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Parametric resonance of flexible footbridges under crowd-induced lateral excitation

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Abstract

The excessive lateral sway motion caused by crowds walking across footbridges has attracted great public attention in the past few years. Three possible mechanisms responsible for such lateral vibrations have been investigated in the literature: direct resonance, dynamic interaction, and internal resonance. In this paper, starting from a critical review of the mechanisms proposed in the literature, a parametric excitation mechanism is analyzed, based on a forcing model whose amplitude is a function of deck oscillations. A stability criterion is identified, depending on the ratio between the structural and excitation frequencies, on the ratio of the structural and pedestrian masses, and on the structural damping. The proposed mechanism can be achieved for very flexible footbridges, with a lateral natural frequency around 0.5 Hz, corresponding to a half of the lateral walking frequency. This situation can occur in modern structures, such as in the case of the London Millennium Footbridge.

1. Introduction

The improved mechanical characteristics of materials, the development of new technologies, the evolution of numerical techniques, or simply aesthetic purposes have enabled engineers to design lighter and more slender footbridges [1]. Pedestrians walking on a footbridge exert a dynamic loading with dominant frequency components around 2 Hz in the vertical direction, and around 1 Hz in the lateral direction: footbridges with natural frequencies close to those values are considered as prone to human-induced vibrations. Grundmann et al. [2] observed that pedestrians are forced to adjust their step length and speed to some extent to the motion of other pedestrians if footbridges are exposed to large pedestrian traffic: this mechanism can be defined as synchronization among pedestrians and it is independent of the footbridge dynamic characteristics. In the case of footbridges with a lateral natural frequency close to the walking frequency, a further synchronization can be achieved between the motion of the footbridge and pedestrians (e.g. Fujino et al. [3]).

The excessive lateral sway motion caused by crowds walking across footbridges has attracted great public attention in the past few years. Three examples of excessive lateral vibrations can be found in recent literature: a cable-stayed pedestrian bridge in Japan, known as the T-bridge, described by Fujino et al. [3], the

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Nomenclature

a, b	coefficients for the general solution of y_0
$\mathscr{C}$	damping operator
D_j	dynamic amplification factor for single-degree-of-freedom systems in the model by Roberts [10]
f_p	force per unit length exerted by pedestrians
F_j	jth modal force
g	gravity acceleration
G	function to describe how much pedestrians synchronize with the bridge's natural frequency in the model by Nakamura [8]
k	proportionality constant in the model by Dallard et al [4]
k_1	dynamic loading factor (α) in the model by Nakamura [8]
k_2	percentage of synchronized pedestrians (λ) in the model by Nakamura [8]
k_3	constant to be derived from experimental tests in the model by Nakamura [8]
L	bridge span length
$\mathscr{L}$	stiffness operator
m_p	pedestrian mass distribution
m_s	structural mass distribution
m_r	mass ratio in the model by Newland [9]
m_{ps}	mass of a single pedestrian
M_{p_j}	jth pedestrian modal mass
M_{s_j}	jth structural modal mass
N_p	number of pedestrians
$\bar{N}$	number of structural modes to be accounted for in the principal transformation
p_k	kth structural principal coordinate
p_{ps_j}	jth modal displacement of the pedestrian centre of mass on a stationary pavement in the model by Newland [9]
q	bridge lateral displacement
q_p	pedestrians' lateral displacement in the model by Roberts [10]
q_{ps}	displacement of the pedestrian centre of mass on a stationary pavement in the model by Newland [9]
$\tilde{t}$	non-dimensional time
v_{ps}	maximum amplitude of lateral pedestrian motion in the model by Roberts [10]
x	abscissa along the bridge deck
y	non-dimensional bridge displacement
y_i	coefficients of the series expansion of y ($i = 0, 1, 2$)
α	dynamic loading factor
α_0	dynamic loading factor on a stationary platform
α_1	dynamic loading factor related to the platform motion
α_{cm}	ratio between the motion amplitude of pedestrian centre of mass and that of the pavement in the model by Newland [9]
β	Portion of the span where pedestrians are uniformly distributed in the model by Roberts [10]
δ	non-dimensional coefficient depending on the frequency ratio
δ_i	coefficients of the series expansion of δ ($i = 0, 1, 2$)
Δ	time-lag between the motion of pedestrian centre of mass and that of the pavement
ε	naturally small parameter
ε_{lim}	limit value of ε defining the transition curve
η	non-dimensional damping
κ	non-dimensional force amplitude
λ	percentage of synchronized pedestrians
ξ_j	jth structural modal damping ratio
φ_k	kth structural mode of vibration
Ω	circular frequency of the lateral force exerted by walking pedestrians
ω_j	jth natural circular frequency

well-known Millennium Footbridge in London, described by Dallard et al. [4,5], and the Solferino Footbridge in Paris, described by Danbon and Grillaud [6]. Other, less documented examples of footbridge lateral vibrations are reported in Refs. [4,7].

The mechanisms investigated in the literature to explain such lateral vibrations can be classified into three classes: direct resonance, activated if the pedestrian excitation is in resonance with a mode of vibration of the bridge [3]; dynamic interaction, based on suitable models of the interaction between the motion of the bridge and the motion of pedestrians walking on it [4,5,8–12]; and internal resonance, due to structural nonlinearities giving rise to internal resonance conditions among different natural modes of vibration of the bridge itself [7,13].

Direct resonance [3] can be achieved if the footbridge has a lateral mode of vibration with a natural frequency within the range of the lateral walking frequency (0.8–1.2 Hz); based on this mechanism, the footbridge response in the lateral direction is harmonic, driven by the resonant walking frequency.

