref. [23]. In so doing the equation for the velocity jump is found to be

$$\dot{q}_n(t_r^+) - \dot{q}_n(t_r^-) = -\frac{p}{\rho AL}\psi_n(vt_r) \quad \text{for any } n = 1 \text{ to } \infty \tag{14}$$

as for q_n this is found to be a continuous function of t at t_r. This means the solution after reattachment (for $t > t_r$) can be obtained by imposing $\dot{q}_n(t_r^+)$ as the initial velocity condition in Eqs. (1) and (2) in order to study the vibration after the reattachment.

With this formula, the beam velocity jump can be expressed as

$$\dot{w}(x, t_r^+) - \dot{w}(x, t_r^-) = -\frac{p}{\rho AL}\sum_{n=1}^{\infty}\psi_n(vt_r)\psi_n(x). \tag{15}$$

The velocity jump of the unsprung mass is calculated in a similar way as

$$\dot{u}(t_r^+) - \dot{u}(t_r^-) = \frac{p}{m_u}. \tag{16}$$

The assumption made is that the reattachment occurs when $w(vt_r, t_r) = u(t_r)$ at a time instant t_r after separation, see condition (3). It is then also assumed that

$$\dot{w}(vt_r, t_r^+) = \dot{u}(t_r^+), \tag{17}$$

which means that after the reattachment the velocities of the unsprung mass and the beam at the contact coordinate are equal. This assumption allows the jump of the modal velocity of the beam at time t_r to be found as

$$\dot{q}_n(t_r^+) = \dot{q}_n(t_r^-) + \frac{\dot{u}(t_r^-) - \dot{w}(vt_r, t_r^-)}{(1/M) + (1/m_u)} \frac{\psi_n(vt_r)}{\rho AL}, \tag{18}$$

