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Journal of Sound and Vibration 311 (2008) 212–223
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

A new criterion of period-doubling bifurcation in maps and its application to an inertial impact shaker

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Received 5 June 2007; received in revised form 24 August 2007; accepted 2 September 2007
Available online 24 October 2007

Abstract

A new critical criterion of period-doubling bifurcations is proposed for high dimensional maps. Without the dependence on eigenvalues as in the classical bifurcation criterion, this criterion is composed of a series of algebraic conditions under which period-doubling bifurcation occurs. The proposed criterion is applied to the analysis of period-doubling bifurcation in a two-degree-of-freedom inertial shaker model. It can be seen in this example that the proposed criterion is preferable to the classical bifurcation criterion in high dimensional maps.

0. Introduction

Period-doubling bifurcation of maps (or discrete-time dynamical systems) generated by iterated maps of nonlinear difference equations have attracted considerable attention in both theoretical studies and practical applications [1–3]. The software package MATCONT [3] allows one to compute period-doubling bifurcation points by using a prediction-correction continuation algorithm based on the Moore–Penrose matrix pseudo-inverse. The analysis of the normal forms of this bifurcation, which determines the type of bifurcation solutions and their stability, is presented in Ref. [1]. It is important to stress that the critical bifurcation conditions in the classical bifurcation theory are stated in terms of the properties of eigenvalues of Jacobian matrix [1,2]. Unfortunately, analytical expressions of all eigenvalues with respect to the bifurcation parameters, in general, are unavailable for a non-constant matrix of high order. This characteristic results in two obvious limitations of those classical critical criteria for high dimensional systems. One is that it is always a common idea to numerically compute all the eigenvalues for checking the eigenvalue assignment by scanning certain range of the system parameters [3]. The other is that it is non-trivial to numerically compute the transversality condition (i.e., the derivative of eigenvalue modulus with respect to the bifurcation parameter) owing to the lack of the analytical expression of the eigenvalue. These limitations becomes the obstacle in analyzing the effect of the parameters to the bifurcation in high dimensional systems, especially for the inverse problem of bifurcation controls (i.e., creation of certain type of bifurcation by control) [4–7]. The detailed

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2007.09.003

discussion [8] on the limitations of the classical bifurcation theory is presented for the inverse problem of Hopf bifurcation control. A new critical criterion of Hopf bifurcation for maps (the so-called Neimark–Sacker bifurcation) [8] was already established on the basis of the Schur–Cohn Stability Criterion [9]. As compared with the methodologies of numerical search for bifurcation points based on the classical bifurcation definitions [1,2], the effect of the bifurcation parameters may be explicitly formulated.

Researches into vibro-impact dynamics are of great significance for noise suppression, reliability analysis and optimization design of mechanical systems whose components collide with each other or other rigid obstacles [10]. The operation principle of vibro-impact systems such as impact dampers, inertial shakers, gears, offshore structures and milling machinery is based on the repeated impacts. These impacts introduce strong nonlinearity [11,12] into the system. The bifurcation phenomena of vibro-impact systems have already been extensively studied in literature, see e.g. grazing bifurcations [13–16], C-bifurcations [17], period-doubling bifurcations [18], Hopf bifurcation in non-resonance case [19] and at resonance points [20] as well as the codimension bifurcations including degenerate Hopf bifurcation [21], interaction of Hopf and period-doubling bifurcations [22], and Hopf–Hopf bifurcation [23].

This paper is motivated by the previous results reported in Refs. [18–23]. In the literature [18–23], the rich and complicated bifurcation phenomena of multi-degree-of-freedom vibro-impact systems have been reported. However, the effect of parameters on the bifurcations are not discussed. A common idea in the existing literature is that the classical critical criterion stated in the definition of bifurcation is employed to search for a bifurcation parameter point. In order to determine the existence of bifurcations, one used to check point by point in certain parameter range if the critical conditions of both eigenvalue assignment and transversality condition are satisfied or not. As a result, one usually has to keep his fingers crossed to search for a proper parameter point at which the bifurcation occurs. Therefore, it is of great significance to develop proper bifurcation critical criteria for high dimensional or complicated dynamical systems such as multi-degree-of-freedom vibro-impact systems.

The main purpose of this paper is to develop a new critical criterion without using eigenvalues for period-doubling bifurcations. The new critical criterion, which is verified in Section 1, is composed of a set of equalities and inequalities which may avoid the difficulty brought about by the requirement of directly computing the eigenvalues. In Section 2, this criterion is used to precisely locate the critical bifurcation parameter region of a four-dimensional inertial shaker. The projection method [24,25] is employed to analyze the stability of the bifurcation solution. Finally, we show the rich dynamic responses of the vibro-impact system by numerical simulations. Section 3 concludes this paper.