Dynamic interaction mechanisms have received great attention in the literature, allowing introduction of the concept of the limit number of pedestrians which could cause serviceability problems for footbridges. Based on experimental tests carried out on the London Millennium Footbridge, Dallard et al. [4,5] proposed a model of the force exerted by pedestrians on the bridge, which is proportional to the bridge velocity and permits definition of a critical condition when the total damping is zero, corresponding to a maximum number of pedestrians; however, a constant appears in the model which has been estimated by back analysis of the data recorded on the Millennium Bridge and cannot be easily generalized to other structural examples. Nakamura [8] proposed an analogous model for the interactive force, which allows schematization of the self-limiting nature of the synchronization phenomenon; however, this model is also based on coefficients which have been estimated from experimental tests and cannot easily be predicted at the design stage. A different model has been proposed by Roberts [10–12], schematizing the interaction between the pedestrians and the footbridge: assuming that synchronization occurs when the displacement of the bridge is greater than the displacement of the pedestrians, a limit number of pedestrians is obtained. Since the interactive force is harmonically varying with frequency of about 1 Hz, such a model can motivate large vibrations only for footbridges with a natural frequency close to that value (i.e. direct resonance conditions). Newland [9] proposed a forcing model as the sum of the forces exerted by pedestrians on a stationary pavement and the inertia forces acting on the pedestrians themselves, assuming that they move with the same law of motion as the floor but in phase delay. In this way, he found a limit condition in which the system becomes unstable. However, assuming that the motion of pedestrians has the same harmonic components as the deck motion, this model can be rightly applied only if the vibration lateral mode of the footbridge has a natural frequency close to the lateral walking frequency (i.e. direct resonance conditions).

Internal resonance [7,13] is possible if the bridge is characterized by a 2:1 ratio between vertical and lateral mode frequencies, with the vertically excited mode close to direct resonance conditions concerning the loading (i.e. with a natural frequency close to 2 Hz).

Some experimental tests have been carried out in order to understand the lateral pedestrian-induced excitation mechanism [3,4,6]. From the analysis of the time histories recorded during the tests performed on the T-Bridge in Japan, characterized by vertical frequencies 0.7, 1.4 and 2 Hz and a lateral frequency 0.9 Hz, Fujino et al. [3] observed a correlation between the lateral displacements and the lateral forces, while no correlation was observed between the lateral and vertical displacements: they concluded that, even if the internal resonance could be achieved for that bridge, its lateral vibrations could be due only to a direct resonance phenomenon; moreover, the great displacements recorded in the lateral direction could be reached only if synchronization occurred among pedestrians. The tests carried out on the London Millennium Bridge by Dallard et al. [4], characterized by vertical frequencies 1.15, 1.54, and 1.89 Hz and lateral frequencies 0.48 and 0.95 Hz, showed that the lateral vibrations had frequency components also corresponding to the first lateral mode (0.48 Hz); tests performed with an increasing number of pedestrians demonstrated that the vibration amplitude dramatically increases when a critical number of pedestrians is reached. The models proposed in the literature [3,4,8–12] may motivate the excitation of the mode at 0.48 Hz only assuming that people were instinctively changing direction slightly every 3–4 footfalls. However, the models based on an interactive force harmonically varying with frequency around 1 Hz [3,9–12] cannot reproduce the increasing trend of oscillatory divergent response, as experimentally observed [5]. Danbon and Grillaud [6] carried out analogous tests on the Solferino footbridge, characterized by a lateral swaying frequency of 0.7 Hz, and concluded that lateral excitation was possible only if the crowd walked very slowly so that the frequency of the force is resonant with the first lateral mode of vibration of the footbridge.

In this paper, starting from an extensive, critical analysis of the excitation mechanisms studied in the literature (Section 2), a new forcing model is proposed based on experimental tests carried out on moving platforms [4]; in this way, the force exerted by pedestrians on the moving bridge is modelled as harmonic with an amplitude depending on the bridge displacement (Section 3). Thus, the equation of motion of the footbridge results as a parametrically excited system of the Mathieu type [14,15], leading to unstable oscillations if the frequency of the lateral crowd-induced excitation is close to twice the lateral frequency of the footbridge. The treatment of the equation of motion through the method of strained parameters allows definition of the transition curves, which divide the stable regions from the unstable ones. The analyzed mechanism could be critical for very flexible footbridges, with horizontal frequencies around a half of the first lateral walking frequency, as in the case of the London Millennium Bridge. A simple criterion defining the limit pedestrian mass is introduced and is compared with analogous formulae available in the literature (Section 4). The proposed model is then applied for the analysis of the pedestrian-induced vibrations of the London Millennium Footbridge (Section 5). Finally, some conclusions are drawn in Section 6.

2. Crowd-induced forces and excitation mechanisms: critical review of the literature

In this section, the force models proposed in the literature for the crowd-induced actions on footbridges are revised (Section 2.1); based on these different forcing models, the possible excitation mechanisms identified in the literature are critically analyzed (Section 2.2).

2.1. Force models

If a light stream of pedestrians is considered, people can move freely and their walking phases are randomly distributed: in this case, the resultant intensity of human-induced forces is low. According to Matsumoto et al. [16], the force per unit length $f_p(x,t)$ exerted by N_p pedestrians can be expressed as

$$f_p(x,t) = \frac{\sqrt{N_p}\alpha g m_{\rm ps}}{L}\cos(\Omega t) \tag{1}$$

where $m_{\rm ps}$ is the mass of a single pedestrian, L is the footbridge span length, α the so-called "dynamic loading factor" and depends on the considered load harmonic and on the load direction [17], g is the gravity acceleration, Ω the dominant walking frequency, which is a function of the walking speed and is commonly assumed around 2 Hz in the vertical direction and around 1 Hz in the horizontal direction. An analogous model is proposed by Roberts [10], who expressed the force $f_p(x,t)$ as

$$f_p(x,t) = \frac{\sqrt{2N_p} m_{\rm ps} v_{\rm ps}\Omega^2}{L}\cos(\Omega t), \tag{2}$$

$v_{\rm ps}$ being the maximum amplitude of pedestrian lateral motion ($v_{\rm ps} \cong 0.025$ m).

If footbridges are exposed to heavy pedestrian traffic (crowd density of the order 0.6–1 pedestrian/m^2), free unconstrained movements are practically impossible and pedestrians are forced to adjust their step length and speed to some extent to the motion of other pedestrians [2]: in such a case, the forces induced by the crowd can be expressed by the following equation (Fig. 1):

$$f_p(x,t) = \lambda\alpha g m_p(x)\cos(\Omega t), \tag{3}$$

where λ is the percentage of synchronized pedestrians ($\lambda = 0.2$ from the tests carried out by Fujino et al. [3] on the T-Bridge), $m_p(x)$ is the distribution of the pedestrian mass walking with frequency Ω along the bridge, commonly assumed as uniform in crowded conditions:

$$m_p(x) = \frac{N_p m_{\rm ps}}{L}. \tag{4}$$

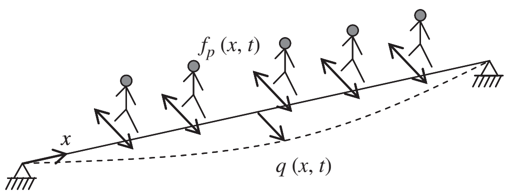

Fig. 1. Schematic representation of a bridge deck.