where

$$M = \frac{\rho AL}{\sum_n \psi_n^2(vt_r)}. \tag{19}$$

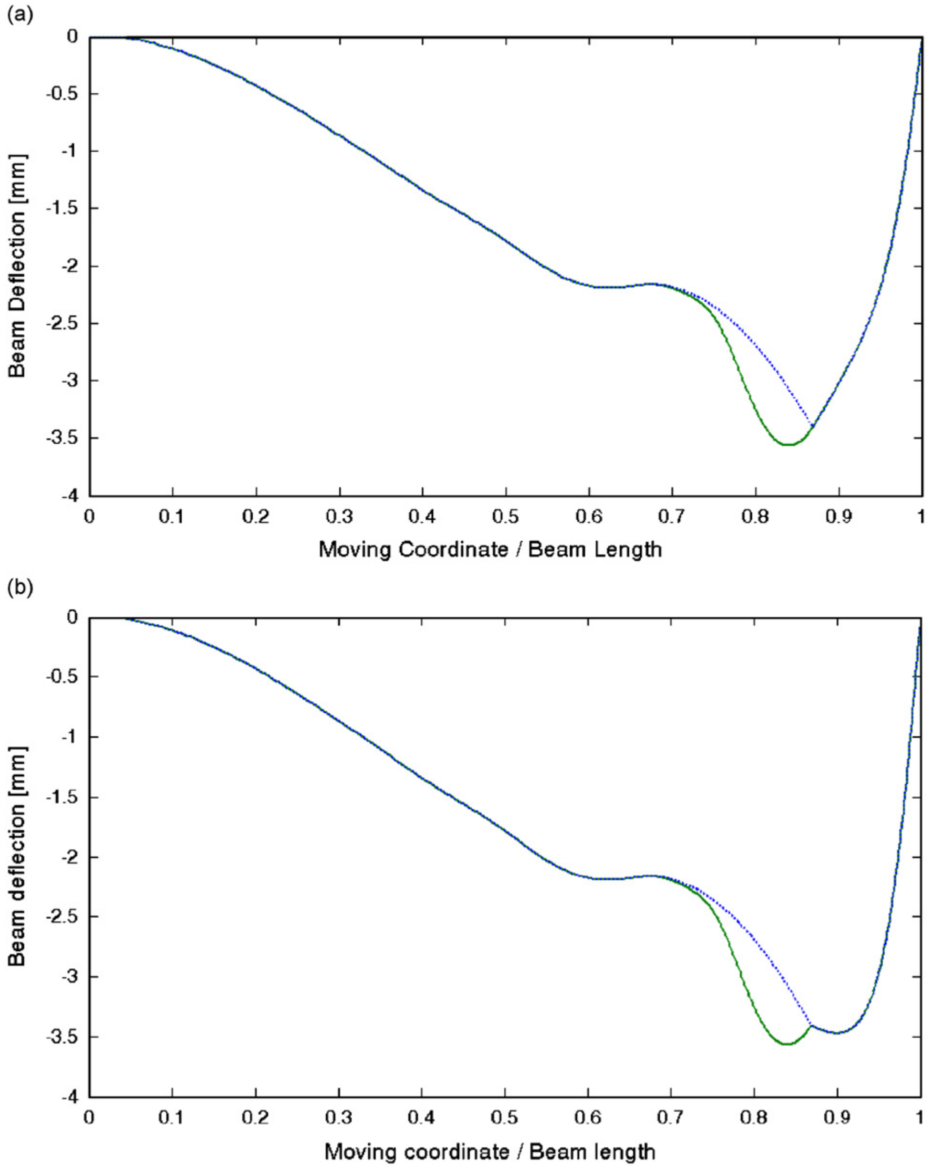

Fig. 3. Deflection of the simply supported beam and displacement of unsprung mass showing separation and reattachment: (a) without impact, (b) without impact --- displacement of the unsprung mass, —— beam deflection.

4. Numerical results

4.1. The influence of impact on the dynamic response

In order to test the separation and reattachment model presented in the paper, an Euler–Bernoulli beam with a moving oscillator shown in Fig. 1 is studied. The beam's geometric and material parameters selected are: length 4.5 m, flexural rigidity of 63,000 Nm^2 and mass per unit length of 20.245 kg/m. All these parameters are the same as in Ref. [14] except for the length of the beam. The critical speed for a simply supported beam with these parameters is about 140 km/h. For this beam with a moving oscillator having a total mass of 70 kg,