1. New critical criterion

Consider a general n-dimentional map

$$x_{k+1} = f_{\mu}(x_k), \tag{1}$$

where $x_{k+1}, x_k \in R^n$ are the state vectors, k is the iterative index and $\mu \in R^m$ is the bifurcation parameter. The definition of period-doubling bifurcation for map (1) is stated as follows:

Definition 1 (*Kuznetsov [1] and Guckenheimer and Holmes [2]*). Assume that f_{μ} has a fixed point x_0. A period-doubling bifurcation takes place at a bifurcation parameter point $\mu = \mu_0$ if and only if the system (1) satisfies the following two conditions:

(C1) Eigenvalue assignment: at $\mu = \mu_0$, the Jacobian matrix $D_{x_k} f_{\mu}(x_0)$ has one real eigenvalue $\lambda_1(\mu)$ with $\lambda_1(\mu_0) = -1$and the rest $\lambda_j(\mu)$, $j = 2, \ldots, n$ inside the unit disk, i.e., $|\lambda_j(\mu_0)| < 1$;
(C2) Transversality condition: $(\partial|\lambda_1(\mu)|)/\partial\mu|_{\mu=\mu_0} \neq 0$.

The conditions (C1) and (C2) denote the critical conditions of period-doubling bifurcation which determine the existence of the bifurcation in maps. Definition 1 is also called as the classical critical criterion. It is clear that the classical critical criterion is stated in terms of the properties of eigenvalues. For instance, the transversality condition means that the real eigenvalue $\lambda_1(\mu_0) = -1$ lying on the unit circle will cross the unit circle at non-zero rate if μ varies nearby μ_0. It may be mentioned that some stability criteria without using eigenvalues, such as the Schur–Cohn Stability Criterion [9,26] and the formulation of the critical constraints for stability limit [27–29], represent the stability conditions with significant computational simplification for multi-parameter linear discrete system. We next show that these stability criteria may be modified to formulate a new bifurcation criterion.

In order to propose the new critical criterion, the characteristic polynomial of map (1) at the fixed point x_0 is defined as below:

$$p_\mu(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n, \tag{2}$$

where $a_0 = 1$ and $a_j = a_j(\mu)$, $j = 1, \ldots, n$. Consider a series of determinants: $\Delta_0^\pm(\mu) = 1$, $\Delta_1^\pm(\mu), \ldots, \Delta_n^\pm(\mu)$, which are defined as below:

$$\Delta_j^\pm(\mu) = \left| \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{j-1} \\ 0 & 1 & a_1 & \cdots & a_{j-2} \\ 0 & 0 & 1 & \cdots & a_{j-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \pm \begin{pmatrix} a_{n-j+1} & a_{n-j+2} & \cdots & a_{n-1} & a_n \\ a_{n-j+2} & a_{n-j+3} & \cdots & a_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & a_n & \cdots & 0 & 0 \\ a_n & 0 & \cdots & 0 & 0 \end{pmatrix} \right|, \quad j = 1, \ldots, n. \tag{3}$$

The following conditions (H1) and (H2) in Proposition 1 can be formulated to establish a critical criterion of period-doubling bifurcation without using eigenvalues.

Proposition 1. *Assume that f_μ has a fixed point x_0. A period-doubling bifurcation takes place at $\mu = \mu_0$ if and only if the following conditions* (H1) *and* (H2) *are satisfied,*

(H1) *Eigenvalue assignment: The following equalities or inequalities hold*:

$$p_{\mu_0}(-1) = 0, \quad p_{\mu_0}(1) > 0, \quad \Delta_{n-1}^\pm(\mu_0) > 0, \quad \Delta_j^\pm(\mu_0) > 0, \quad j = n-2, n-4, \ldots, 1$$
$$(\text{or } 2), \text{ when } n \text{ is odd } (\text{or even, respectively}); \tag{4}$$

(H2) *Transversality condition*:

$$\frac{\sum_{i=1}^n a_i'(-1)^{n-i}}{\sum_{i=1}^n (n-j+1)(-1)^{n-j}a_{j-1}} \neq 0, \tag{5}$$

where a_i' stands for the derivative of $a_i(\mu)$ with respect to μ at $\mu = \mu_0$.

Proof. Notice that if the conditions (H1) and (H2) are equivalent to (C1) and (C2), then they constitute a new critical criterion of period-doubling bifurcations for maps. Firstly, we show that (H1) is equivalent to (C1). In (H1), $p_{\mu_0}(-1) = 0$ indicates that the Jacobian matrix $D_{x_k}f_\mu(x_0)$ has one real eiganvalue -1 at $\mu = \mu_0$ and vice versa. Thus, the characteristic polynomial (2) of map f_μ at $\mu = \mu_0$ can be rewritten as

$$p_\mu(\lambda) = (\lambda + 1)\tilde{p}_\mu(\lambda), \tag{6}$$

where

$$\tilde{p}_\mu(\lambda) = \lambda^{n-1} + b_1\lambda^{n-2} + \cdots + b_{n-2}\lambda + b_{n-1}. \tag{7}$$

All what we have to do further is prove that all the roots of the polynomial equation $\tilde{p}_{\mu_0}(\lambda) = 0$ remain inside the unit disk. Now we introduce a set of new determinants $\tilde{\Delta}_j^\pm(\mu)$ for $\tilde{p}_\mu(\lambda)$, $j = n-1, n-2, \ldots, 1, 0$, similar to Eq. (3). Furthermore, the following Lemma 1 is needed to serve our purpose. □

Lemma 1 (*Brown [29]*). *The necessary and sufficient conditions that the roots of $\tilde{p}_{\mu_0}(\lambda) = 0$ remain inside the unit circle are given by one of the following conditions* (L1) *and* (L2)

(L1) $(-1)^{n-1}\tilde{p}_{\mu_0}(-1)>0$, $\tilde{p}_{\mu_0}(1)>0$ *as well as* $\tilde{\Delta}_j^{\pm}(\mu)>0$, $j=n-2,n-4,\ldots,2$ *or* 1, *or the equivalent*,
(L2) $(-1)^{n-1}\tilde{p}_{\mu_0}(-1)>0$, $\tilde{p}_{\mu_0}(1)>0$, $\tilde{\Delta}_{n-2}^{-}(\mu)>0$ *as well as* $\tilde{\Delta}_j^{\pm}(\mu)>0$, $j=n-3,n-5,\ldots,1$ *or* 2.

