Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 310 (2008) 998–1013
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

A new method for the pole estimation of linear time-invariant systems using singular value decomposition

Xiaobo Liu

General Motors Corporation, Noise & Vibration Center, 3300 GM Road, Milford, MI 48380, USA

Received 15 September 2006; received in revised form 13 August 2007; accepted 17 August 2007
Available online 27 September 2007

Abstract

A new method is presented for the pole estimation of linear time invariant (LTI) systems. For a single input case, this method gives estimation of each pole of the system. For a multiple input case, it estimates the repeated system poles with multiplicity equal to the number of inputs. The new method employs the singular value decomposition technique (SVD), and is shown to be numerically robust against measurement noises.
© 2007 Elsevier Ltd. All rights reserved.

1. Introduction

System identification is a process to curve fit a mathematical model of a physical system by using experimental data. Depending upon the nature and purpose of the identification problem, the mathematical model to be derived can be of different forms, as well as the experimental data to be used for curve fitting. A large class of physical systems are described or approximated by linear time invariant (LTI) models, therefore, the LTI system identification problem is of particular interests and importance, and has been a subject of extensive research in the past few decades (see, for example, Refs. [1–8] and the references therein). In such a case, we often seek to derive a transfer function or state space description of the underlying physical system, by using experimental data such as frequency responses, impulse responses, forced responses or free decays.

A most important step in a LTI system identification process is to estimate the system poles, which includes the estimation of system order, or equivalently the number of system modes well excited and observable in the experimental setup, as well as the complex values of the poles corresponding to these modes. It is a relatively easy task in the deterministic case. In the stochastic case where measurement noises are presented, it becomes quite more complex. In this case, different system orders are usually assumed and a series of parameter estimation is performed, and a stability diagram is generated which reflects the variation of the estimated parameters with respect to increments of the assumed system orders. However, with existing methods, sometimes it can be difficult to distinguish between actual system poles and spurious ones on the stability diagram. To improve the quality of parameter estimation, in this paper we present a new method by

E-mail address: xiaobo.liu@gm.com

0022-460X/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsv.2007.08.013

employing the singular value decomposition (SVD) technique [9]. The new method estimates each pole for a single input LTI system. For a multiple input system, it estimates the repeated system poles with multiplicity equal to the number of inputs. This method utilizes the column vectors of an orthogonal matrix obtained from SVD for parameter estimation. It gives another form of stability diagram, which reflects the variation of estimates with respect to the column index. The order of parameter estimation is fixed at every step, and the system poles are guaranteed to appear when the vector index exceeds the real system order. As shown by examples, the new method is numerically robust against measurement noises.

The rest of the paper is organized as follows. In Section 2, we review some LTI system theory related to the pole estimation problem. In Section 3, we give two propositions which characterize the system poles. On top of these propositions, we propose the new method in Section 4. A couple of examples are given in Section 5 to illustrate the use and effectiveness of the new method; Finally, conclusions are drawn in Section 6.

2. Description of LTI systems

A $m \times n$ linear time invariant system is described by a minimal state space realization [10]

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} = \mathbf{C}\mathbf{x}, \end{cases} \tag{1}$$

where $\mathbf{A} \in R^{k\times k}$, $\mathbf{B} \in R^{k\times n}$, $\mathbf{C} \in R^{m\times k}$, k is the system order, n is the number of inputs, and m is the number of outputs. The state space identification problem consists of determining the matrix triplet (**A**, **B**, **C**) from measured data. The transfer function matrix of the system is

$$\mathbf{\Pi}(s) = C(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \sum_{k=0}^{\infty} \mathbf{C}\mathbf{A}^k\mathbf{B}s^{-(k+1)}. \tag{2}$$

The eigenvalues of state matrix **A** are the poles of the system, and the coefficient matrices $\mathbf{C}\mathbf{A}^k\mathbf{B}$ in the above equation are called the system's Markov parameters. Let $s = \mathrm{j}\omega$, then we have an expression of the system's frequency response function matrix $\mathbf{\Pi}(\mathrm{j}\omega)$. The time domain counterpart of $\mathbf{\Pi}(\mathrm{j}\omega)$ is the matrix of impulse response functions, which in discrete time is expressed as

$$\mathbf{H}(\alpha) = \mathbf{C}\mathrm{e}^{\mathbf{A}\alpha\Delta t}\mathbf{B}, \tag{3}$$

where Δt is the sampling period. Let

$$\tilde{\mathbf{A}} = \exp(\mathbf{A}\Delta t), \tag{4}$$

then Eq. (3) is equivalent to

$$\mathbf{H}(\alpha) = \mathbf{C}\tilde{\mathbf{A}}^{\alpha}\mathbf{B}, \tag{5}$$

The eigenvalues $\lambda_j|_{1\leqslant j\leqslant k}$ of **A** are related to the eigenvalues $\tilde{\lambda}_j|_{1\leqslant j\leqslant k}$ of $\tilde{\mathbf{A}}$ as

$$\lambda_j = \ln(\tilde{\lambda}_j)/\Delta t, \tag{6}$$

and both matrices share the same set of eigenvectors. For matrix **A**, let $\omega_{\max}$ be the maximum of the absolute values of its eigenvalues' imaginary parts. If the condition $0 \leqslant \omega_{\max}\Delta t \leqslant \pi$ is satisfied, then there is a one-to-one correspondence between the eigenvalues of the two matrices $\tilde{\mathbf{A}}$ and **A**. This fact is just a reiteration of Shannon's sampling theorem. In such a case, the eigenstructure of the system matrix **A** can be uniquely recovered from that of $\tilde{\mathbf{A}}$. For such a reason, and because the Markov parameters of the system $(\tilde{\mathbf{A}}, \mathbf{B}, \mathbf{C})$ are the measurable impulse responses of the original system (**A**, **B**, **C**), we can focus on the identification of $(\tilde{\mathbf{A}}, \mathbf{B}, \mathbf{C})$ and return to the original state space system as needed.

3. Characteristics of system poles

In this section we give two propositions which characterize the poles of a LTI system. The propositions relate the system poles to the roots of a polynomial equation formed by a vector in the null space of a Hankel matrix.

Proposition 1. *Suppose an $m \times n$ system $(\tilde{\mathbf{A}}, \mathbf{B}, \mathbf{C})$ has a transfer function matrix*

$$\mathbf{\Pi}(s) = \sum_{i=1}^{k} \frac{\mathbf{\Delta}_i}{s - \tilde{\lambda}_i}, \tag{7}$$

where the $\tilde{\lambda}_i$'s are the poles of the system, and the $\mathbf{\Delta}_i$'s are the residue matrices. Let

$$\mathbf{H}(\alpha) = \mathbf{C}\tilde{\mathbf{A}}^{\alpha}\mathbf{B}, \quad \alpha = 0, 1, \ldots, \infty, \tag{8}$$

be the Markov parameters. Assume that each matrix $\mathbf{\Delta}_i|_{1 \leqslant i \leqslant k}$ is of full column rank, which implies that the order of the system is nk. Let

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{H}(\beta + p) & \mathbf{H}(\beta + p - 1) & \cdots & \mathbf{H}(\beta) \\ \mathbf{H}(\beta + 1 + p) & \mathbf{H}(\beta + p) & \cdots & \mathbf{H}(\beta + 1) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{H}(\beta + \alpha + p) & \mathbf{H}(\beta + \alpha + p - 1) & \cdots & \mathbf{H}(\beta + \alpha) \end{bmatrix}, \tag{9}$$

be a so-called Hankel matrix [2–6] for some integers β, p, α with $p, \alpha \geqslant nk$. Also, for a $n(p+1)$ dimensional vector

$$\mathbf{x} = [\, x_1 \quad x_2 \quad \ldots \quad x_{n(p+1)} \,]^{\mathrm{T}}, \tag{10}$$

let $\Gamma = \{\lambda_j\}_{1 \leqslant j \leqslant n(p+1)-1}$ be the set of roots to the polynomial equation

$$x_1 \lambda^{n(p+1)-1} + x_2 \lambda^{n(p+1)-2} + \cdots + x_{n(p+1)-1}\lambda + x_{n(p+1)} = 0. \tag{11}$$

If the vector $\mathbf{x}$ is in the null space of the matrix $\mathbf{\Omega}$, or equivalently

$$\mathbf{\Omega}\mathbf{x} = \mathbf{0}, \tag{12}$$

then

$$\sqrt[n]{\tilde{\lambda}_i} \in \Gamma \tag{13}$$

for any pole $\tilde{\lambda}_i$ of the system $(\tilde{\mathbf{A}}, \mathbf{B}, \mathbf{C})$.