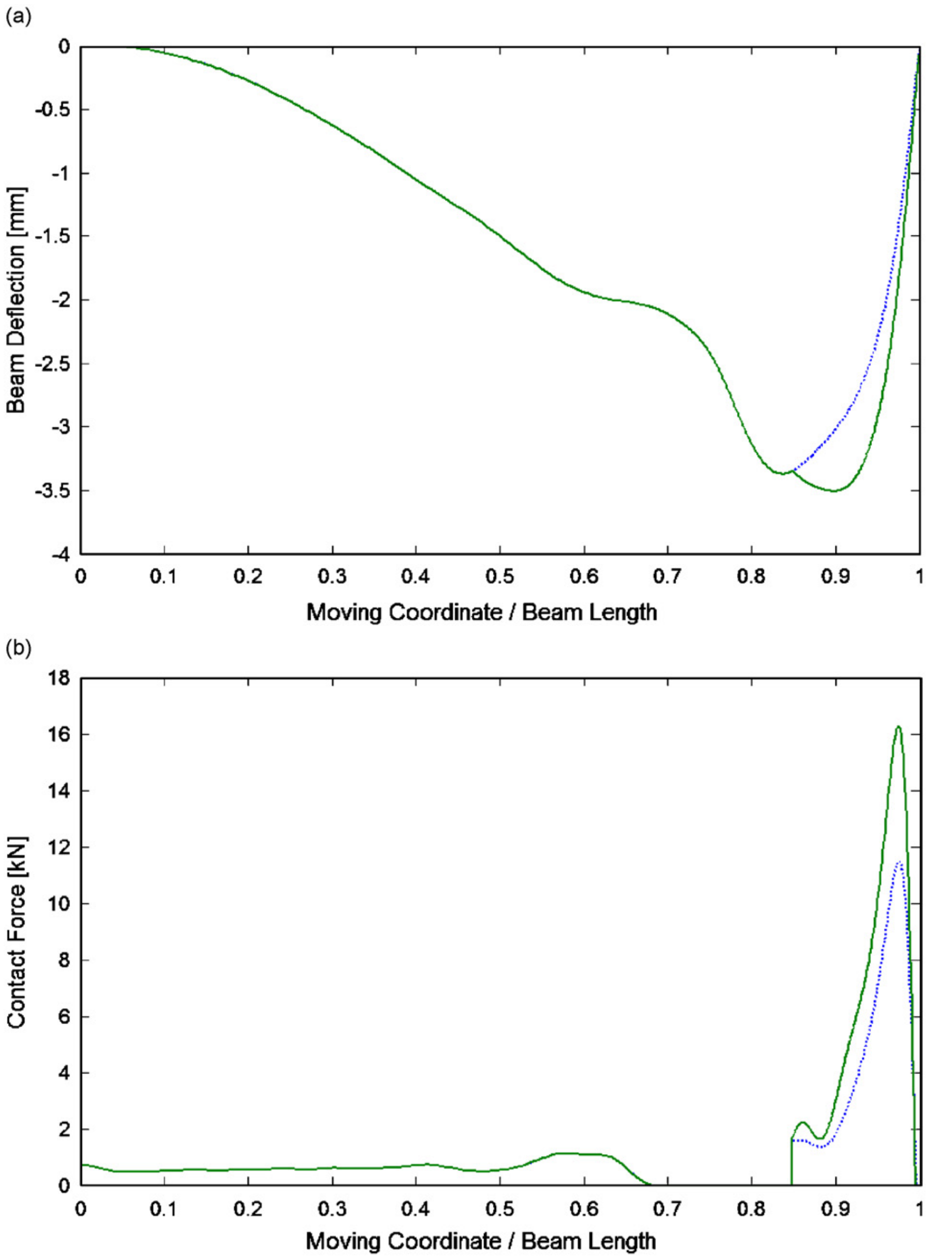

Fig. 4. (a) Deflection of the fixed-pinned beam. --- No impact, —— with impact. (b) Contact force along the fixed-pinned beam. --- No impact, —— with impact.

a speed of 160 km/h is found to be approximately the minimum speed at which separation might occur. The oscillator parameters are chosen as: $m_s = 50$ kg, $m_u = 20$ kg, $k = 10$ kN/m, $c = 200$ Ns/m. For this particular case the spring is considered to be soft as its eigenfrequency is about 2.25 Hz, and the static spring deflection is 0.05 m.

The effect of the impact modelled with Eqs. (11) and (12) on the beam dynamics is shown in Figs. 3a and b where both the beam deflection $w(x, t)$ and the trajectory of the unsprung mass $u(t)$ are plotted. In Fig. 3a, the body reattaches to the beam, trading its dynamics for the dynamics of the beam like in Ref. [14]. In Fig. 3b, the impact produces a discontinuity in the first derivative of the beam deflection altering the dynamics of the beam. For this case $v = 360$ km/h. No change in the system dynamics occurs before the reattachment.

For the same oscillator moving at the same speed 360 km/h and the same beam but with different boundary conditions, the effect of the impact on the beam dynamic response is also analysed. The boundary conditions studied are fixed-pinned, simply supported and fixed-free.