Such a force model is based on experimental measurements carried out on stationary platforms. In case of great lateral oscillations of the footbridge, this model might not be appropriate, since the motion of the centre of mass of pedestrians, and thus the force exerted on the pavement, will depend on the motion of the pavement itself. Experimental measurements involving pedestrians walking on a moving platform have shown that the dynamic force exerted by pedestrians is a function of the deck motion amplitude (Fig. 2, [4]). Starting from the results of experiments carried out at Imperial College, the dynamic loading factor α can be approximately expressed as a function of the deck motion amplitude $q(t)$ as follows:

$$\alpha = \alpha_0 + \alpha_1 q(t), \tag{5}$$

where α_0 is the dynamic loading factor on a stationary platform ($\alpha_0 = 0.04$), and α_1 can be estimated from the data shown in the Fig. 2 ($\alpha_1 \cong 2\,\text{m}^{-1}$). It should be noted that Fig. 2 has been obtained from tests performed on a harmonically moving platform with pedestrians walking with the same frequency as the platform.

Based on the measurements carried out on the Millennium Bridge, Dallard et al. [4] proposed a different model for the lateral force exerted by pedestrians on footbridges, in which the forces are proportional to the bridge velocity:

$$f_p(x,t) = k\frac{N_p}{L}\dot{q}(x,t), \tag{6}$$

where k is a proportionality constant, which has been estimated from a back analysis of data recorded on the Millennium bridge ($k = 300\,\text{Ns/m}$).

An alternative model, proposed by Newland [9], expresses the forces induced by pedestrians as the sum of forces exerted by pedestrians on a stationary pavement and of forces due to the motion of the pavement, assuming that the centre of mass of pedestrians moves with the same law of motion as the floor, having a suitable phase delay:

$$f_p(x,t) = -\lambda m_p(x)\ddot{q}_{ps}(x,t) - \lambda\alpha_{cm} m_p(x)\ddot{q}(x, t-\Delta), \tag{7}$$

where $q_{ps}(x,t)$ is the displacement of the centre of mass of pedestrians on a stationary pavement and $q(x,t)$ is the displacement of the pavement, Δ is the time-lag between the motion of pedestrian centre of mass and that of the pavement, and α_{cm} the ratio between the motion amplitude of pedestrian centre of mass and that of the pavement (α_{cm} can approximately be taken equal to $\frac{2}{3}$). It should be noted that, even if the contribution is

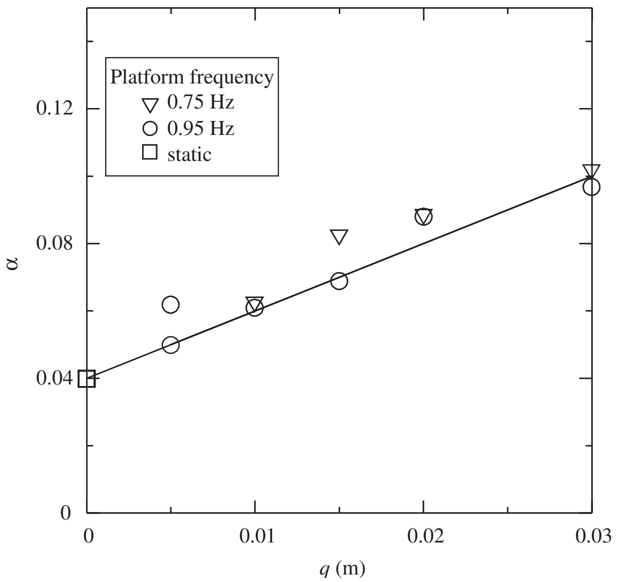

Fig. 2. Dynamic loading factor for harmonically moving platform with different frequencies [4] (▽ $n = 0.75$ Hz, ○ $n = 0.95$ Hz, and □ static).

twofold in Eq. (7), it is doubtful whether such a model can be applied when pedestrian and bridge motions have different harmonic contents.

2.2. Excitation mechanisms

The excitation mechanisms analyzed in the literature in order to explain the excessive lateral sway motion induced by a crowd crossing a footbridge can mainly be classified into three classes: direct resonance [3], dynamic interaction between the footbridge and the crowd [4,8–12], and internal resonance [7,13].

In order to deal with the direct resonance and the dynamic interaction mechanisms, let us consider the equation of motion of a footbridge, modelled as a linear mono-dimensional damped dynamical system (Fig. 1):

$$m_s(x)\frac{\partial^2 q(x,t)}{\partial^2 t} + \mathscr{C}\left[\frac{\partial q(x,t)}{\partial t}\right] + \mathscr{L}[q(x,t)] = f_p(x,t), \tag{8}$$

where $m_s(x)$ is the structural mass, $\mathscr{C}$ the damping operator, $\mathscr{L}$ the stiffness operator, and $f_p(x,t)$ the crowd-induced force. Eq. (8) is usually solved applying the principal transformation:

$$q(x,t) = \sum_{k=1}^{\bar{N}} \varphi_k(x) p_k(t), \tag{9}$$

where $\varphi_k(x)$ is the kth mode of vibration, $p_k(t)$ the corresponding principal coordinate and $\bar{N}$ a suitable subset of the structural modes. Substituting Eq. (9) into Eq. (8), under the hypothesis of classical damping, the equation of motion of the jth principal coordinate is expressed as

$$\ddot{p}_j(t) + 2\xi_j\omega_j\dot{p}_j(t) + \omega_j^2 p_j(t) = \frac{1}{M_{s_j}}F_j(t), \tag{10}$$

where ξ_j is the jth modal damping ratio, ω_j the jth natural circular frequency, M_{s_j} the jth modal mass, and $F_j(t)$ the jth modal force:

$$F_j(t) = \int_0^L f_p(x,t)\varphi_j(x)\,\mathrm{d}x. \tag{11}$$

2.2.1. Direct resonance

The simplest excitation mechanism able to motivate the crowd-induced excessive lateral sway motion is direct resonance, in which a part of the crowd crossing the footbridge moves with a frequency equal to one of the lateral frequencies of the bridge [3]. This mechanism can be achieved only if the footbridge has a lateral frequency around 1 Hz, which is the dominant frequency exerted by pedestrians in the lateral direction.