In particular, if $n = 1$, which is the single input case, the proposition states that the system poles themselves are contained in the set Γ.

Proof. Suppose the characteristic equation of the matrix $\tilde{\mathbf{A}}$ takes the form

$$\begin{aligned} \det(\tilde{\lambda}\mathbf{I} - \tilde{\mathbf{A}}) &= \tilde{\lambda}^{k} + a_{k-1}\tilde{\lambda}^{k-1} + \cdots + a_0 \\ &= 0, \end{aligned} \tag{14}$$

from the Caley–Hamilton theorem we have

$$\tilde{\mathbf{A}}^{k} + a_{k-1}\tilde{\mathbf{A}}^{k-1} + \cdots + a_0\mathbf{I} = \mathbf{0}. \tag{15}$$

Furthermore, through consecutively multiplying Eq. (15) by $\tilde{\mathbf{A}}$ we get

$$\begin{cases} \tilde{\mathbf{A}}^{k} + a_{k-1}\tilde{\mathbf{A}}^{k-1} + \cdots + a_0 I = 0 \\ \tilde{\mathbf{A}}^{k+1} + a_{k-1}\tilde{\mathbf{A}}^{k} + \cdots + a_0 \tilde{\mathbf{A}} = 0 \\ \qquad\vdots \\ \tilde{\mathbf{A}}^{k+\eta} + a_{k-1}\tilde{\mathbf{A}}^{k+\eta-1} + \cdots + a_0 \tilde{\mathbf{A}}^{\eta} = 0 \\ \qquad\vdots \end{cases} \tag{16}$$

Pre-multiplying Eq. (16) by **C** and post-multiplying by **B**, and noting Eq. (8), we obtain

$$\begin{cases} \mathbf{H}(k) + a_{k-1}\mathbf{H}(k-1) + \cdots + a_0\mathbf{H}(0) = \mathbf{0} \\ \mathbf{H}(k+1) + a_{k-1}\mathbf{H}(k) + \cdots + a_0\mathbf{H}(1) = \mathbf{0} \\ \qquad\vdots \\ \mathbf{H}(k+\eta) + a_{k-1}\mathbf{H}(k+\eta-1) + \cdots + a_0\mathbf{H}(\eta) = \mathbf{0} \\ \qquad\vdots \end{cases} \tag{17}$$

or equivalently

$$\begin{bmatrix} \mathbf{H}(k) & \mathbf{H}(k-1) & \cdots & \mathbf{H}(0) \\ \mathbf{H}(k+1) & \mathbf{H}(k) & \cdots & \mathbf{H}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}(k+\eta) & \mathbf{H}(k+\eta-1) & \cdots & \mathbf{H}(\eta) \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n\times n} \\ a_{k-1}\mathbf{I}_{n\times n} \\ \vdots \\ a_0\mathbf{I}_{n\times n} \end{bmatrix} = 0. \tag{18}$$

Define

$$\mathbf{\Theta} = \begin{bmatrix} \mathbf{I}_{n\times n} & \mathbf{0} & \cdots & \mathbf{0} \\ a_{k-1}\mathbf{I}_{n\times n} & \mathbf{I}_{n\times n} & \cdots & \mathbf{0} \\ \vdots & a_{k-1}\mathbf{I}_{n\times n} & \cdots & \vdots \\ a_0\mathbf{I}_{n\times n} & & \cdots & \mathbf{0} \\ \mathbf{0} & a_0\mathbf{I}_{n\times n} & \cdots & \mathbf{I}_{n\times n} \\ \mathbf{0} & \mathbf{0} & \cdots & a_{k-1}\mathbf{I}_{n\times n} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & a_0\mathbf{I}_{n\times n} \end{bmatrix}_{n(p+1)\times n(p+1-k)}. \tag{19}$$

It is straightforward to verify that

$$\mathbf{\Omega\Theta} = \mathbf{0}. \tag{20}$$

Because the rank of the matrix $\mathbf{\Omega}$ is equal to the system order nk, and the $n(p+1-k)$ columns of $\mathbf{\Theta}$ are linearly independent, so the columns of $\mathbf{\Theta}$ span the null space of $\mathbf{\Omega}$. Therefore, if

$$\mathbf{\Omega x} = \mathbf{0}, \tag{21}$$

is assumed, let $q = p + 1 - k$, we have

$$
\begin{aligned}
x &= \begin{bmatrix}
\mathbf{I}_{n\times n} & \mathbf{0} & \cdots & \mathbf{0} \\
a_{k-1}\mathbf{I}_{n\times n} & \mathbf{I}_{n\times n} & \cdots & \mathbf{0} \\
\vdots & a_{k-1}\mathbf{I}_{n\times n} & \cdots & \vdots \\
a_0\mathbf{I}_{n\times n} & & \cdots & \mathbf{0} \\
\mathbf{0} & a_0\mathbf{I}_{n\times n} & \cdots & \mathbf{I}_{n\times n} \\
\mathbf{0} & \mathbf{0} & \cdots & a_{k-1}\mathbf{I}_{n\times n} \\
\vdots & \vdots & \cdots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & a_0\mathbf{I}_{n\times n}
\end{bmatrix}
\left\{\begin{array}{c}
\gamma_{1,1} \\ \vdots \\ \gamma_{1,n} \\ \gamma_{2,1} \\ \vdots \\ \gamma_{2,n} \\ \vdots \\ \gamma_{q,1} \\ \vdots \\ \gamma_{q,n}
\end{array}\right\} \\
&= \left\{\begin{array}{c}
\gamma_{1,1} \\ \vdots \\ \gamma_{1,n} \\ a_{k-1}\gamma_{1,1} \\ \vdots \\ a_{k-1}\gamma_{1,n} \\ \vdots \\ a_0\gamma_{1,1} \\ \vdots \\ a_0\gamma_{1,n} \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0
\end{array}\right\}
+ \left\{\begin{array}{c}
0 \\ \vdots \\ 0 \\ \gamma_{2,1} \\ \vdots \\ \gamma_{2,n} \\ a_{k-1}\gamma_{2,1} \\ \vdots \\ a_{k-1}\gamma_{2,n} \\ \vdots \\ a_0\gamma_{2,1} \\ \vdots \\ a_0\gamma_{2,n} \\ 0 \\ \vdots \\ 0
\end{array}\right\}
+ \cdots +
\left\{\begin{array}{c}
0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ \gamma_{q,1} \\ \vdots \\ \gamma_{q,n} \\ a_{k-1}\gamma_{q,1} \\ \vdots \\ a_{k-1}\gamma_{q,n} \\ \vdots \\ a_0\gamma_{q,1} \\ \vdots \\ a_0\gamma_{q,n}
\end{array}\right\}. \qquad (22)
\end{aligned}
$$