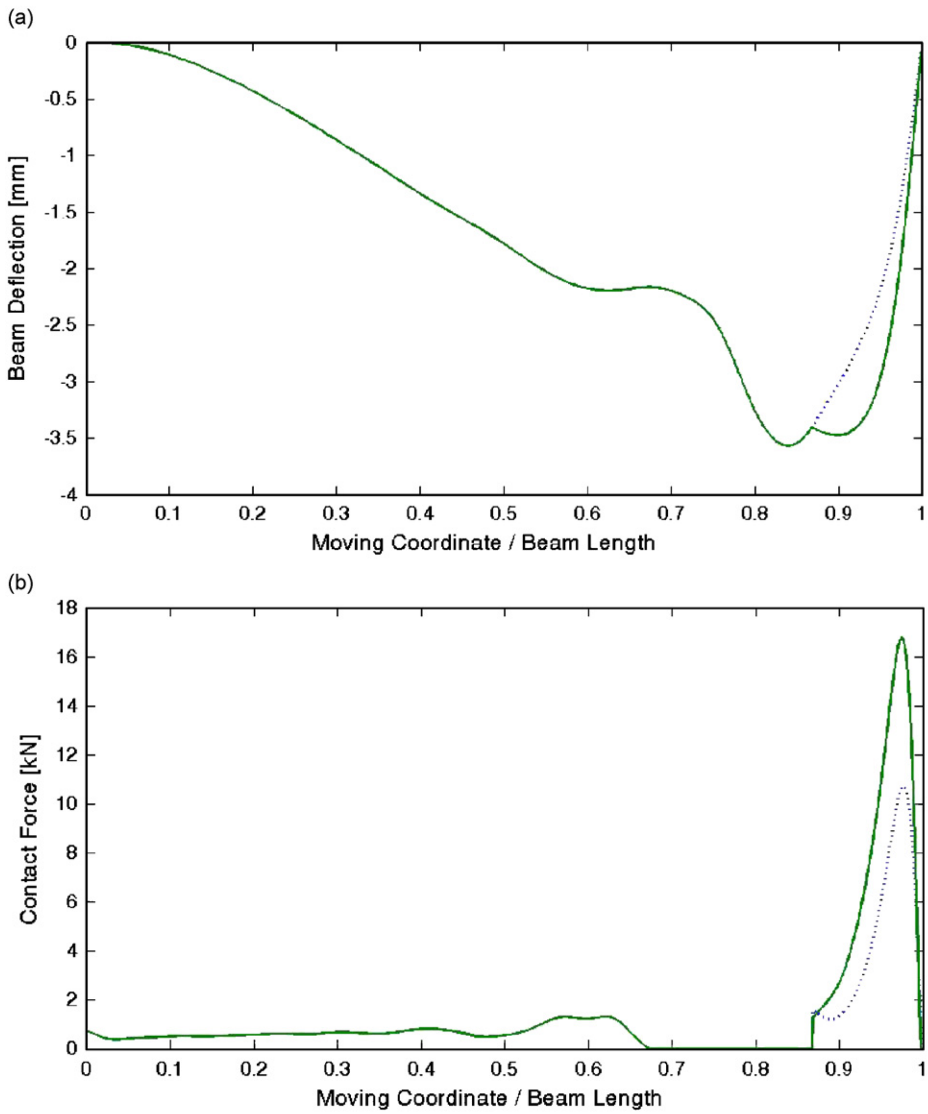

Fig. 5. (a) Deflection of the simply supported beam. --- No impact, —— with impact. (b) Contact force along the simply supported beam. --- No impact, —— with impact.

It can be seen that for the case of a fixed-pinned and a simply supported beam the impact produces an important difference in the beam deflection (Figs. 4a and 5a). The contact force variation along the moving coordinate for the same two cases is shown in Figs. 4b and 5b respectively, with the contact force being zero during the separation. When the impact is taken into account, the contact force near the end of the beam becomes higher.

The differences between the beam deflections determined using the separation model with and without impact for a cantilever beam are less significant, as shown in Fig. 6. It is also possible to see that the duration of separation is shorter.