Substituting Eq. (3) with $\Omega = \omega_j$ into Eq. (11), the modal force is obtained; the structural response is then given by

$$q(x,t) = \frac{\lambda\alpha g}{M_{s_j}\omega_j^2}\frac{1}{2\xi_j}\varphi_j(x)\left[\int_0^L m_p(x)\varphi_j(x)\,\mathrm{d}x\right]\cos\left(\omega_j t - \frac{\pi}{2}\right). \tag{12}$$

Fujino et al. [3] have adopted this model for the lateral vibration of the T-Bridge. Theoretical displacements corresponding to measurements have been found assuming that 20% of pedestrians are synchronized with the natural frequency of the footbridge ($\lambda = 0.2$). This mechanism cannot motivate great amplitude oscillations for structures with natural frequencies in the lateral direction far away from 1 Hz. Dealing with structures with a lateral frequency around 1 Hz corresponding to a skew-symmetric vibration mode, such a model can motivate large oscillations only assuming that the pedestrian mass is not uniformly distributed (otherwise the integral in Eq. (12) is nil).

2.2.2. Dynamic interaction

Four different models based on dynamic interaction between pedestrians and the footbridge have been proposed in the literature: Dallard et al. [4] and Nakamura [8] based their models on interaction forces proportional to the bridge velocity; Roberts [11] proposed a model characterized by the joint solution of the equations of motion of the bridge and the pedestrians, identifying a critical condition when the pedestrian motion is larger than the bridge motion; Newland [9] proposed a model of interaction forces based on the superimposition of forces on a stationary pavement and forces due to the pavement motion.

Dallard et al. [4] adopted a pedestrian forcing model given by Eq. (6). Assuming that the bridge response is dominated by the jth mode of vibration, with a modal force obtained substituting Eq. (6) into Eq. (11), the equation of motion of the jth principal component can be expressed as follows:

$$\ddot{p}_j(t) + \left[2\xi_j\omega_j - \frac{1}{M_{s_j}} k \frac{N_p}{L}\int_0^L \varphi_j^2(x)\,\mathrm{d}x\right]\dot{p}_j(t) + \omega_j^2 p_j(t) = 0. \tag{13}$$

The force transmitted by pedestrians is then modelled as a source of negative damping for the structure, and the jth principal coordinate becomes unstable if the total damping becomes negative; the stability criterion is then

$$\xi_j > \frac{kN_p \int_0^L \varphi_j^2(x)\,\mathrm{d}x}{2M_j\omega_j L}. \tag{14}$$

Assuming a uniform distribution of pedestrian mass in Eq. (4), Eq. (14) can be written as follows:

$$\frac{M_{p_j}}{M_{s_j}} < 2\xi_j\omega_j \frac{m_{\mathrm{ps}}}{k}, \tag{15}$$

where M_{p_j} is the jth pedestrian modal mass given by

$$M_{p_j} = \int_0^L m_p(x)\varphi_j^2(x)\,\mathrm{d}x. \tag{16}$$

Nakamura [8] proposed a refined forcing model analogous to the previous one, representing the self-limiting nature of the pedestrian synchronization. The modal force is expressed as follows:

$$F_j(t) = k_1 k_2 \frac{\dot{p}_j(t)}{k_3 + |\dot{p}_j(t)|} G(n_j) M_{p_j} g, \tag{17}$$

where k_1 is the dynamic loading factor (α_0 in Eq. (5)) ($k_1 = 0.04$), k_2 is the percentage of synchronized pedestrians (λ in Eq. (4)) ($k_2 = 0.2$), k_3 is a constant that has to be derived from experiments so that it corresponds to the measured data ($k_3 = 0.001$ for the London Millennium Footbridge), $G(n_j)$ is a function able to describe how much pedestrians synchronize with the bridge's natural frequency ($G(n_j) = 1$ in absence of experimental data), and M_{p_j} is the pedestrians' modal mass (Eq. (16)).

The model proposed by Roberts [10,11] is based on the joint solution of the motion equations of the footbridge (10) and of pedestrians, this latter being expressed as follows:

$$m_p(x)\ddot{q}_p(x,t) = -f_p(x,t), \tag{18}$$

where $q_p(x,t)$ represents the pedestrians' lateral displacement. The critical condition is defined when the motion of pedestrians is larger than the motion of the bridge, the stability condition is then:

$$\frac{M_{p_j}}{M_{s_j}} < \left(\frac{\omega_j}{\Omega}\right)^2 \frac{1}{D_j}, \tag{19}$$

D_j being the dynamic amplification factor for single-degree-of-freedom systems:

$$D_j = \sqrt{\frac{1}{[1-(\Omega/\omega_j)^2]^2 + (2\xi_j(\Omega/\omega_j))^2}}. \tag{20}$$

Considering the situation in which pedestrians are uniformly distributed over a portion of length βL (in one continuous or several discrete lengths) and sinusoidal mode shapes, the ratio M_{p_j}/M_{s_j} can be expressed as [11]

$$\frac{M_{p_j}}{M_{s_j}} = \frac{2}{1+\beta^2}\frac{N_p m_{\mathrm{ps}}}{L m_s}. \tag{21}$$

In order to take into account the fact that it is virtually impossible for a large group of pedestrians to maintain a precise walking frequency over an extended period of time, Roberts [10] recommends that Eq. (19) should be used in conjunction with an average value of the dynamic amplification factor D_j for a frequency ratio Ω/ω_j in the range 0.8–1.2. In addition, Roberts [12] analyses separately the cases of random and of partially synchronized pedestrian loading, taking into account the possible variation of parameters which can influence the stability condition; the pedestrian lateral force amplitude and the probability of synchronization of pedestrian motion with the motion of the bridge prove to affect the pedestrian limit number.