Next, we show that $\tilde{p}_{\mu_0}(\lambda)=0$ satisfies the conditions stated in Lemma 2 when (H1) holds. By expanding the RHS of Eq. (6) and comparing all the coefficients with Eq. (2), it is easy to obtain,

$$a_m = b_m + b_{m-1}, \tag{8}$$

where $m=1,\ldots,n$, $b_0=1$ and $b_j=1$ if $j>(n-1)$ or $j<0$. We substitute Eq. (8) into a set of determinants $\Delta_j^{\pm}(\mu)$, $j=n-1,n-2,\ldots,2$ or 1, and do the elementary row operations for each row of $\Delta_j^{\pm}(\mu)$ as follows: starting from the last row of $\Delta_j^{\pm}(\mu)$, multiply -1 with the mth row and add it to the $(m-1)$th row, and then multiply -1 with the new $(m-1)$th row and add it to the $(m-2)$th row. Repeating this operation until the first row, we can obtain:

$$\Delta_j^{\pm}(\mu_0) = \tilde{\Delta}_j^{\pm}(\mu_0), \quad j=n-1,n-2,\ldots,2 \text{ or } 1. \tag{9}$$

Substitution of Eq. (9) into the inequalities in (H1) gives $\tilde{\Delta}_j^{\pm}(\mu)>0$, $j=n-2,n-4,\ldots,2$ or 1. Furthermore, the following two formulations proven in Ref. [29] are needed to serve our purpose:

$$\tilde{\Delta}_{n-1}^{+}(\mu_0) = \tilde{p}_{\mu_0}(1)\tilde{\Delta}_{n-2}^{-}(\mu_0) \text{ and } \tilde{\Delta}_{n-1}^{-}(\mu_0) = (-1)^{n-1}\tilde{p}_{\mu_0}(-1)\tilde{\Delta}_{n-2}^{-}(\mu_0). \tag{10}$$

It is clear that (H1) guarantees $(-1)^{n-1}\tilde{p}_{\mu_0}(-1)>0$ and $\tilde{p}_{\mu_0}(1)>0$ when Eq. (9) is substituted into Eq. (10). With the satisfaction of the conditions in (L1), all roots of the polynomial equation $\tilde{p}_{\mu_0}(\lambda)=0$ lie inside the unit disk. We have deduced (C1) from (H1). Inversely, if (C1) holds, it is easy to check that we have $p_{\mu_0}(-1)=0$ and the rest inequalities in (H1) except for $p_{\mu_0}(1)>0$ by successively applying Eqs. (6), (9), Lemma 1 and Eq. (10). It follows from Eq. (6) that $p_{\mu_0}(1)=2\tilde{p}_{\mu_0}(1)$. This shows $p_{\mu_0}(1)>0$ due to $\tilde{p}_{\mu_0}(1)>0$, which is guaranteed by Lemma 1. Therefore, one can ascertain further that that (H1) is equivalent to (C1) in Definition 1.

As to the equivalence of (H2) and (C2), we differentiate both sides of $p_\mu(\lambda)=0$ with respect to μ directly and then substitute $\lambda(\mu_0)=-1$ into the results. Then the equivalence of (H2) and (C2) can be easily checked out.

It should be stressed that different from the conditions (C1) and (C2), the conditions (H1) and (H2) independent of the cumbersome computations of all eigenvalues, are formulated as a set of simple equalities or inequalities. To clarify these, taking the four-dimensional system (1) as an example, we have the following Corollary 1 according to Proposition 1.

Corollary 1. *For system* (1) *with* $n=4$, *a period-doubling bifurcation takes place at* $\mu=\mu_0$ *if and only if the following conditions are satisfied*:

(E1) *Eigenvalue assignment*: $1-a_1+a_2-a_3+a_4=0$, $1+a_1+a_2+a_3+a_4>0$, $|a_2-a_2a_4+a_4-a_4^3+a_4a_1^2-a_1a_3|<a_2a_4-a_2a_4^2+1-a_4^2+a_1a_3a_4-a_3^2$, $|a_3-a_4a_1|<1-a_4^2$;
(E2) *Transversality condition*:

$$\frac{a_4'-a_3'+a_2'-a_1'}{4-3a_1+2a_2-a_3}\neq 0.$$

Assume that there are two system parameters related to this kind of bifurcations. If Definition 1 is used to search for a critical parameter point, we have to resort to the cumbersome computations of all eigenvalues point by point by sweeping the parameter plane. As a matter of fact, it is extremely difficult to luckily encounter the accurate parameter point in a plane. However, in virtue of the algorithm in Corollary 1, the inequalities play an important role in explicitly locating the critical bifurcation parameter region and rapidly picking off the insignificant parameter domain. Thus, the exhausted numerical search for a critical parameter point may be avoided and the significant computational simplification may be achieved. In particular, it should be mentioned that for the transversality condition only the computation of the derivatives of the coefficients of the characteristic polynomial (2) with respect to the bifurcation parameter is needed and the difficulty in obtaining the derivatives of eigenvalue modulus is overcome.