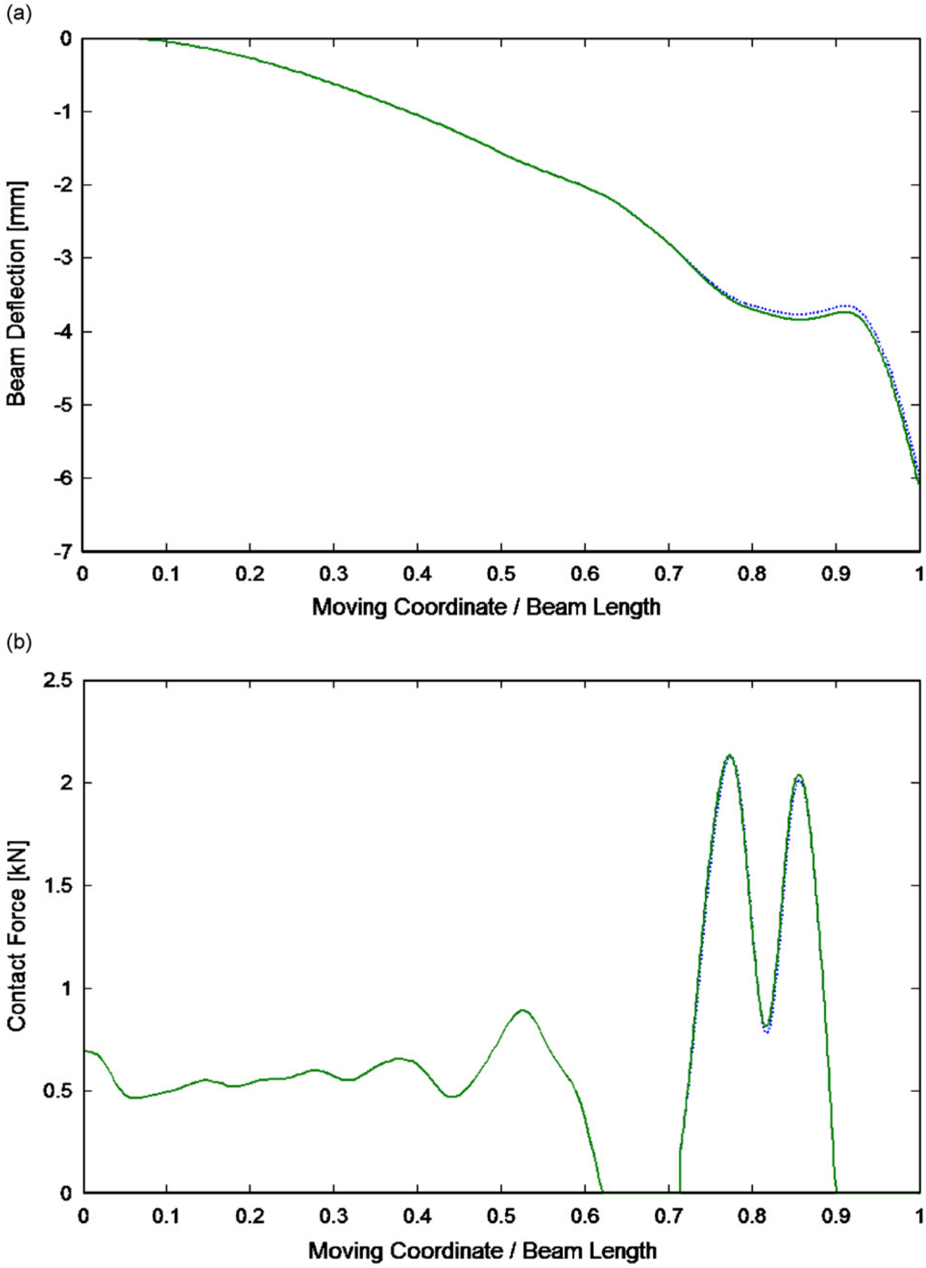

Fig. 6. (a) Deflection of the fixed-free beam. --- No impact, —— with impact. (b) Contact force along the fixed-free beam. --- No impact, —— with impact.

For these three particular examples, it can be concluded that as the rigidity of the supporting structure decreases due to the boundary conditions, the effect of the impact becomes smaller. It also becomes smaller for a lower moving speed, as shown by the beam deflection along the moving coordinate in Fig. 7. The moving oscillator speed for this example is 260 km/h.