Newland [9] adopted the forcing model described by Eq. (7). Assuming that the bridge response is dominated by the jth mode of vibration and obtaining the modal force substituting Eq. (7) into Eq. (11), the equation of motion of the jth principal component can be expressed as follows:

$$\ddot{p}_j(t) + m_r\ddot{p}_j(t-\Delta) + 2\xi_j\omega_j\dot{p}_j(t) + \omega_j^2 p_j(t) = -\frac{1}{\alpha_{\mathrm{cm}}} m_r \ddot{p}_{ps_j}(t), \tag{22}$$

where $p_{ps_j}(t)$ is the modal displacement of the pedestrian centre of mass on a stationary pavement, and m_r is the mass ratio, given by

$$m_r = \alpha_{\mathrm{cm}}\lambda\frac{M_{p_j}}{M_{s_j}}. \tag{23}$$

Neglecting the contribution of the right-hand member of Eq. (22) owing to the system linearity, the analysis of stability permits one to deduce the following limit criterion:

$$\xi_j^2 > \frac{1}{4}\left[2 + \left(\frac{\Omega}{\omega_j}\right)^2 (m_r^2 - 1) - \left(\frac{\omega_j}{\Omega}\right)^2\right]. \tag{24}$$

The worst condition for the system (i.e. the highest limit damping) happens when

$$\left(\frac{\Omega}{\omega_j}\right)^2 = \frac{1}{1-m_r^2} \quad \xi_j^2 > \frac{1}{2}\left(1 - \sqrt{1-m_r^2}\right). \tag{25}$$

In case of small mass ratio, Eq. (25) can be written as

$$\frac{\Omega}{\omega_j} = 1 + \frac{m_r^2}{4} \quad \xi_j > \frac{m_r}{2}, \tag{26}$$

leading to the stability criterion:

$$\frac{M_{p_j}}{M_{s_j}} < \frac{2\xi_j}{\alpha_{\mathrm{cm}}\lambda}. \tag{27}$$

Due to the particular force model assumed, this method can be rightly applied when the synchronization between pedestrian and footbridge motions occurs, that is for bridges with a natural frequency around 1 Hz, since the pedestrian mass must able to follow, although out-of-phase, bridge oscillations (and pedestrian lateral vibrations have to fall in the range 0.8–1.2 Hz [1]).

2.2.3. Internal resonance

Fujino et al. [13] studied both from an experimental and analytical point of view the model of a cable-stayed bridge characterized by vibration modes in an integer frequency ratio 2:1:1 (first vertical girder mode, first lateral girder mode and first symmetric in-plane cable mode), observing that auto-parametric resonance could be achieved. They observed that the T-Bridge was characterized by such a frequency ratio but, from experimental measurements on the bridge [3], they found that lateral displacements were not correlated with vertical displacements, so that large vibration amplitudes could not be explained by the auto-parametric resonance.

Blekherman [7] also considers nonlinear auto-parametric resonance as a reason for excessive lateral vibrations induced by walking pedestrians on bridges with an integer ratio between vertical and lateral mode frequencies, when the vertically excited mode is close to direct resonance conditions concerning the loading (i.e. close to 2 Hz). He proposes a physical model, not directly related to footbridges, made up of an elastic pendulum of variable length, leading to a classic parametrically excited system. He states that swaying of pedestrian bridges can be treated as a two-step process: the first step consists of the achievement of a jump phenomenon, which is possible if a load parameter (the vertical load) passes through a critical value; then the second step is the process of interaction between applied forces and the lateral mode of vibration.

3. The proposed excitation mechanism: model and solution

Forces exerted by pedestrians are essentially due to the motion of their centre of mass [18]. In case of a stationary pavement, the motion of the pedestrian centre of mass is almost harmonic [19] and, thus, also the interactive force is harmonic. In case of lateral oscillations of the platform, the motion of the centre of mass, and then the force exerted on the pavement, necessarily depends on the motion of the pavement itself. Tests carried out on harmonically moving platforms [4,20] showed a magnification effect of the force exerted by the pedestrians when the motion of the platform is almost in resonance with the walking frequency. To the best of the authors' knowledge, tests carried out in case of generic motion of the platform are not available in the literature. In this study, the force exerted by pedestrians on a moving footbridge is modelled by a harmonic formulation (Eq. (3)) in which, however, the dependence of the dynamic loading factor on the deck motion amplitude is taken into account (Eq. (5)). As observed above, the dynamic load relation (5) has been obtained from tests carried out on harmonically moving platforms. Due to the current incompleteness of testing, it seems reasonable to generalize these results to a footbridge characterized by a generic law of motion. This assumption appears at least justifiable for structures which exhibit lateral natural frequencies around 1 Hz in their spectrum (i.e. with one lateral mode approximately in resonance with the main lateral walking frequency). The following expression is then considered:

$$f_p(x,t) = \lambda[\alpha_0 + \alpha_1 q(x,t)]g m_p(x)\cos(\Omega t). \tag{28}$$

The proposed model is suitable to schematize groups of pedestrians, such as those usually adopted in dynamic loading tests (e.g. London Millennium Footbridge [4] and Solferino bridge [6]). Moreover, it justifies the experimental evidence that the increase in the bridge motion leads to an increase in the amplitude of the periodic forces laterally exerted by a single pedestrian (e.g. from the typical Eurocode value of 70 N as far as 300 N [10]).

Substituting Eq. (28) into the equation of motion of the footbridge (Eq. (8)), the following equation is obtained:

$$m_s(x)\frac{\partial^2 q(x,t)}{\partial^2 t} + \mathscr{C}\left[\frac{\partial q(x,t)}{\partial t}\right] + \mathscr{L}[q(x,t)] - \lambda\alpha_1 g m_p(x)\cos(\Omega t)q(x,t) = \lambda\alpha_0 g m_p(x)\cos(\Omega t). \tag{29}$$

Applying the principal transformation (Eq. (9)) to Eq. (29), and assuming that the system is classically damped and the pedestrian mass $m_p(x)$ is distributed proportionally to the structural mass $m_s(x)$, Eq. (29) becomes

$$\ddot{p}_j(t) + 2\omega_j\xi_j\dot{p}_j(t) + \left[\omega_j^2 - g\lambda\alpha_1\frac{M_{p_j}}{M_{s_j}}\cos\,\Omega t\right]p_j(t) = \left[\frac{g\lambda\alpha_0}{M_{s_j}}\int_0^L m_p(x)\varphi_j(x)\,\mathrm{d}x\right]\cos\,\Omega t. \tag{30}$$