Furthermore, let $r = n(p+1)-1$, we have

$$
\begin{aligned}
&x_1\lambda^r + x_2\lambda^{r-1} + \cdots + x_r\lambda + x_{r+1} \\
&\quad = \gamma_{1,1}(\lambda^r + a_{k-1}\lambda^{r-n} + \cdots + a_0\lambda^{r-kn}) + \cdots + \gamma_{1,n}(\lambda^{r-n+1} + a_{k-1}\lambda^{r-2n+1} + \cdots + a_0\lambda^{r-(k+1)n+1}) \\
&\quad + \gamma_{2,1}(\lambda^{r-n} + a_{k-1}\lambda^{r-2n} + \cdots + a_0\lambda^{r-(k+1)n}) + \cdots + \gamma_{2,n}(\lambda^{r-2n+1} + a_{k-1}\lambda^{r-3n+1} + \cdots + a_0\lambda^{r-(k+2)n+1}) \\
&\quad + \cdots + \gamma_{q,1}(\lambda^{r-(q-1)n} + a_{k-1}\lambda^{r-qn} + \cdots + a_0\lambda^{r-(q+k-1)n}) + \cdots + \gamma_{q,n}(\lambda^{r-qn+1} + a_{k-1}\lambda^{r-(q+1)n+1} + \cdots + a_0\lambda^{r-(q+k)n+1}) \\
&\quad = \sum_{\mu=1}^{q}\sum_{v=1}^{n}\gamma_{\mu,v}(\lambda^{r-n(\mu-1)-(v-1)} + a_{k-1}\lambda^{r-n\mu-(v-1)} + \cdots + a_0\lambda^{r-n(\mu+k-1)-(v-1)}) \\
&\quad = (\lambda^{nk} + a_{k-1}\lambda^{n(k-1)} + \cdots + a_0)\sum_{\mu=1}^{q}\sum_{v=1}^{n}\gamma_{\mu,v}\lambda^{r-n(\mu+k-1)-(v-1)}.
\end{aligned} \tag{23}
$$

Therefore, the roots of the equation

$$\lambda^{nk} + a_{k-1}\lambda^{n(k-1)} + \cdots + a_0 = 0, \tag{24}$$

belongs to $\Gamma = \{\lambda_j\}_{1\leqslant j\leqslant r}$, which is the set of roots to Eq. (11). Because the roots of Eq. (24) are just the nth complex roots of poles of the system ($\tilde{\mathbf{A}}$, $\mathbf{B}$, $\mathbf{C}$), we conclude that

$$\sqrt[n]{\tilde{\lambda}_i} \in \Gamma, \tag{25}$$

for every pole of the system $\tilde{\lambda}_i$, where $1\leqslant i\leqslant k$. $\square$

The above proposition requires that the residue matrix corresponding to each pole of the system has full column rank. This condition is relaxed in the following proposition, which characterizes only those system poles with full column ranked residue matrices.

Proposition 2. *For the* $m\times n$ *system* $(\tilde{\mathbf{A}}, \mathbf{B}, \mathbf{C})$, *with* $\mathbf{\Omega}$, $\mathbf{X}$, *and* Γ *as defined in Proposition* 1, *if*

$$\mathbf{\Omega x} = \mathbf{0}, \tag{26}$$

then for any pole $\tilde{\lambda}_i$ *of the system with the corresponding residue matrix* $\mathbf{\Delta}_i$ *being of full column rank*,

$$\sqrt[n]{\tilde{\lambda}_i} \in \Gamma. \tag{27}$$

Proof. Partition the transfer function matrix $\mathbf{\Pi}(s)$ as

$$\mathbf{\Pi}(s) = [\,\mathbf{\Pi}_1(s) \quad \mathbf{\Pi}_2(s) \quad \ldots \quad \mathbf{\Pi}_n(s)\,], \tag{28}$$

where

$$\mathbf{\Pi}_\varepsilon(s)|_{1\leqslant\varepsilon\leqslant n} = \sum_{i=1}^{k}\frac{\mathbf{\Delta}_{i,\varepsilon}}{s-\tilde{\lambda}_i}, \tag{29}$$

in which $\mathbf{\Delta}_{i,\varepsilon}$ is the εth column of $\mathbf{\Delta}_i$. Each $\mathbf{\Pi}_\varepsilon(s)$ is a vector of transfer functions with respect to εth the input. We denote the Markov parameters of the single input system $\mathbf{\Pi}_\varepsilon$ as $\mathbf{h}_\varepsilon$.

Eq. (26) is equivalent to

$$\begin{bmatrix} \mathbf{h}_1(\beta+p) & \mathbf{h}_1(\beta+p-1) & \cdots & \mathbf{h}_1(\beta) \\ \mathbf{h}_1(\beta+1+p) & \mathbf{h}_1(\beta+p) & \cdots & \mathbf{h}_1(\beta+1) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{h}_1(\beta+\alpha+p) & \mathbf{h}_1(\beta+\alpha+p-1) & \cdots & \mathbf{h}_1(\beta+\alpha) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_{1+n} \\ \vdots \\ x_{1+pn} \end{Bmatrix}$$

$$+\begin{bmatrix} \mathbf{h}_2(\beta+p) & \mathbf{h}_2(\beta+p-1) & \cdots & \mathbf{h}_2(\beta) \\ \mathbf{h}_2(\beta+1+p) & \mathbf{h}_2(\beta+p) & \cdots & \mathbf{h}_2(\beta+1) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{h}_2(\beta+\alpha+p) & \mathbf{h}_2(\beta+\alpha+p-1) & \cdots & \mathbf{h}_2(\beta+\alpha) \end{bmatrix} \begin{Bmatrix} x_2 \\ x_{2+n} \\ \vdots \\ x_{2+pn} \end{Bmatrix}$$

$$+\cdots+\begin{bmatrix} \mathbf{h}_n(\beta+p) & \mathbf{h}_n(\beta+p-1) & \cdots & \mathbf{h}_n(\beta) \\ \mathbf{h}_n(\beta+1+p) & \mathbf{h}_n(\beta+p) & \cdots & \mathbf{h}_n(\beta+1) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{h}_n(\beta+\alpha+p) & \mathbf{h}_n(\beta+\alpha+p-1) & \cdots & \mathbf{h}_n(\beta+\alpha) \end{bmatrix} \begin{Bmatrix} x_n \\ x_{n+n} \\ \vdots \\ x_{n+pn} \end{Bmatrix} = \mathbf{0}. \tag{30}$$

Let $\mathbf{V}_\varepsilon$ be the vector space spanned by the columns of the matrix

$$\begin{bmatrix} \mathbf{h}_\varepsilon(\beta+p) & \mathbf{h}_\varepsilon(\beta+p-1) & \cdots & \mathbf{h}_\varepsilon(\beta) \\ \mathbf{h}_\varepsilon(\beta+1+p) & \mathbf{h}_\varepsilon(\beta+p) & \cdots & \mathbf{h}_\varepsilon(\beta+1) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{h}_\varepsilon(\beta+\alpha+p) & \mathbf{h}_\varepsilon(\beta+\alpha+p-1) & \cdots & \mathbf{h}_\varepsilon(\beta+\alpha) \end{bmatrix} \tag{31}$$

for $1\leqslant\varepsilon\leqslant n$. If there exists a system pole $\tilde{\lambda}_i$ such that the corresponding residue matrix $\mathbf{\Delta}_i$ is of full column rank, then

$$\mathbf{V}_\varepsilon \cap \mathbf{V}_\eta\Big|_{\substack{1\leqslant\varepsilon,\eta\leqslant n \\ \varepsilon\neq\eta}} = \mathbf{0}. \tag{32}$$

Therefore from Eq. (30) we obtain

$$\begin{bmatrix} \mathbf{h}_\varepsilon(\beta+p) & \mathbf{h}_\varepsilon(\beta+p-1) & \cdots & \mathbf{h}_\varepsilon(\beta) \\ \mathbf{h}_\varepsilon(\beta+1+p) & \mathbf{h}_\varepsilon(\beta+p) & \cdots & \mathbf{h}_\varepsilon(\beta+1) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{h}_\varepsilon(\beta+\alpha+p) & \mathbf{h}_\varepsilon(\beta+\alpha+p-1) & \cdots & \mathbf{h}_\varepsilon(\beta+\alpha) \end{bmatrix} \begin{Bmatrix} x_\varepsilon \\ x_{\varepsilon+n} \\ \vdots \\ x_{\varepsilon+pn} \end{Bmatrix} = \mathbf{0} \tag{33}$$

for $1\leqslant\varepsilon\leqslant n$. Also, because such a $\tilde{\lambda}_i$ is a pole of every single input system $\mathbf{\Pi}_\varepsilon(s)|_{1\leqslant\varepsilon\leqslant n}$, by using Proposition 1, we can conclude that

$$x_\varepsilon\tilde{\lambda}_i^p + x_{\varepsilon+n}\tilde{\lambda}_i^{p-1} + \cdots + x_{\varepsilon+(p-1)n}\tilde{\lambda}_i + x_{\varepsilon+pn} = 0 \tag{34}$$