The separation process is also influenced by the impact at high travelling speeds. Fig. 8 shows plots of the separation regions (SR) when the impact is taken into account alongside the SRs when the impact is not considered. For this example, the effect of the impact influences the occurrence of the separation.

4.2. Influence of the parameters upon the separation

An analytical study of the effect of beam and oscillator parameters on the separation process proves to be difficult. A practical approach though limited is to vary the parameters, one at a time and graphically analyse what differences these variations make to the separation process.

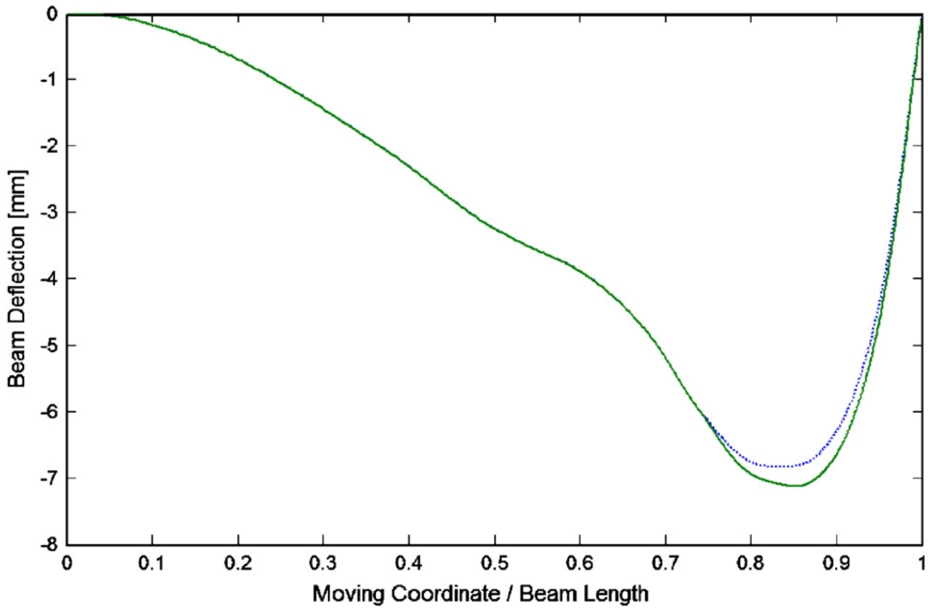

Fig. 7. Deflection of the simply supported beam at $v = 260$ km/h. --- No impact, — with impact.

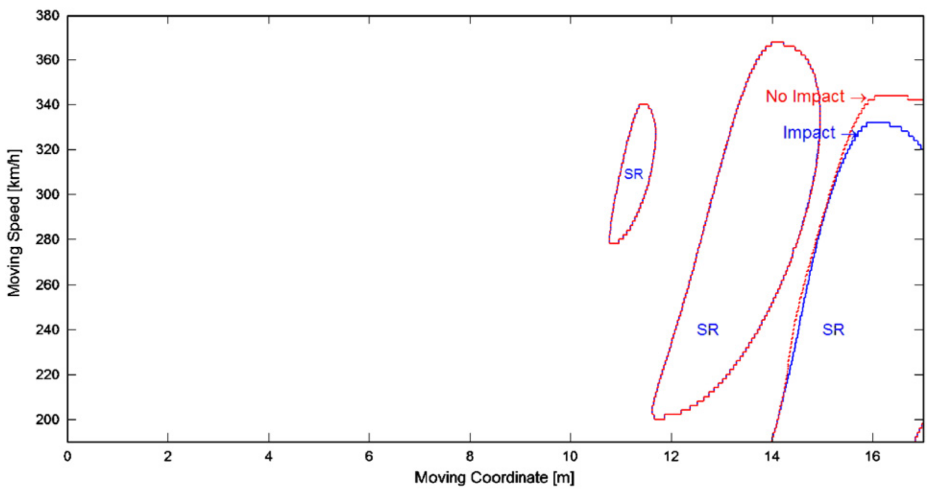

Fig. 8. Separation regions for a travelling oscillator for the model with impact and without impact. --- No impact, — with impact.