Furthermore, it is easy to check that the inequalities in (H1) satisfy with the stability condition (L2) stated in Lemma 1 if the condition $p_{\mu_0}(-1) = 0$ in (H1) becomes $(-1)^n p_{\mu_0}(-1) > 0$. This implies that Proposition 1 is feasible to simultaneously locate the set of critical bifurcation parameters as well as the stability parameter range of the fixed point x_0. It should be mentioned that Definition 1 is incapable of regulating the stability parameter range nearby the critical parameter points.

2. Application to an inertial shaker model

2.1. Mechanical model of inertial shaker and its Poincare map

The inertial shaker model [22] is shown schematically in Fig. 1. A vibrating platform with mass M is connected to the foundation with a linear spring with stiffness K and a linear viscous dashpot with damping constant C. The platform is subjected to a harmonic excitation with amplitude F_0, excitation frequency ω and phase angle δ. The rigid-body cast with mass m is in the gravitation field without other forces when no impact occurs. Consequently, the cast bounces on the flat horizontal surface of the platform. Let y and x denote the displacements of mass m and mass M, respectively. An impact between the masses m and M occurs while $y = x$. In this paper, we consider a combination of smooth motions governed by linear differential equation interrupted by a continuous sequence of contacts at non-zero relative speed.

The impact dynamics of the inertial shaker system can be described by the following mathematical problem:

$$\begin{cases} M\ddot{x} + C\dot{x} + Kx = F_0 \sin(wt + \delta) \\ \ddot{y} = -g \end{cases} \quad (x \neq y) \tag{11}$$

and

$$\begin{cases} M\dot{x}_- + m\dot{y}_- = M\dot{x}_+ + m\dot{y}_+ \\ \dot{x}_+ - \dot{y}_+ = -R(\dot{x}_- - \dot{y}_-) \end{cases} \quad (x = y), \tag{12}$$

where $\dot{x}_+$ and $\dot{x}_-$, $\dot{y}_+$ and $\dot{y}_-$ are the velocities at contact points just after and before impacts; R indicates the constant coefficient of restitution.

Scaling $x = s\bar{x}$, $y = s\bar{y}$, $\theta = wt$ with $s = F_0 \Big/ \left(K\sqrt{(1-z^2)^2 + (2\zeta z)^2} \right)$, we transform the system (11) and (12) into the following non-dimensional form by dropping the bar for convenience:

$$\begin{cases} \ddot{x} + \dfrac{2\zeta}{z}\dot{x} + \dfrac{1}{z^2}x = \dfrac{\sqrt{(1-z^2)^2 + (2\zeta z)^2}}{z^2}\sin(\theta + \delta) \quad (x \neq y), \\ \ddot{y} = -e_1 \end{cases} \tag{13}$$

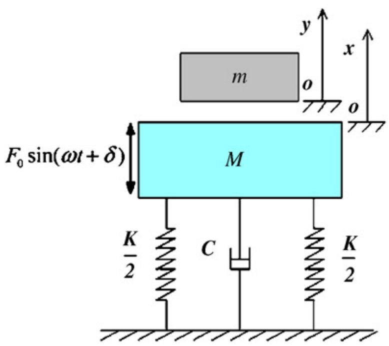

Fig. 1. Schematic of an inertial shaker model.

$$\dot{x}_+ = \frac{1-\eta R}{1+\eta}\dot{x}_- + \frac{\eta(1+R)}{1+\eta}\dot{y}_- \quad (x=y), \tag{14}$$

$$\dot{y}_+ = \frac{1+R}{1+\eta}\dot{x}_- + \frac{\eta - R}{1+\eta}\dot{y}_- \quad (x=y), \tag{15}$$

where $e_1 = Mg\sqrt{(1-z^2)^2 + (2\zeta z)^2}\Big/(F_0 z^2)$, $z = w/w_n$, $w_n = \sqrt{K/M}$, $\zeta = C/(2Mw_n)$, $\eta = m/M$ and $\beta = F_0/Mg$.

If the motion of a vibro-impact system between the impacts holds all smoothness properties, a general way to characterize the repeated impact behavior is to study the so-called Poincare map derived from the equations of motion and the boundary condition. Each iteration of the map corresponds to one of the repeated impacts. By choosing a Poincare section, $\sigma \subset R^4 \times S$, where $\sigma = \{(x, \dot{x}, y, \dot{y}, \theta) \in R^4 \times S, x = y, \dot{x} = \dot{x}_+, \dot{y} = \dot{y}_+\}$, one can establish the four-dimensional Poincare map for system (13)–(15) as follows [13]:

$$X_{k+1} = f(\mu, X_k), \tag{16}$$

where $X_k = (x_k, \dot{x}_k, \dot{y}_k, \tau_k)^{\mathrm{T}}$; μ denotes one or more parameters among $z, \zeta, \eta, R, \beta; f : \Omega \to \Re^3 \times S^1$ stands for the map function, and $\Omega \subset \Re^3 \times S^1$ is a connected open domain. Map (16) can be rewritten as