for $1\leqslant\varepsilon\leqslant n$. Since

$$\begin{aligned} f(\lambda) &= x_1\lambda^{n(p+1)-1} + x_2\lambda^{n(p+1)-2} + \cdots + x_{n(p+1)-1}\lambda + x_{n(p+1)} \\ &= (x_1\lambda^{np} + x_{1+n}\lambda^{n(p-1)} + \cdots + x_{1+(p-1)n}\lambda^n + x_{1+pn})\lambda^{n-1} \\ &\quad + (x_2\lambda^{np} + x_{2+n}\lambda^{n(p-1)} + \cdots + x_{2+(p-1)n}\lambda^n + x_{2+pn})\lambda^{n-2} \\ &\quad + \cdots + (x_{n-1}\lambda^{np} + x_{(n-1)+n}\lambda^{n(p-1)} + \cdots + x_{(n-1)+(p-1)n}\lambda^n + x_{(n-1)+pn})\lambda \\ &\quad + (x_n\lambda^{np} + x_{2n}\lambda^{n(p-1)} + \cdots + x_{pn}\lambda^n + x_{(p+1)n}). \end{aligned} \tag{35}$$

Noting Eq. (34), we can conclude that

$$f(\sqrt[n]{\tilde{\lambda}_i}) = 0, \tag{36}$$

or equivalently,

$$\sqrt[n]{\tilde{\lambda}_i} \in \Gamma. \qquad \square \tag{37}$$

4. Estimation of system poles

From the conclusions obtained in the previous section, for an $m \times n$ LTI system ($\tilde{\mathbf{A}}$, $\mathbf{B}$, $\mathbf{C}$), given any vector in the null space of the Hankel matrix, we can solve for the complex roots of a polynomial equation formed by this vector as in Eq. (11), and take the nth power of the roots to form a set of estimates, the poles of the system will be contained in this set. Suppose a series of estimates have been obtained from different vectors in the null space of the Hankel matrix, then the variation of these estimates with respect to the vector indices will constitute a form of stability diagram, on which the poles of the system will appear stable and can thus be differentiated out. However, a problem remains to be solved should this approach be used for the estimation of the system poles: in real situations the measured Hankel matrix is always contaminated by measurement noises, so it is not possible to precisely determine vectors in its null space. To fix this problem, we now introduce a robust numerical method to obtain optimum approximations of the desired vectors. This method employs the singular value decomposition (SVD) technique.

Starting with the Hankel matrix $\mathbf{\Omega}$ as in Eq. (9), we can perform singular value decomposition [9]

$$\mathbf{\Omega}^{\mathrm{T}}\mathbf{\Omega} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^{\mathrm{T}}, \tag{38}$$

in which $\mathbf{U}$ is an orthogonal matrix of dimension $n(p+1) \times n(p+1)$, and $\mathbf{\Sigma}$ is a diagonal matrix

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_{n(p+1)} \end{bmatrix}, \tag{39}$$

where $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{n(p+1)} \geqslant 0$. If a $\sigma_j|_{1 \leqslant j \leqslant n(p+1)}$ is non-zero, it is called a singular value. The number of singular values is the same as the rank of the matrix $\mathbf{\Omega}$. Only when there is no measurement noise, this number is equal to the order of the underlying LTI system, and in such a case we can obtain the null space of $\mathbf{\Omega}$ as spanned by the columns of $\mathbf{U}$ corresponding to the zero diagonal entries of $\mathbf{\Sigma}$. If measurement noises are presented, $\mathbf{\Omega}$ is often full ranked and we cannot determine a vector orthogonal to its rows. However, we can find vectors which approximately satisfy the orthogonality condition. Let $\mathbf{u}_i|_{1 \leqslant i \leqslant n(p+1)}$ be the ith column of $\mathbf{U}$, it is easy to verify that

$$|\mathbf{\Omega}\mathbf{u}_i|^2 = \sigma_i, \tag{40}$$

thus σ_i is a measure of orthogonality between $\mathbf{u}_i$ and the row space of $\mathbf{\Omega}$. If for a vectors $\mathbf{u}_i$, the corresponding σ_i is small enough, then this vector is approximately orthogonal to the rows of $\mathbf{\Omega}$ and can be used for pole estimation. In practice, we can do a parameter estimation using each column vector of the matrix $\mathbf{U}$, and construct a stability diagram to reflect the variation of the estimated parameters with respect to the vector index. When the vector index increases passing through the real order of the underlying system, the corresponding singular value becomes small, and the system poles are expected to come into view and can thus be differentiated out.

In summary, we propose the following steps to estimate the poles of the $m \times n$ LTI system ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$):

(1) Form the Hankel matrix $\mathbf{\Omega}$ as in Eq. (9), and perform SVD as in Eq. (38) to obtain matrix $\mathbf{U}$.
(2) For each column vector $\mathbf{u}_i|_{1 \leqslant i \leqslant n(p+1)}$ of $\mathbf{U}$, let $\mathbf{x} = \mathbf{u}_i$ and find $\Gamma_i = \{\lambda_{i,j}\}_{1 \leqslant j \leqslant n(p+1)-1}$ which is the set of roots of Eq. (11).
(3) Form $\Gamma_i' = \{\ln(\lambda_{i,j}^n)/\Delta t\}_{1 \leqslant j \leqslant n(p+1)-1}$ for $1 \leqslant i \leqslant n(p+1)$.

(4) Construct a stability diagram to reflect the variation of Γ_i' with respect to the vector index i. Identify system poles as those stable parameters as the vector index increases passing through the system order.

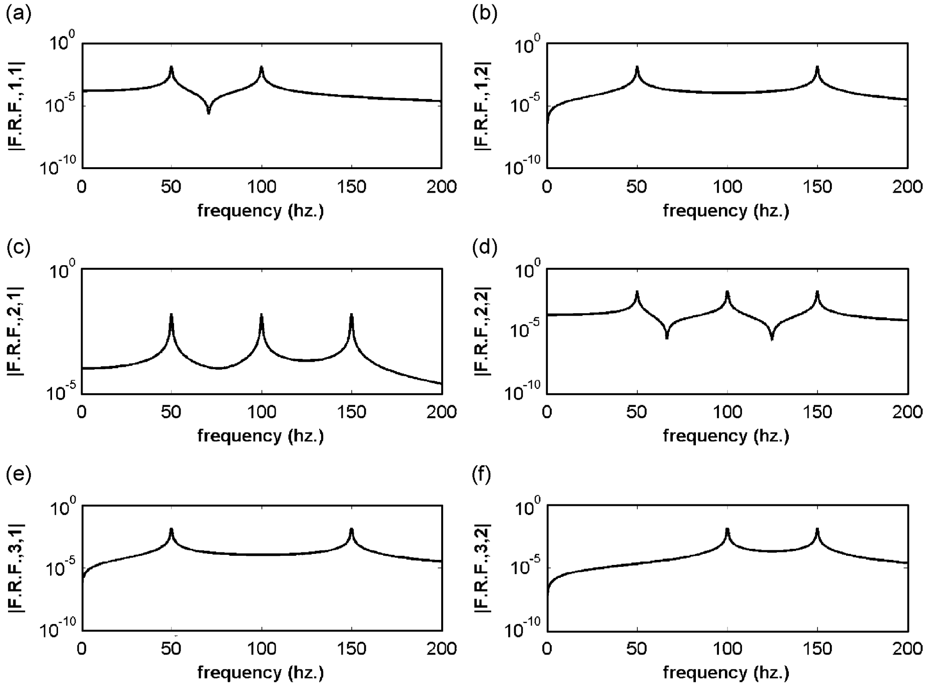

Fig. 1. Frequency response function magnitudes. (a) $\mathbf{\Pi}_{1,1}(\mathrm{j}2\pi f)$, (b) $\mathbf{\Pi}_{1,2}(\mathrm{j}2\pi f)$, (c) $\mathbf{\Pi}_{2,1}(\mathrm{j}2\pi f)$, (d) $\mathbf{\Pi}_{2,2}(\mathrm{j}2\pi f)$, (e) $\mathbf{\Pi}_{3,1}(\mathrm{j}2\pi f)$, (f) $\mathbf{\Pi}_{3,2}(\mathrm{j}2\pi f)$.