Let us introduce the following non-dimensional quantities:

$$y = \frac{p_j}{L}, \quad \tilde{t} = \frac{1}{2}\Omega t. \tag{31}$$

Using the chain rule for differentiation ($[\mathrm{d}/\mathrm{d}t] = [\mathrm{d}/\mathrm{d}\tilde{t}][\mathrm{d}\tilde{t}/\mathrm{d}t]$) and multiplying both terms of Eq. (30) by $(4/\Omega^2 L)$, Eq. (30) can be re-written as follows:

$$\ddot{y} + 2\eta\dot{y} + [\delta - 2\varepsilon\cos 2\tilde{t}]y = \kappa\cos 2\tilde{t}, \tag{32}$$

where η, δ, ε, κ are given by

$$\eta = \frac{2\omega_j}{\Omega}\xi_j \quad \delta = \frac{4\omega_j^2}{\Omega^2} \quad \varepsilon = \frac{2g\lambda\alpha_1}{\Omega^2}\frac{M_{p_j}}{M_{s_j}} \quad \kappa = \frac{4g\lambda\alpha_0}{\Omega^2 M_{s_j} L}\int_0^L m_p(x)\varphi_j(x)\,\mathrm{d}x. \tag{33}$$

Since the pedestrians' modal mass is usually small compared with the structural modal mass, the parameter ε is naturally small; moreover, the damping parameter η is also assumed small of the same order as ε:

$$\eta = \varepsilon\tilde{\eta}. \tag{34}$$

Substituting Eq. (34) into Eq. (32), it becomes

$$\ddot{y} + 2\varepsilon\tilde{\eta}\dot{y} + [\delta - 2\varepsilon\cos 2\tilde{t}]y = \kappa\cos 2\tilde{t}. \tag{35}$$

Eq. (35) describes the parametric excitation of a single-degree-of-freedom system and is practically coincident to the classic Mathieu equation (e.g. [14,15]). Depending on the values of the parameters, the solution of Eq. (35) can be stable or unstable. The transition curves separate the regions of stability from those of instability in the space of parameters $(\delta, \varepsilon, \eta)$; along these curves the solution is periodic with period π or 2π. Such transition curves can be analytically obtained using perturbation methods, among which the method of strained parameters [14]. An alternative method, based on the Floquet theory, has been recently proposed by Seyranian and Mailybaev [21], to obtain a first-order approximation of the instability domain of oscillatory systems with small viscous damping and small periodic parametric excitation.

Without any loss of generality working in the elastic linear field, the problem of the free vibrations is considered since the external forces do not affect the stability of the system. Let us expand the solution y and δ in powers of ε:

$$\begin{aligned} \delta &= \delta_0 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + \cdots, \\ y &= y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots. \end{aligned} \tag{36}$$

Substituting Eq. (36) into Eq. (35) and equating the coefficient of each power of ε, the following perturbation equations are obtained:

$$\text{order } \varepsilon^0 \quad \ddot{y}_0 + \delta_0 y_0 = 0, \tag{37}$$

$$\text{order } \varepsilon^1 \quad \ddot{y}_1 + \delta_0 y_1 = -2\tilde{\eta}\dot{y}_0 - \delta_1 y_0 + 2y_0\cos 2\tilde{t}, \tag{38}$$

$$\text{order } \varepsilon^2 \quad \ddot{y}_2 + \delta_0 y_2 = -2\tilde{\eta}\dot{y}_1 - \delta_1 y_1 - \delta_2 y_0 - 2y_1\cos 2\tilde{t}. \tag{39}$$

The general solution of Eq. (37) is [14,15]:

$$y_0 = a\cos\sqrt{\delta_0}\tilde{t} + b\sin\sqrt{\delta_0}\tilde{t}. \tag{40}$$

Let us consider the principal resonance condition, imposing $\delta_0 = 1$. Substituting Eq. (40) into Eq. (38) with $\delta_0 = 1$, and eliminating the terms that produce a secular term in y_1 demands that

$$\delta_1 = \pm\sqrt{1 - 4\tilde{\eta}^2}. \tag{41}$$

Substituting Eq. (41) into Eq. (36), one obtains:

$$\delta = 1 \pm \sqrt{\varepsilon^2 - 4\eta^2}. \tag{42}$$

Eq. (42) can be solved with respect to ε in order to obtain the transition curve to the first-order approximation:

$$\varepsilon_{\lim} = \sqrt{(\delta - 1)^2 + 4\eta^2} = \sqrt{(\delta - 1)^2 + 4\xi_j^2\delta}. \tag{43}$$

The y_1 solution is derived substituting Eq. (40) into Eq. (38). Substituting y_1 into Eq. (39) and eliminating the terms that produce a secular term in y_2 demands that

$$\delta_2 = -\frac{1}{8}. \tag{44}$$

Substituting Eqs. (41) and (44) into Eq. (36), one obtains:

$$\delta = 1 \pm \sqrt{\varepsilon^2 - 4\eta^2} - \frac{\varepsilon^2}{8}. \tag{45}$$

Eq. (45) can be solved with respect to ε to define the transition curve to the second-order approximation:

$$\varepsilon_{\lim} = 2\sqrt{2}\sqrt{5 - \delta - \sqrt{-4\eta^2 - 8\delta + 24}} = 2\sqrt{2}\sqrt{5 - \delta - \sqrt{-4\xi_j^2\delta - 8\delta + 24}}. \tag{46}$$

Let us now consider the secondary resonance condition, imposing $\delta_0 = 4$. Substituting Eq. (40) into Eq. (38) with $\delta_0 = 4$ gives

$$\ddot{y}_1 + 4y_1 = \sin(2t)(4\tilde{\eta}a - \delta_1 b) + \cos(2t)(-4\tilde{\eta}b - \delta_1 a) + a\cos(4t) + a + b\sin(4t). \tag{47}$$

Eliminating the terms that produce a secular term in y_1 demands that

$$16\tilde{\eta}^2 + \delta_1^2 = 0. \tag{48}$$

Eq. (48) can never be satisfied; thus, no periodic solution is possible around the secondary resonance conditions. The same result has been obtained by Seyranian and Mailybaev [21] using the Floquet theory. As already observed in Refs. [15,21], it has been verified that the presence of a damping term of the same magnitude order as the periodic parametric excitation stabilizes the system around the secondary resonance. The secondary resonance is possible only assuming that the damping is smaller than the parametric excitation term (i.e. of order ε^2) [15]; however, such a condition is not of technical interest, since it could be verified for systems with very small unrealistic structural damping or in case of very great parametric excitation (unrealistic pedestrian mass).