$$\begin{cases} x_{k+1} = f_1(x_k, \dot{x}_k, \dot{y}_k, \tau_k) = \hat{f}_1(\tilde{\theta}(x_k, \dot{x}_k, \dot{y}_k, \tau_k), x_k, \dot{x}_k, \tau_k), \\ \dot{x}_{k+1} = f_2(x_k, \dot{x}_k, \dot{y}_k, \tau_k) = \hat{f}_2(\tilde{\theta}(x_k, \dot{x}_k, \dot{y}_k, \tau_k), x_k, \dot{x}_k, \dot{y}_k, \tau_k), \\ \dot{y}_{k+1} = f_3(x_k, \dot{x}_k, \dot{y}_k, \tau_k) = \hat{f}_3(\tilde{\theta}(x_k, \dot{x}_k, \dot{y}_k, \tau_k), x_k, \dot{x}_k, \dot{y}_k, \tau_k), \\ \tau_{k+1} = f_4(x_k, \dot{x}_k, \dot{y}_k, \tau_k) = \tilde{\theta}(x_k, \dot{x}_k, \dot{y}_k, \tau_k) + \tau_k \,(\mathrm{mod}\, 2\pi), \end{cases} \tag{17}$$

where $x_k, \dot{x}_k, \dot{y}_k, \tau_k$ are the state variables just after kth impact and $x_{k+1}, \dot{x}_{k+1}, \dot{y}_{k+1}, \tau_{k+1}$ stand for the state variables just after $(k+1)$th impact. $\tilde{\theta}$ is the smallest root of the equation below:

$$G(\theta, x_k, \dot{x}_k, \tau_k) = x(\theta, x_k, \dot{x}_k, \tau_k) - y(\theta, \dot{y}_k, \tau_k) = 0. \tag{18}$$

By virtue of implicit function theorem, $\tilde{\theta}$ can be represented as the implicit function of $\tilde{\theta}(x_k, \dot{x}_k, \dot{y}_k, \tau_k)$ with respect to $x_k, \dot{x}_k, \dot{y}_k, \tau_k$. It is now obvious that the established Poincare map (17) is implicit. This implies what brings about the difficulty in bifurcation analysis of this kind of vibro-impact systems.

The vibro-impact system (11) and (12) can exhibit periodic-impact behavior under suitable system parameters. The periodic-impact motions correspond to the fixed points of map (16). Suppose that there exits a fixed point $X_0(\mu) = (x_0, \dot{x}_0, \dot{y}_0, \tau_0)^{\mathrm{T}}$ for map (16) satisfying $f(\mu, X_0) = X_0$ as in Proposition 1. The fixed point $X_0(\mu)$ stands for the period motion of system (13)–(15) with only one impact during one excitation period. We finally discuss the existence of the fixed point $X_0(\mu)$. Notice that one of the coordinate elements of the fixed point $X_0(\mu)$ is τ_0 in the following form [22]:

$$\tau_0 = \arccos(\pi e_1((1 - 2\eta R - R)/(1+R) - 2\eta(\gamma(D_0 - 1) + \zeta C_0)/(\gamma(C_0^2 + (D_0-1)^2)))), \tag{19}$$

where $C_0 = \mathrm{e}^{-\zeta(2\pi/z)} \sin \gamma(2\pi/z)$, $D_0 = \mathrm{e}^{-\zeta(2\pi/z)} \cos \gamma(2\pi/z)$ and $\gamma = \sqrt{1-\zeta^2}$. If τ_0 exists, the absolute value of $\cos(\tau_0)$ should be less than 1. Therefore, for period-doubling bifurcation in the vibro-impact system (13)–(15), the following inequality (20) should be the additional critical condition rather than the ones in (E1) and (E2):

$$|\pi e_1((1 - 2\eta R - R)/(1+R) - 2\eta(\gamma(D_0 - 1) + \zeta C_0)/(\gamma(C_0^2 + (D_0-1)^2)))| \leqslant 1. \tag{20}$$

2.2. Existence of period-doubling bifurcation

In this subsection, we investigate the existence of the period-doubling bifurcation of map (16) at the fixed point $X_0(\mu)$ based on the proposed critical criterion stated in Corollary 1.

Assume that $z = 1.7075883$, $\zeta = 0.09649167$, $R = 0.9$, $\eta \in (0.01, 1)$, $\beta \in (0.01, 5)$. By setting $\mu = (\eta, \beta)$, a critical parameter point $\mu = \mu_0$ for period-doubling bifurcation is to be determined and the effect of the

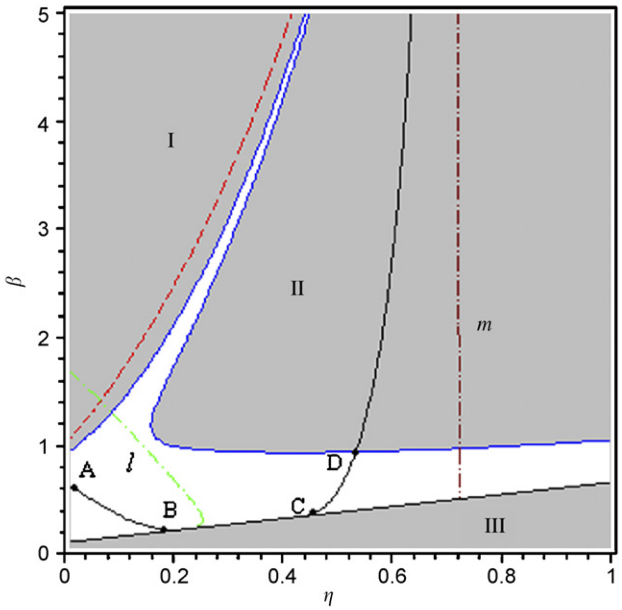

Fig. 2. Bifurcation plot of Poincare map (16).