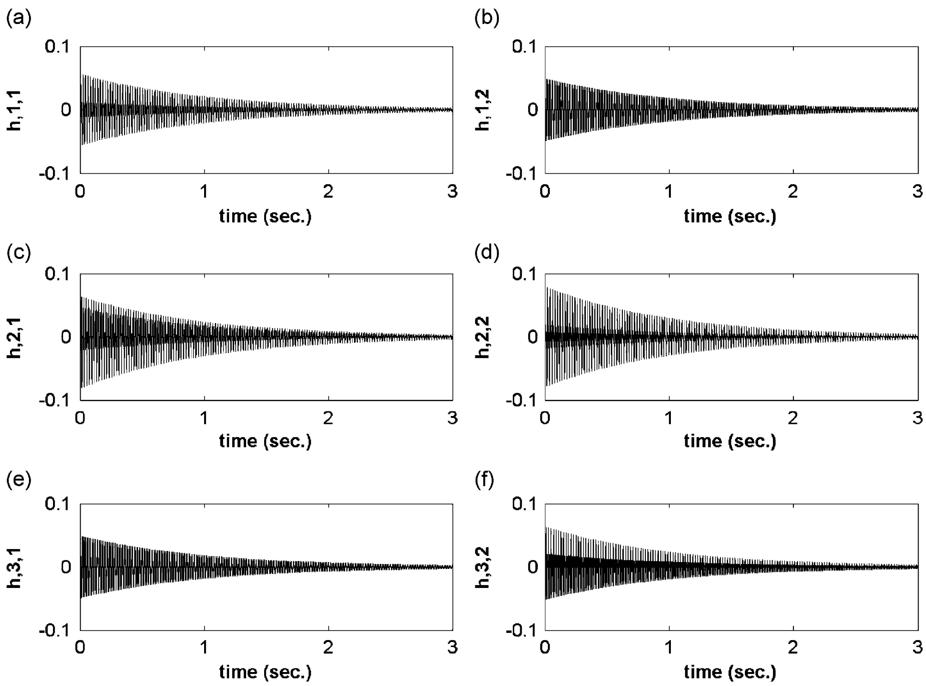

Fig. 2. Impulse response functions. (a) $h_{1,1}(t)$, (b) $h_{1,2}(t)$, (c) $h_{2,1}(t)$, (d) $h_{2,2}(t)$, (e) $h_{3,1}(t)$, (f) $h_{3,2}(t)$.

Note that in this new approach, the order of parameter estimation is fixed at every step. As shown by examples in the next section, this method gives robust estimates of system poles against measurement noises.

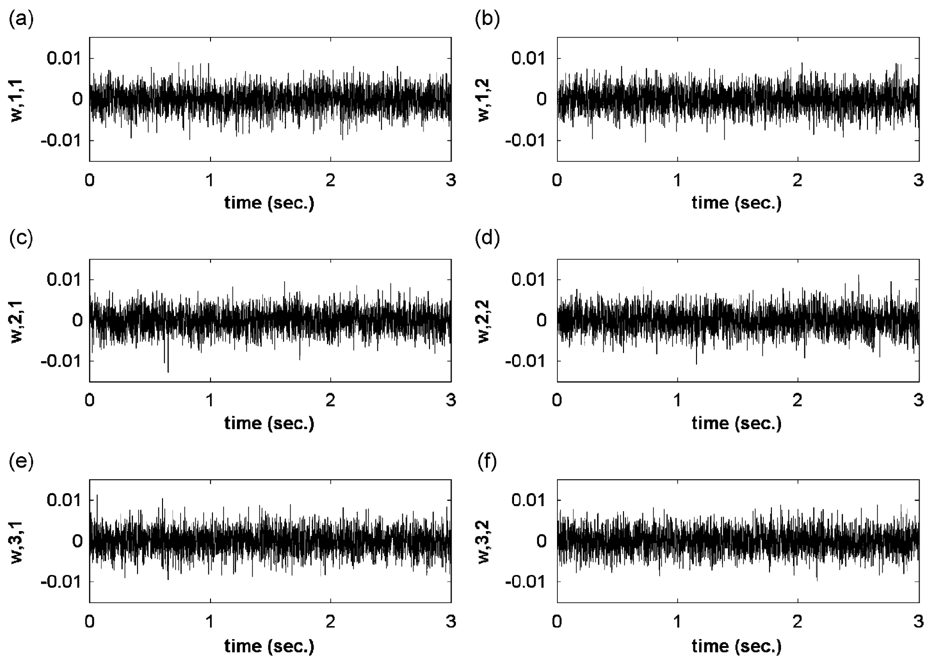

Fig. 3. Measurement noises. (a) $w_{1,1}(t)$, (b) $w_{1,2}(t)$, (c) $w_{2,1}(t)$, (d) $w_{2,2}(t)$, (e) $w_{3,1}(t)$, (f) $w_{3,2}(t)$.

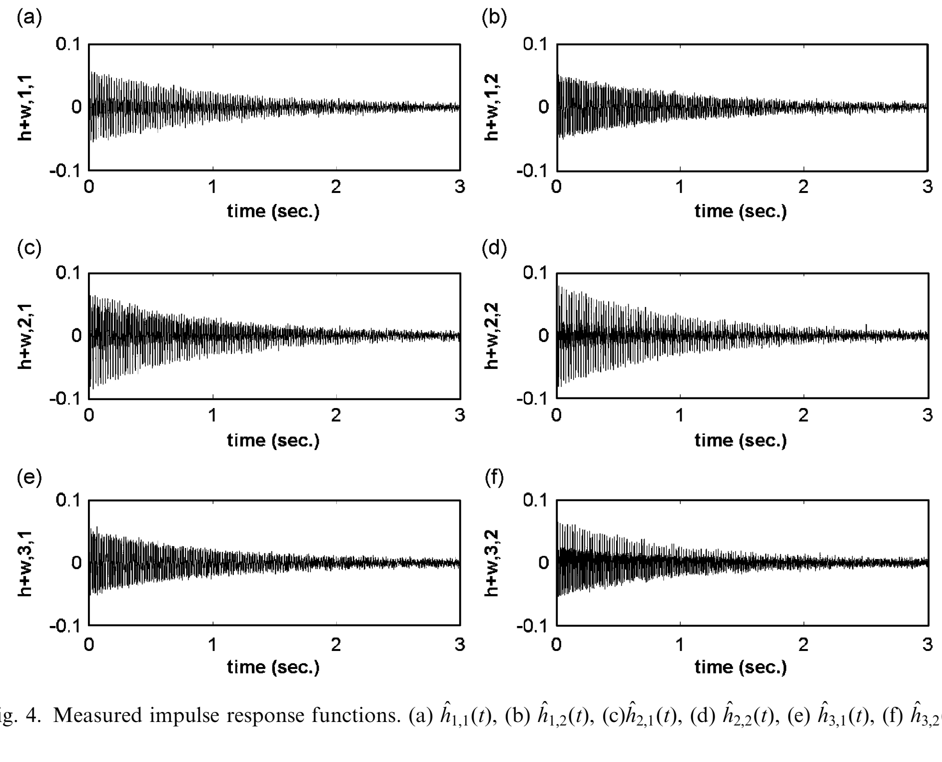

Fig. 4. Measured impulse response functions. (a) $\hat{h}_{1,1}(t)$, (b) $\hat{h}_{1,2}(t)$, (c)$\hat{h}_{2,1}(t)$, (d) $\hat{h}_{2,2}(t)$, (e) $\hat{h}_{3,1}(t)$, (f) $\hat{h}_{3,2}(t)$.