Fig. 3(a) shows the transition curves around the principal resonance at the first-order approximation (Eq. (43)) (dashed lines), compared with the second-order approximation (Eq. (46)) (solid lines), corresponding to $\xi_j = 0.005$. It is worth observing that the transition from stable to unstable conditions is highly influenced by the value of δ: the limit value of ε rapidly increases when δ deviates from the unity value. From a comparison between the first- and the second-order approximations, it can be deduced that the two

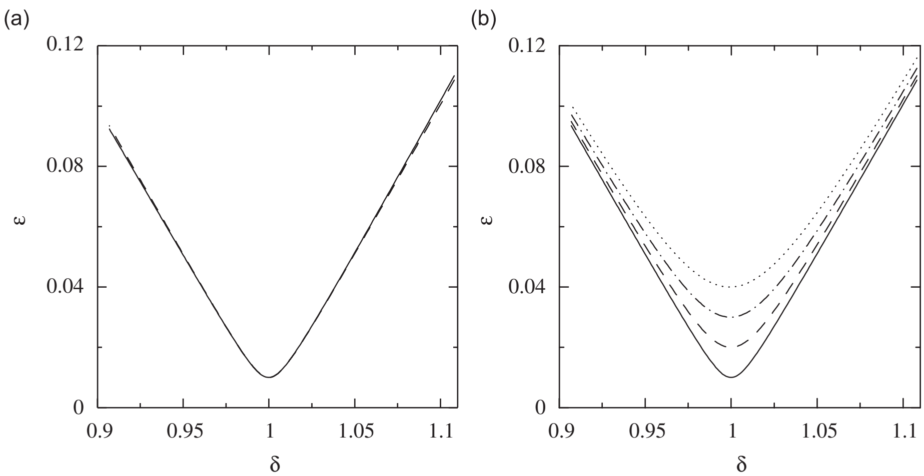

Fig. 3. (a) First- (– –) and second- (—) order transition curves for $\xi = 0.005$ and (b) first-order transition curve for different values of ξ (— $\xi = 0.005$, – – $\xi = 0.01$, –·– $\xi = 0.015$, ··· $\xi = 0.02$).

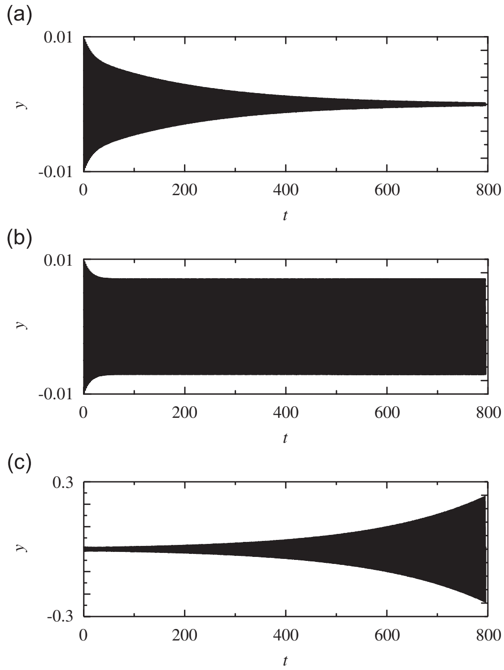

Fig. 4. Time histories of the response corresponding to $\xi = 0.007$, $\delta = 1$ for different values of ε: (a) $\varepsilon = 0.8\ \varepsilon_{\text{lim}}$, (b) $\varepsilon = \varepsilon_{\text{lim}}$, and (c) $\varepsilon = 1.2\ \varepsilon_{\text{lim}}$.

approximations are almost coincident for any value of δ. Fig. 3(b) shows the variation of the first-order transition curve with the damping ratio ($\xi_j = 0.005$ solid lines, $\xi_j = 0.01$ dashed lines, $\xi_j = 0.015$ dash–dotted lines, $\xi_j = 0.02$ dotted lines). The transition curve is highly influenced by the damping ratio, especially for δ close to 1. For δ deviating from unity, the influence of such a parameter on the stability limit is less sensitive.

In order to test the precision of the analytical transition curves, numerical simulations have been carried out and the Mathieu equation (Eq. (32)) has been solved by the central finite difference method for various values of the parameters, corresponding to stable and unstable situations.

Fig. 4 shows the time histories of the free non-dimensional response $y(t)$ assuming $\xi = 0.007$ and $\delta = 1$, due to the initial conditions $y_0 = 0.01$, $\dot{y}_0 = 0$ and $\varepsilon = 0.8\ \varepsilon_{\text{lim}}$ (Fig. 4a), $\varepsilon = \varepsilon_{\text{lim}}$ (Fig. 4b) and $\varepsilon = 1.2\ \varepsilon_{\text{lim}}$ (Fig. 4c), being $\varepsilon_{\text{lim}} = 0.014$. The numerical simulations confirm the stability conditions defined by the transition curves: for $\varepsilon < \varepsilon_{\text{lim}}$ (Fig. 4a) the system is stable and the response tends to zero; $\varepsilon = \varepsilon_{\text{lim}}$ (Fig. 4b) corresponds to the transition between stability and instability and the solution is periodic; for $\varepsilon > \varepsilon_{\text{lim}}$ (Fig. 4c) the system is unstable and the response tends to grow without limits (oscillatory divergence).

Fig. 5 shows the time histories of the response corresponding to $\delta = 1$ and $\varepsilon = 1.2\ \varepsilon_{\text{lim}}$ for different values of the damping: Figs. 5a–c correspond to $\xi = 0.005$ ($\varepsilon_{\text{lim}} = 0.01$), $\xi = 0.007$ ($\varepsilon_{\text{lim}} = 0.014$), and $\xi = 0.01$ ($\varepsilon_{\text{lim}} = 0.02$), respectively. It can be observed that the rate of increase of the response depends on the damping: in particular, the response increases much more rapidly in case of high damping (Fig. 5c). This fact can be explained observing that, in case of high damping, the limit parametric stiffness leading to instability is much greater than in case of low damping; thus, it affects the equations of motion in a stronger way.