parameter μ on the bifurcation will be discussed. The Jacobian matrix $Df_X(\mu, X)$ of the established map (17) at the fixed point $X_0(\mu)$ is derived first. Its characteristic polynomial can be written into the form (2) as follows:

$$p_\mu(\lambda) = a_0(\mu)\lambda^4 + a_1(\mu)\lambda^3 + a_2(\mu)\lambda^2 + a_3(\mu)\lambda + a_4(\mu). \tag{21}$$

According to Corollary 1, if $\mu = \mu_0$ is a critical parameter point of period-doubling bifurcation, the conditions in (E1) and (E2) are satisfied at $\mu = \mu_0$. It is possible that the critical parameter μ_0 is not unique. Certain areas consisting of all the critical parameter points may exist in the two-dimensional parameter plane (η, β). Inversely, by utilizing the conditions in (E1) and (E2) as well as inequality (20), we can explicitly locate the critical bifurcation parameter region, i.e., a set of μ_0 in the parameter plane (η, β).

Maple software is employed here to solve the equalities and inequalities in (E1), (E2) and Eq. (20) to obtain the bifurcation plot. As shown in Fig. 2, the points on the open arcs AB and CD in the blank region consist of the critical parameter points at which all conditions in (E1), (E2) and Eq. (20) are satisfied. The gray regions I and II indicate the insignificant region in which at least one condition in E(1) fails. In the gray region III, the condition (20) fails (correspondingly, the fixed point does not exist). The blank region is divided into three parts by the arcs AB and CD. The LHS of arc AB and the RHS of arc CD stand for the stability region of the fixed point $X_0(\mu)$ based on the Lemma 1 as all of the inequalities in (E1) and $1 - a_1 + a_2 - a_3 + a_4 > 0$ hold. The other blank region surrounded by the arcs AB and CD represents the potential parameter region where there may exist the solutions of period-doubling bifurcation. The points on the dash-dot lines l and m do not satisfy the transversality condition (E2). Notice that one of extended parts of arc CD embeds in the gray region II. Thus, it should be stressed that it is not enough to find a critical parameter point only by computing $p_{\mu_0}(-1) = 0$.

It should be stressed that the black arcs AB and CD represent the equality in (H1):

$$p_{\mu_0}(-1) = 0, \tag{22}$$

i.e., $1 - a_1 + a_2 - a_3 + a_4 = 0$. The green dash-dot line l represents the equality:

$$\frac{a_4' - a_3' + a_2' - a_1'}{4 - 3a_1 + 2a_2 - a_3} = 0, \tag{23}$$

where a_i' stands for the derivative of $a_i(\eta, \beta)$ with respect to η. The brown dash-dot line m represents the equality:

$$\frac{a_4' - a_3' + a_2' - a_1'}{4 - 3a_1 + 2a_2 - a_3} = 0, \tag{24}$$

where a_i' stands for the derivative of $a_i(\eta, \beta)$ with respect to β.

As already stated, the points in the open arc AB and CD are the critical bifurcation parameter points. As an example, we choose one of the points on the open arc CD, $\mu_0 = (\eta_0, \beta_0) = (0.5, 0.6)$. It follows from Corollary 1 that a period-doubling bifurcation occurs at μ_0.

In summary, the inequalities in Corollary 1 may exclude some insignificant parameter regions in the parameter plane and pick out the feasible parameter domain. According to Corollary 1, it is very convenient to construct the bifurcation plot where the stability domain and the bifurcation domain in the parameter plane are clear. Moreover, without requirement of computing the derivatives of eigenvalue modulus, Corollary 1 suggests the significant computational simplification for transversality condition, i.e., the sensitiveness analysis of bifurcation parameter.

2.3. Stability analysis of bifurcation solutions

In this subsection, our task is to determine the stability of the bifurcated solutions in the vibro-impact system (11) and (12) or the counterpart of map (16). The stability of bifurcation solutions depends on the nonlinear property of map f in Eq. (16). Some methodologies such as the center manifold reduction and normal form theory [1,30] and the frequency-domain reduction [31] are capable of determining stability analytically. As an example, the Projection Method [24,25] is addressed here.

It is useful for the subsequent developments to introduce the Taylor series of the four-dimensional nonlinear map (16):

$$X_{k+1} = AX_k + Q(X_k, X_k) + C(X_k, X_k, X_k) + \cdots, \tag{25}$$

where A is the Jacobian matrix at the fixed point $X_0(\mu)$ at $\mu = \mu_0$; $Q(X_k, X_k)$ and $C(X_k, X_k, X_k)$ are the symmetric bilinear vector and the symmetric tri-linear vector, respectively; the other terms of higher order are not explicitly written in Eq. (25).