5. Examples

Example 1. A 3×2 LTI system is described by the following transfer function matrix:

$$\mathbf{\Pi}(s) = \begin{bmatrix} \frac{10}{(s-\lambda_1)(s-\lambda_2)} + \frac{20}{(s-\lambda_3)(s-\lambda_4)} & \frac{25138.164(s+1)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_5)(s-\lambda_6)} \\ \frac{10}{(s-\lambda_1)(s-\lambda_2)} + \frac{15715.698(s+1)}{(s-\lambda_3)(s-\lambda_4)(s-\lambda_5)(s-\lambda_6)} & \frac{10}{(s-\lambda_1)(s-\lambda_2)} + \frac{20}{(s-\lambda_3)(s-\lambda_4)} + \frac{30}{(s-\lambda_5)(s-\lambda_6)} \\ \frac{25138.164(s+1)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_5)(s-\lambda_6)} & \frac{15715.698(s+1)}{(s-\lambda_3)(s-\lambda_4)(s-\lambda_5)(s-\lambda_6)} \end{bmatrix}, \tag{41}$$

where

$$\begin{aligned} \lambda_1 &= -1 + \mathrm{j}100\pi, \quad \lambda_2 = \lambda_1^*, \\ \lambda_3 &= -1 + \mathrm{j}200\pi, \quad \lambda_4 = \lambda_3^*, \\ \lambda_5 &= -1 + \mathrm{j}300\pi, \quad \lambda_6 = \lambda_5^*. \end{aligned} \tag{42}$$

Each $\lambda_i|_{1 \leqslant i \leqslant 6}$ is a pole of the system, and the residue matrices can be evaluated as

$$\mathbf{\Delta}_i = (s - \lambda_i)\mathbf{\Pi}(s)|_{s=\lambda_i} \tag{43}$$

for $i = 1, 2, \ldots, 6$. It is easy to verify that each residue matrix has full column rank. Every pole $\lambda_i|_{1 \leqslant i \leqslant 6}$ is repeated and has a multiplicity of two, and the system is of order 12. The frequency response functions of the system are shown in Fig. 1, and the impulse responses in the time interval of 0–3 s are shown in Fig. 2. Suppose we have measured the impulse response functions, we now use the proposed method to estimate the system poles. If the impulse responses contain no measurement noises, the poles can be identified precisely. Here we simulate a general case in which measurement noises are presented. The noise signals and the measured impulse responses are shown in Figs. 3 and 4, respectively. In the simulation, we pick the sampling frequency to be $f_s = 1000\,\text{Hz}$, and the sampling period is $\Delta t = 1/f_s$. To proceed, we choose $p = 34$, $\alpha = 2500$, and

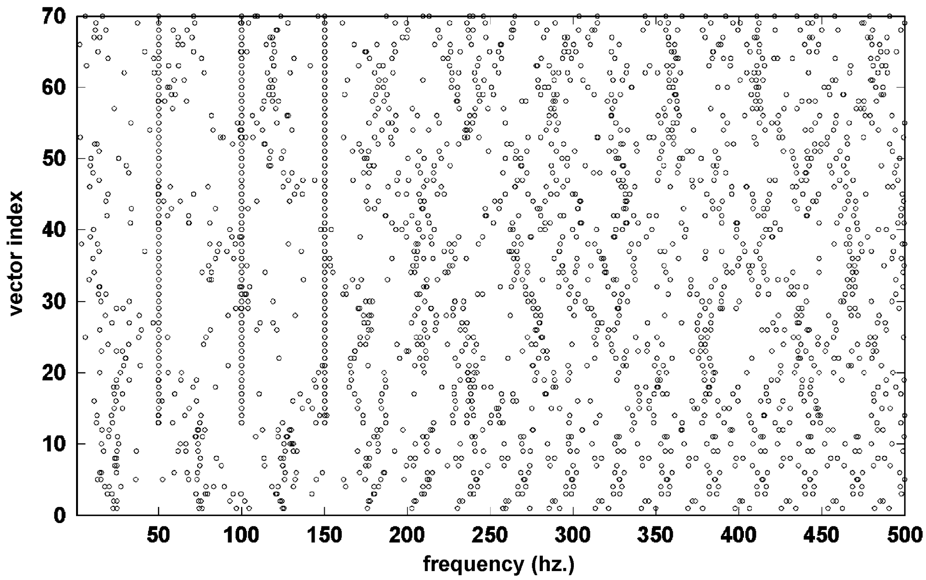

Fig. 5. Stability diagram. Frequency location of poles as a function of vector index. Unchanging frequency locations as index increases indicates a stable estimate of pole position.

construct an Hankel matrix as

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{H}(p) & \cdots & \mathbf{H}(1) & \mathbf{H}(0) \\ \mathbf{H}(p+1) & \cdots & \mathbf{H}(2) & \mathbf{H}(1) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{H}(\alpha+p) & \cdots & \mathbf{H}(\alpha+1) & \mathbf{H}(\alpha) \end{bmatrix}. \tag{44}$$

The Hankel matrix has full column rank because of measurement noise. Perform singular value decomposition

$$\mathbf{\Omega}^{\mathrm{T}}\mathbf{\Omega} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^{\mathrm{T}}, \tag{45}$$

where $\mathbf{U}$ is a 70×70 orthogonal matrix, and $\mathbf{\Sigma}$ is a 70×70 diagonal matrix with positive diagonal entries. For each column $\mathbf{u}_i|_{1 \leqslant i \leqslant 70} = [u_{1,i} \quad u_{2,i} \quad \cdots \quad u_{70,i}]^{\mathrm{T}}$ of $\mathbf{U}$, we find the set of roots $\Gamma_i = \{\lambda_{i,j}\}_{1 \leqslant j \leqslant 69}$ of the

Table 1
Results of pole estimation by using the last 15 column vectors of **U**

Vector index	Estimated poles		
70	−1.1964±j314.0345 −1.0868±j314.0972	−1.0166±j628.3304 −1.0758±j628.7170	−0.9982±j942.4672 −0.9651±j942.4692
69	−1.0029±j313.9789 −1.0167±j314.0516	−1.2478±j626.8193 −1.0821±j627.9977	−0.9598±j942.4711 −0.9935±j942.5752
68	−1.0830±j313.9560 −0.8758±j314.0170	−1.0051 ±j628.1814 −0.8400±j628.2651	−1.0012±j942.4106 −0.9994±j942.4668
67	−1.1122±j314.1464 −0.9238±j314.1751	−0.8850±j628.1957 −0.9323±j628.2329	−1.0190±j942.4241 −1.0114±j942.4417
66	−0.9987±j314.1254 −1.0415±j314.1408	−1.0402±j628.2734 −1.0020±j628.3541	−0.9776±j942.3683 −1.0545±j942.4326
65	−0.9523±j314.0542 −1.0384±j314.0928	−0.5270±j627.5201 −1.1212±j628.1005	−1.0849±j942.3777 −0.9425±j942.4426
64	−1.1230±j314.1294 −0.7604±j315.4042	−1.0380±j628.1152 −1.0353±j628.4873	−0.9494±j942.4487 −0.9140±j942.5306
63	−0.8708±j313.9145 −1.0254±j313.9493	−1.0249±j628.2390 −1.0111±j628.2754	−1.2086±j942.2728 −0.9856±j942.4449
62	−1.2160±j314.1277 −1.1890±j314.2177	−1.0094±j628.2301 −0.8390±j628.4039	−0.8234±j942.2723 −0.9370±j942.4120
61	−1.0042±j314.1444 −0.8102±j314.5570	−0.9928±j628.1636 −0.7002±j628.5356	−0.7589±j942.2541 −0.9104±j942.4181
60	−1.0490±j314.0133 −1.1232±j314.5206	−0.9057±j628.3118 −1.0303±j628.3187	−0.9279±j942.2791 −0.9666±j942.4099
59	−1.0511±j314.0598 −1.1002±j314.1890	−0.9525±j628.2578 −0.9912±j628.3033	−0.9758±j942.2372 −0.9713±j942.4070
58	−0.7630±j313.8828 −1.1730±j314.0065	−0.8710±j628.2306 −1.2827±j628.4673	−1.0191±j942.3993 −0.9783±j942.4357
57	−0.9929±j314.1404 −0.8230±j314.3440	−0.9879±j628.2307 −0.9320±j628.4295	−0.9586±j942.3126 −0.9616±j942.4220
56	−0.9669±j314.0413 −0.9999±j314.1162	−1.4389±j628.3003 −0.9796±j628.4822	−0.9535±j942.3795 −0.8047±j942.5550

polynomial equation

$$u_{1,i}\lambda_i^{69} + u_{2,i}\lambda_i^{68} + \cdots + u_{69,i}\lambda_i + u_{70,i} = 0, \tag{46}$$

and form a set $\Gamma'_i = \{\ln(\lambda_{i,j}^2)/\Delta t\}_{1\leqslant j\leqslant 69}$. Fig. 5 is a stability diagram which shows the variations of the $1/2\pi$ scaled imaginary parts of the parameters in Γ'_i, with respect to the column index of the vector $\mathbf{u}_i$ in matrix $\mathbf{U}$ in the parameter range $(0, f_s/2]$. It is clear that the true frequencies at 50, 100 and 150 Hz appear very stable on this diagram, when the column index is greater than the real system order 12. Thus the system poles can be