The moving speed is known to have a noticeable influence on the onset of separation. Two spring stiffness constants are considered now. In the first case the spring constant is $k = 10\,\text{kN/m}$, considered soft, while in the second case $k = 200\,\text{kN/m}$ considered hard. Fig. 9 shows that for a stiffer moving structure, a new SR appears at high speeds.

The ratio m_s/m_u at the same total mass m_s+m_u pushes the separation towards higher speed region (Fig. 10) when a soft spring is used ($k = 10\,\text{kN/m}$). The same trend is followed for a hard spring ($k = 200\,\text{kN/m}$), except in this case a second SR appears. The arrow on the plots shows the direction of increasing ratio of m_s/m_u.

The total mass of the oscillator has the same influence over the separation process as the single moving mass [14]. As the ratio $(m_s+m_u)/\rho AL$ increases, separation starts at lower speeds irrespective of the value of the spring stiffness.

The influence of the spring stiffness over the separation process shows a complicated pattern. This influence for a constant speed of 250 km/h is plotted in Fig. 11. The influence of the impact is seen to be small as the SRs with and without impact overlap.

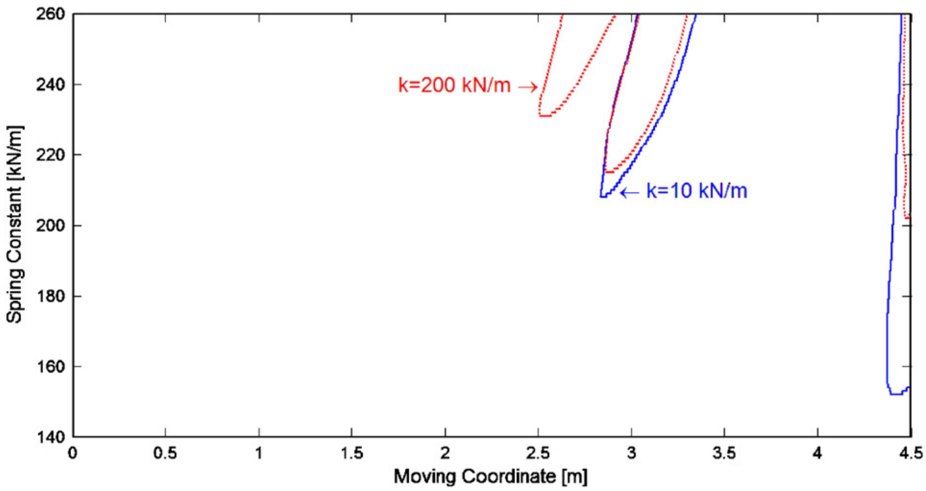

Fig. 9. Moving speed influence on the separation process. --- Spring stiffness 200 kN/m, —— spring stiffness 10 kN/m.