Notice that after determining $\mu = \mu_0$ and $X_0(\mu_0)$, the eigenvalues and eigenvectors of the Jacobian matrix A is easily obtained. Let l and r denote the left and right eigenvectors of the eigenvalue $\lambda_1(0) = -1$, respectively. Moreover, the eigenvector r with the first element being 1 is orthogonal to l, i.e., $lr = 1$. For convenience of statement, we further define an index α_2 as follows:

$$\alpha_2 = 2l[\tilde{C}(r, r, r) - 2\tilde{Q}(r, (\tilde{A}^{\mathrm{T}}\tilde{A} + l^{\mathrm{T}}l)^{-1}\tilde{A}^{\mathrm{T}}\tilde{Q}(r, r))], \tag{26}$$

where $\tilde{Q}(x, x) = AQ(x, x) + Q(Ax, Ax)$, $\tilde{C}(x, x, x) = AC(x, x, x) + 2Q(Ax, Q(x, x)) + C(Ax, Ax, Ax)$ and $\tilde{A} = A^2 - I$. In what follows, the following Lemma 2 can be used to determine the stability of the bifurcated solutions.

Lemma 2 (*Abed and Fu [24] and Abed et al. [25]*). *If at $\mu = \mu_0$, the Jacobian matrix of* (16) *at the fixed point $X_0(\mu_0)$ has one eigenvalue $\lambda_1(\mu)$ satisfying $\lambda_1(\mu_0) = -1$ and $\lambda_1'(\mu)|_{\mu=\mu_0} \neq 0$, and the rest satisfying $|\lambda_j(\mu_0)| < 1$, $j = 2, 3, \ldots n$, then at a small neighborhood of $\mu = \mu_0$, the system* (16) *bifurcates from $X_0(\mu)$ to the supercritical period-doubling bifurcation trajectory (i.e., stable fixed point of order two) when $\alpha_2 < 0$, or unstable fixed point of order two when $\alpha_2 > 0$.*

In the example considered here, we obtain $\alpha_2 = -0.3541833$. According to Lemma 2, we assert that the period-doubling bifurcation of map (16) at $\mu = \mu_0$ gives rise to a pair of stable fixed points which map to each other as the original fixed point $X_0(\mu)$ loses its stability.

2.4. Simulations

In this section, one can graphically show the theoretical results as well as the rich dynamic behaviors of the vibro-impact system (11) and (12) as the parameters vary further. It should be stressed that we employ the original Poincare map (17) without any approximation rather than (25) for the simulations in the sequel. By this way, the independence between the theoretical analysis and numerical validation is guaranteed.

For example, by setting $\mu = \mu_0 + \Delta\mu = (\eta_0 - 0.005, \beta_0)$, we simulate map (17) starting from the initial point $X = X_0(\mu) + \Delta X$ with $\Delta X = (0.001, 0.001, 0, 0)^{\mathrm{T}}$. After 1500 impacts in the simulation, the results in Poincare section are shown in Fig. 3. It is clear that the iteration near X_0 converges to the stable fixed points of order two which correspond to the stable periodic-impact motion with two impacts during tow excitation periods for the inertial impact shaker in Fig. 1. The numerical results are consistent with the proposed theory.

Vibro-impact system may give rise to rich and complicated dynamic behaviors as the bifurcation parameter varies [9,13]. In the blank region surrounded with arc AB and CD in Fig. 2, $\mu = (\eta_0 - 0.008, \beta_0)$ is chosen in our numerical analysis. As shown in Fig. 4, the stable fixed points of order four bifurcate from the fixed points of order two.

When η decreases up to $(\eta_0 - 0.0111)$, the system (17) exhibits more complicated dynamical behaviors. The fixed points of order four lose their stability and the iteration converges to four invariant limit circles, as

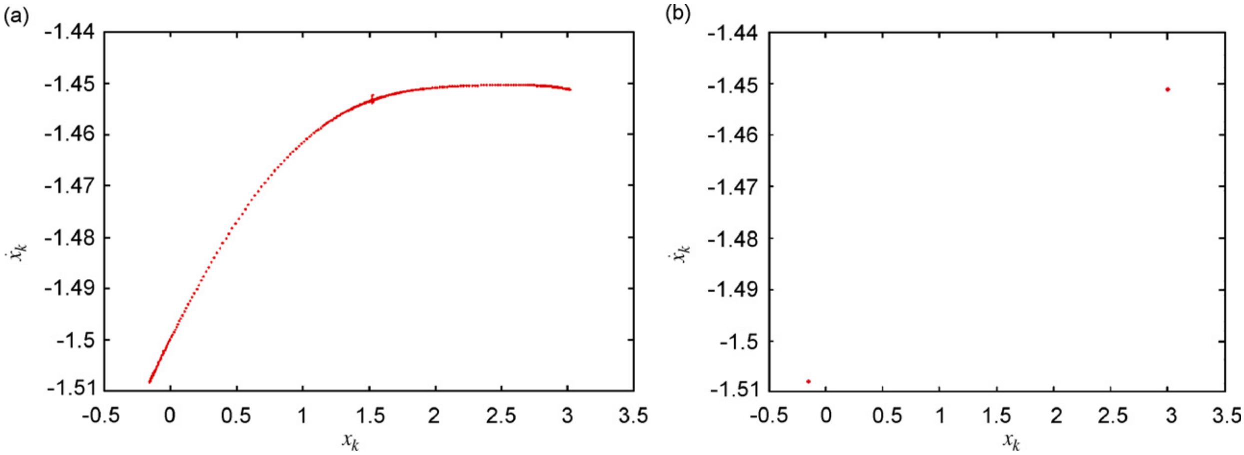

Fig. 3. The bifurcation solution of Eq. (17): stable fixed point of order two: (a) the transient response from the fixed point to the bifurcation solution; and (b) the results by canceling previous 500 impact points among 1500 impacts.