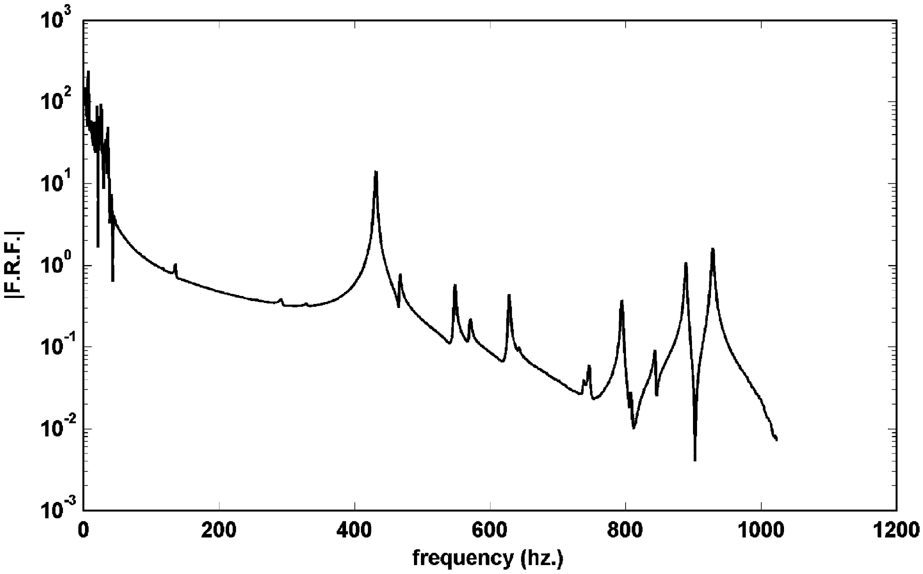

Fig. 6. Measured frequency response function magnitude.

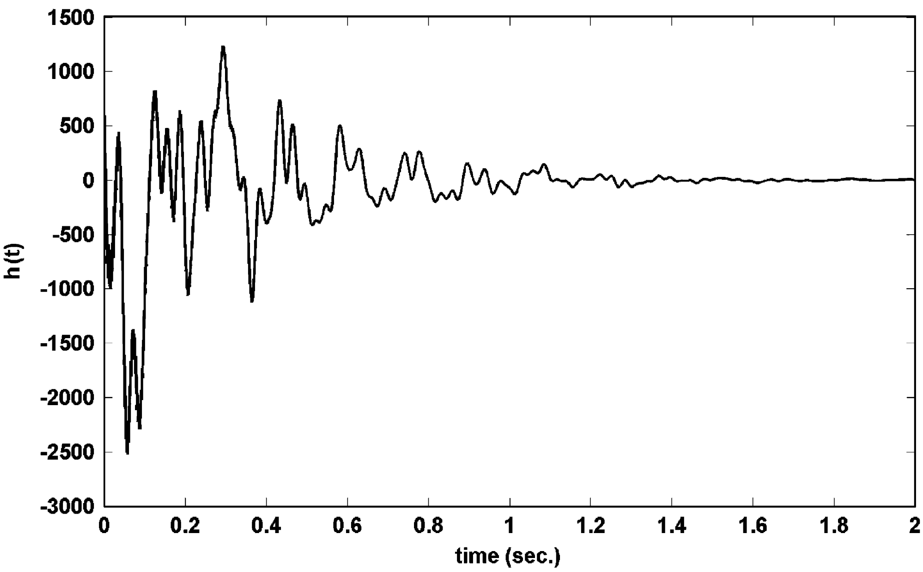

Fig. 7. Impulse response function $h(t)$.

differentiated out from the stability diagram. The poles estimated using the last 15 columns of **U** are listed in Table 1. It is shown that the proposed method gives robust estimates of system poles.

Example 2. A measured frequency response function is shown in Fig. 6. By using the inverse FFT, the impulse response function $h(t)$ is obtained and is shown in Fig. 7.

We now use the proposed method to estimate the system poles. To proceed, we choose $p = 49$, $\alpha = 3500$, and construct a Hankel matrix as

$$\mathbf{\Omega} = \begin{bmatrix} h(p) & \cdots & h(1) & h(0) \\ h(p+1) & \cdots & h(2) & h(1) \\ \vdots & \ddots & \ddots & \vdots \\ h(\alpha+p) & \cdots & h(\alpha+1) & h(\alpha) \end{bmatrix}. \tag{47}$$

Perform singular value decomposition

$$\mathbf{\Omega}^{\mathrm{T}}\mathbf{\Omega} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^{\mathrm{T}}, \tag{48}$$

where **U** is a 50×50 orthogonal matrix, and $\mathbf{\Sigma}$ is a 50×50 diagonal matrix. For each column $\mathbf{u}_i|_{1 \leqslant i \leqslant 50} = [\, u_{1,i} \quad u_{2,i} \quad \cdots \quad u_{50,i}\,]^{\mathrm{T}}$ of **U**, we find the set of roots $\Gamma_i = \{\lambda_{i,j}\}_{1 \leqslant j \leqslant 49}$ of the polynomial equation

$$u_{1,i}\lambda_i^{49} + u_{2,i}\lambda_i^{48} + \cdots + u_{49,i}\lambda_i + u_{50,i} = 0 \tag{49}$$

and form a set $\Gamma_i' = \{\ln(\lambda_{i,j})/\Delta t\}_{1 \leqslant j \leqslant 49}$. Fig. 8 is a stability diagram which shows the variations of the $1/2\pi$ scaled imaginary parts of the parameters in Γ_i', with respect to the column index of the vector $\mathbf{u}_i$ in matrix **U**. Fifteen system poles are identified from this stability diagram. The poles are also listed in Table 2. It is shown that the proposed method gives robust estimates of system poles.

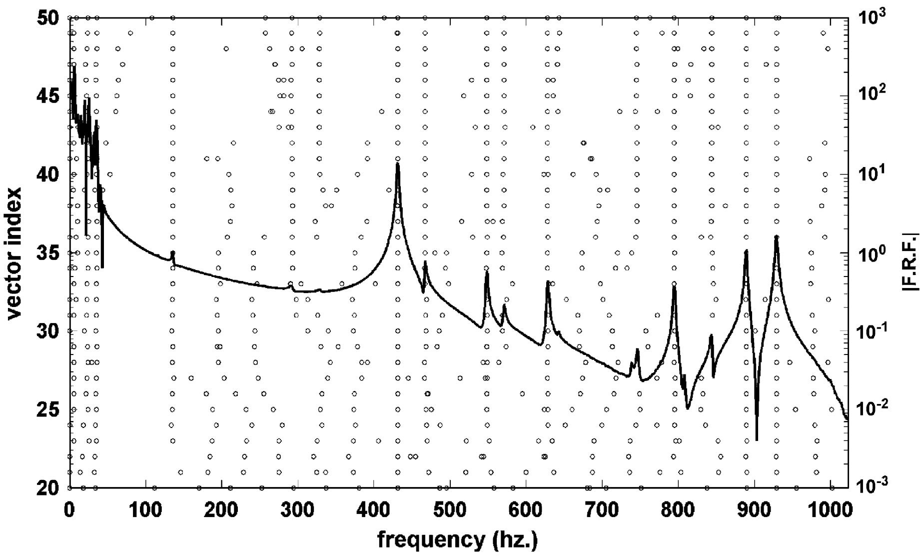

Fig. 8. Stability diagram. Frequency location of poles as a function of vector index. Unchanging frequency locations as index increases indicates a stable estimate of pole position.