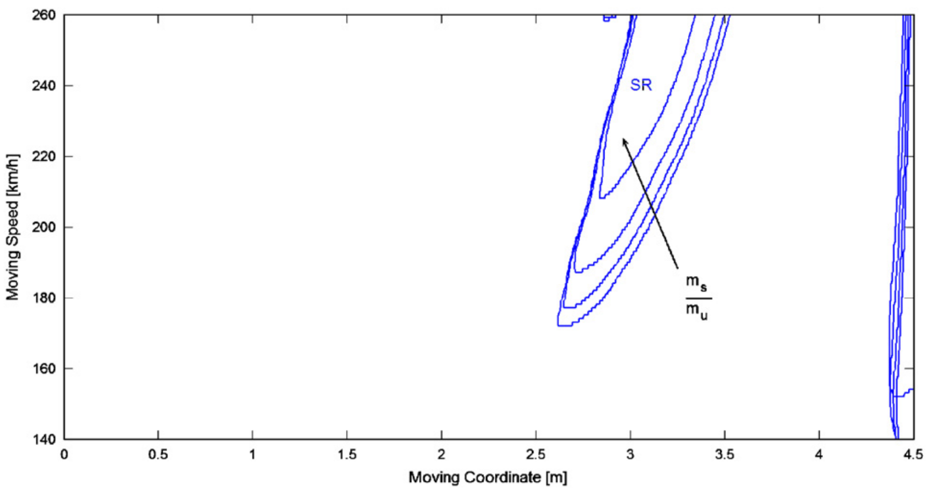

Fig. 10. Mass ratio influence for $k = 10\,\text{kN/m}$. $m_u/m_s = 20/50,\ 30/40,\ 40/30,\ 50/20$.

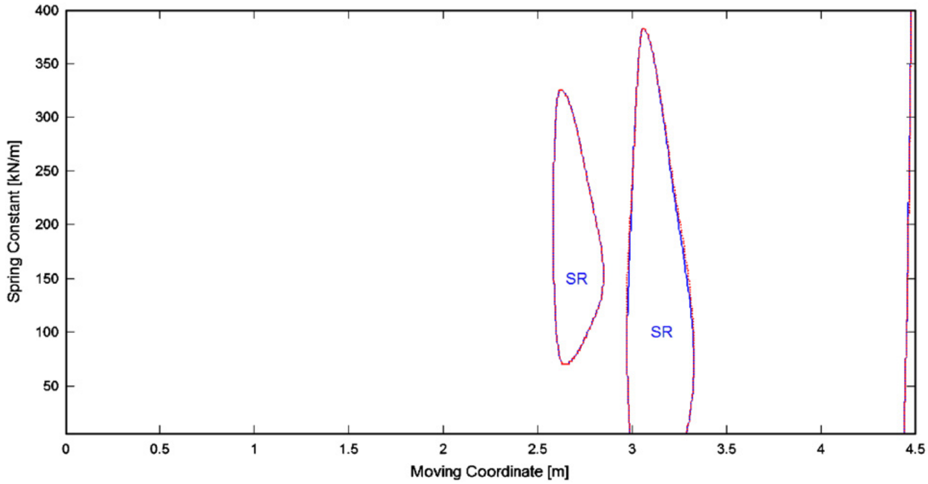

Fig. 11. The influence of the spring stiffness upon the separation process. --- No impact, —— with impact.

5. Conclusions

The dynamic response of a beam under a moving structure can be influenced by the occurrence of the separation phenomenon. Separation can easily occur at high moving speeds not only for the moving mass problem already presented in Ref. [14] but also for the moving oscillator problem studied in this paper and more possibly for more complicated structures [12].

Particularly for a moving oscillator like the one studied in the present paper, the stiffness of the spring connecting the two masses as well as the m_s/m_u ratio can influence to a large extent the onset of separation. It is more appropriate that for those studies involving separation the moving mass model should be replaced with the moving oscillator model.

When the spring stiffness becomes very high, the oscillator can usually be replaced with a single mass, without affecting the dynamic response of the beam. However that statement can be misleading when separation is considered. The onset of separation is influenced by very small perturbations in the contact force, a case when the asymptotic behaviour of the oscillator may not work [11].

The influence of the impact on the deflection of the beam is less important when the contact point approaches the end of a beam constrained to zero displacement by its boundary conditions. In this case the contact force after the impact becomes more important.

In many cases, the reattachment conditions can also influence the dynamics of the structure under the moving loads. The reattachment model presented in this paper, though based upon two simplifying assumptions, provides a way of assessing its influence on the dynamic response. Some examples where the response is considerably influenced by the impact model are presented herein. A more in-depth study of the reattachment phenomenon including restitution factor or a more complicated impact model may be required for some particular cases where the local dynamic response in the structure is very important.

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