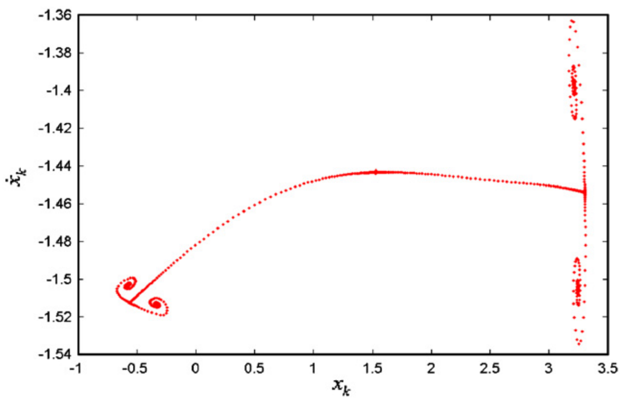

Fig. 4. Stable fixed point of order four in map (17) at $\mu = (\eta_0 - 0.008, \beta_0)$.

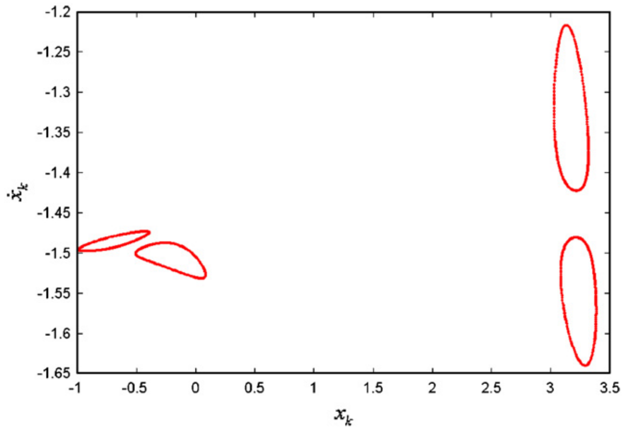

Fig. 5. Four invariant circles as the previous 1000 impact points are not shown.

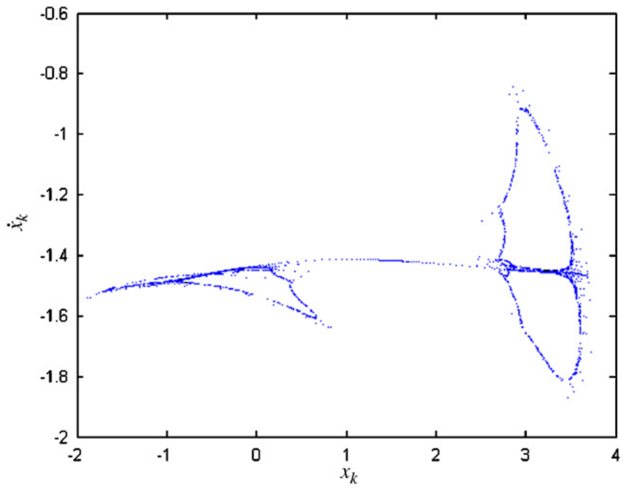

Fig. 6. Topologically varied circles in map (17) at $\mu = (\eta_0 - 0.016, \beta_0)$.

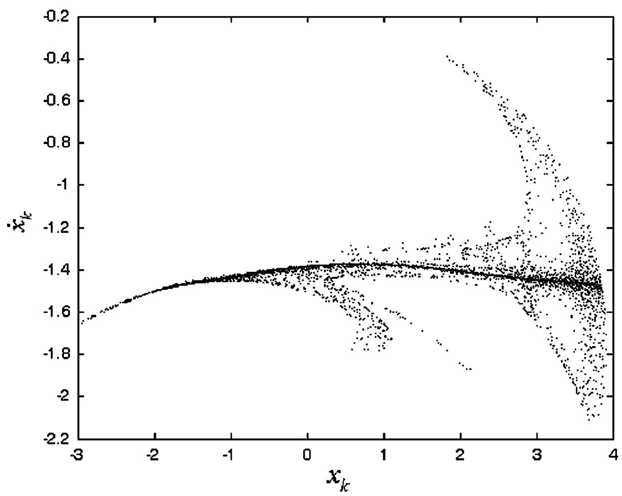

Fig. 7. Chaotic behavior in map (17) at $\mu = (\eta_0 - 0.025, \beta_0)$.

shown in Fig. 5. The dynamic behaviors evolve into chaos as the limit circles vary topologically (see Figs. 6 and 7).

3. Conclusions

In this paper, a new critical criterion of period-doubling bifurcation has been proposed for maps in a general sense. It has a great potential to simplify the computation of determining the critical parameter points. It also reveals the effect of parameters to the bifurcation. The feasibility is illustrated by the bifurcation analysis of the implicit Poincare map of a multiple-parameter inertial impact shaker.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (No. 10672054), and by the national 863 Program of China under Grant No. 2006AA04Z123.

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