Table 2
Results of pole estimation by using the last 10 column vectors of **U**

Vector index	Estimated poles ($\times 10^3$)
50	$-0.0091\pm j5.8358$, $-0.0072\pm j5.5873$, $-0.0047\pm j5.3012$, $-0.0078\pm j4.9934$, $-0.0107\pm j4.6783$ $-0.0072\pm j3.9479$, $-0.0090\pm j3.5858$, $-0.0072\pm j3.4450$, $-0.0059\pm j2.9342$, $-0.0069\pm j2.7102$ $-0.0114\pm j2.0681$, $-0.0135\pm j1.8365$, $-0.0058\pm j0.8526$, $-0.0029\pm j0.2261$, $-0.0050\pm j0.0575$
49	$-0.0087\pm j5.8363$, $-0.0073\pm j5.5874$, $-0.0064\pm j5.3017$, $-0.0080\pm j4.9928$, $-0.0124\pm j4.6815$ $-0.0070\pm j3.9479$, $-0.0090\pm j3.5862$, $-0.0071\pm j3.4450$, $-0.0059\pm j2.9342$, $-0.0069\pm j2.7102$ $-0.0120\pm j2.0669$, $-0.0137\pm j1.8358$, $-0.0056\pm j0.8522$, $-0.0040\pm j0.2256$, $-0.0001\pm j0.0364$
48	$-0.0087\pm j5.8363$, $-0.0072\pm j5.5874$, $-0.0054\pm j5.3040$, $-0.0061\pm j4.9920$, $-0.0106\pm j4.6825$ $-0.0071\pm j3.9477$, $-0.0082\pm j3.5859$, $-0.0071\pm j3.4448$, $-0.0061\pm j2.9343$, $-0.0070\pm j2.7102$ $-0.0175\pm j2.0639$, $-0.0178\pm j1.8341$, $-0.0058\pm j0.8522$, $-0.0038\pm j0.2247$, $-0.0002\pm j0.0362$
47	$-0.0087\pm j5.8363$, $-0.0073\pm j5.5875$, $-0.0097\pm j5.3044$, $-0.0073\pm j4.9936$, $-0.0117\pm j4.6835$ $-0.0069\pm j3.9476$, $-0.0081\pm j3.5864$, $-0.0071\pm j3.4448$, $-0.0061\pm j2.9343$, $-0.0070\pm j2.7102$ $-0.0200\pm j2.0644$, $-0.0219\pm j1.8363$, $-0.0053\pm j0.8535$, $-0.0036\pm j0.2176$, $-0.0099\pm j0.0354$
46	$-0.0087\pm j5.8363$, $-0.0073\pm j5.5874$, $-0.0063\pm j5.3022$, $-0.0080\pm j4.9935$, $-0.0121\pm j4.6822$ $-0.0061\pm j3.9483$, $-0.0105\pm j3.5875$, $-0.0065\pm j3.4452$, $-0.0062\pm j2.9344$, $-0.0070\pm j2.7102$ $-0.0171\pm j2.0608$, $-0.0182\pm j1.8295$, $-0.0052\pm j0.8528$, $-0.0041\pm j0.2190$, $-0.0062\pm j0.0363$
45	$-0.0088\pm j5.8363$, $-0.0074\pm j5.5874$, $-0.0067\pm j5.3008$, $-0.0082\pm j4.9937$, $-0.0126\pm j4.6822$ $-0.0058\pm j3.9482$, $-0.0101\pm j3.5880$, $-0.0066\pm j3.4451$, $-0.0062\pm j2.9344$, $-0.0070\pm j2.7102$ $-0.0152\pm j2.0613$, $-0.0149\pm j1.8315$, $-0.0054\pm j0.8527$, $-0.0032\pm j0.2215$, $-0.0016\pm j0.0362$
44	$-0.0087\pm j5.8363$, $-0.0073\pm j5.5874$, $-0.0053\pm j5.3018$, $-0.0076\pm j4.9936$, $-0.0138\pm j4.6706$ $-0.0066\pm j3.9476$, $-0.0085\pm j3.5869$, $-0.0070\pm j3.4449$, $-0.0061\pm j2.9342$, $-0.0070\pm j2.7102$ $-0.0114\pm j2.0639$, $-0.0126\pm j1.8346$, $-0.0054\pm j0.8528$, $-0.0030\pm j0.2211$, $-0.0019\pm j0.0361$
43	$-0.0087\pm j5.8363$, $-0.0072\pm j5.5874$, $-0.0058\pm j5.3020$, $-0.0080\pm j4.9933$, $-0.0122\pm j4.6827$ $-0.0079\pm j3.9469$, $-0.0093\pm j3.5951$, $-0.0057\pm j3.4445$, $-0.0064\pm j2.9346$, $-0.0070\pm j2.7102$ $-0.0077\pm j2.0616$, $-0.0098\pm j1.8356$, $-0.0056\pm j0.8526$, $-0.0062\pm j0.2273$, $-0.0014\pm j0.0723$
42	$-0.0087\pm j5.8363$, $-0.0073\pm j5.5874$, $-0.0058\pm j5.3010$, $-0.0079\pm j4.9932$, $-0.0106\pm j4.6839$ $-0.0073\pm j3.9478$, $-0.0085\pm j3.5870$, $-0.0071\pm j3.4449$, $-0.0060\pm j2.9342$, $-0.0070\pm j2.7102$ $-0.0103\pm j2.0646$, $-0.0124\pm j1.8364$, $-0.0050\pm j0.8527$, $-0.0032\pm j0.2186$, $-0.0018\pm j0.0358$
41	$-0.0088\pm j5.8362$, $-0.0074\pm j5.5872$, $-0.0054\pm j5.3002$, $-0.0077\pm j4.9932$, $-0.0093\pm j4.6854$ $-0.0073\pm j3.9478$, $-0.0087\pm j3.5871$, $-0.0071\pm j3.4450$, $-0.0060\pm j2.9342$, $-0.0070\pm j2.7102$ $-0.0097\pm j2.0647$, $-0.0120\pm j1.8368$, $-0.0041\pm j0.8525$, $-0.0021\pm j0.2273$, $-0.0097\pm j0.0575$

6. Conclusions

We have presented a new method for the pole estimation of linear time invariant systems. This method employs the singular value decomposition technique, it is easy to be implemented and is robust against measurement noises. We have illustrated its use and effectiveness through examples.

Acknowledgments

The author would like to acknowledge the reviewer's valuable comments.

References

[1] R.J. Allemang, D.L. Brown, A unified matrix polynomial approach to modal identification, *Journal of Sound and Vibration* 211 (3) (1998) 301–322.
[2] D.J. Ewins, *Modal Testing: Theory and Practice*, Research and Studies Press Ltd., UK, 1984.
[3] B.L. Ho, R.E. Kalman, Effective construction of linear state variable models from input/output data, *Proceedings of the Third Annual Allerton Conference on Circuit and System Theory*, Monticello, IL, 1965, pp. 449–459.

[4] J.N. Juang, R.S. Pappa, An eigensystem realization algorithm for modal parameter identification and model reduction, *Journal of Guidance, Control and Dynamics* 8 (5) (1985) 620–627.
[5] J.N. Juang, *Applied System Identification*, Prentice-Hall, Englewood Cliffs, NJ, 1994.
[6] X. Liu, A state space method for modal identification of mechanical systems from time domain responses, *Shock and Vibration* 12 (4) (2005) 273–282.
[7] L. Ljung, *System Identification: Theory for the User*, Prentice-Hall, Englewood Cliffs, NJ, 1999.
[8] Y. Zhang, Z. Zhang, X. Xu, H. Hua, Modal parameter identification using response data only, *Journal of Sound and Vibration* 282 (2005) 367–380.
[9] G.H. Golub, C.F. Von Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, MD, 1996.
[10] C.T. Chen, *Linear System Theory and Design*, Oxford University Press, New York, 1999.