4. Discussion of technical aspects

The transition curves can be interpreted from a technical point of view taking into account that the parameter ε is proportional to the ratio between the modal mass of the synchronized pedestrians and the structural modal mass, while the parameter δ is a function of the ratio between the natural frequency of the footbridge and the pedestrians' lateral walking frequency (Eq. (33)). In particular, it is worth noting that the ε parameter hardly reaches values greater than 0.1 (see Fig. 3). Thus, only that part of the transition curves

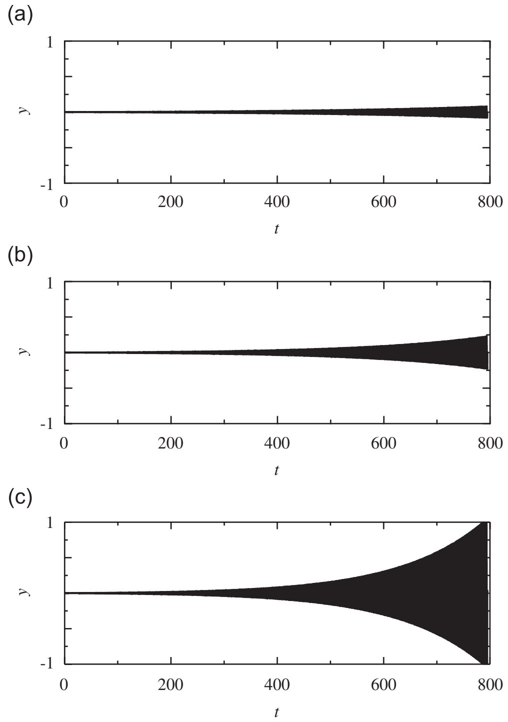

Fig. 5. Time histories of the response corresponding to $\delta = 1$, $\varepsilon = 1.2\ \varepsilon_{\lim}$ for different values of ξ: (a) $\xi = 0.005$, (b) $\xi = 0.007$, and (c) $\xi = 0.01$.

corresponding to $\varepsilon < 0.1$ is of technical interest: this region, corresponds to values of δ in the interval (0.9–1.1). Therefore, the instability mechanism analyzed here is of technical interest only for footbridges with lateral frequencies very close to the principal resonance condition, corresponding to $\omega_j = \Omega/2$ (Ω being in the range 0.8–1.2 Hz [1]) and, thus, natural frequencies n_j around 0.5 Hz. Moreover, since the first- and second-order approximations of the transition curves almost coincide, the first-order approximation can be adopted (Eq. (43)). The limit condition can be written as a function of the structural damping, the mass ratio and the frequency ratio as follows:

$$\frac{M_{p_j}}{M_{s_j}} < \frac{\Omega^2}{2g\lambda\alpha_1}\sqrt{\left(4\frac{\omega_j^2}{\Omega^2} - 1\right)^2 + 16\frac{\omega_j^2}{\Omega^2}\xi_j^2}. \tag{49}$$

The worst condition (the minimum in the transition curve) occurs when the frequency of the pedestrian motion is exactly double the structural natural frequency, $\Omega = 2\omega_j$; in this case, the stability criterion is given by

$$\frac{M_{p_j}}{M_{s_j}} < \frac{4\omega_j^2}{g\lambda\alpha_1}\xi_j. \tag{50}$$

Assuming $\omega_j = \pi$ (i.e. setting the lateral walking frequency to the mean value, 1 Hz) and $\alpha_1 = 2$ (see Section 2.1, Fig. 2), Eq. (50) becomes

$$\frac{M_{p_j}}{M_{s_j}} < \frac{2\pi^2}{g\lambda}\xi_j \tag{51}$$

or, alternatively, it can express the minimum value of damping coefficient to avoid instability conditions:

$$\xi_j > \frac{\lambda g}{2\pi^2}\frac{M_{p_j}}{M_{s_j}}, \tag{52}$$

where the percentage of synchronized pedestrians λ typically assumes values in the range (0.2–0.4) [3,9]. In case of uniform structural and pedestrian mass distribution, the ratio between the modal pedestrian and structural masses, M_{p_j}/M_{s_j} in Eqs. (49)–(52), can be replaced by the ratio between the people and footbridge per-unit-length masses, m_p/m_s.

From the analysis of Eqs. (51) and (52) it can be deduced that footbridges more prone to the analyzed unstable mechanism are light (small m_s), lowly damped (small ξ) in crowded conditions (great m_p). The stability criterion expressed by Eq. (51) is comparable with analogous formulae of the literature, in particular with Roberts' (Eq. (19)) and Newland's (Eq. (27)) expressions. A formal comparison highlights that the present proposal (Eq. (51)) is more conservative than the ones given by Newland (Eq. (27)) and by Roberts (Eq. (19)). On the contrary, the formulation by Dallard et al. (Eq. (15)) seems hardly comparable with the previous ones since it is deeply linked to the experimental measurements on the London Millennium Footbridge, and cannot be directly applied to other bridges.

However, it should be noted that the three criteria, (19), (27) and (51), admit different application fields according to frequency values and oscillation shapes of the footbridge. The mechanisms analyzed by Roberts [10–12] and Newland [9] appear possible for footbridges with the smallest lateral frequency around 1 Hz (certainly not lower than 0.7 Hz; see, e.g. trials on Solferino footbridge [6]); in these cases the criterion (51) proposed in this paper cannot be applied. On the contrary, if the footbridge is very flexible (with a first lateral frequency around 0.5 Hz) the mechanism of parametric excitation can occur and the criterion (51) becomes generally more urgent with regard to the others. In these cases, indeed, the contemporary presence of higher lateral modes of frequency around 1 Hz is very probable (e.g. it happens for the London Millennium Footbridge), even if they show different modal shapes and different modal masses (e.g. skew-symmetric rather than symmetric, with the application of the synchronized load on a portion of the structure). So, in case of very flexible footbridges, both the dynamic interaction and the parametric excitation mechanisms could be activated, and the application of the distinct stability criteria must be conducted using different values of the involved parameters.

5. A real case study: the London Millennium Footbridge

During the opening day, a maximum crowd density of between 1.3 and 1.5 persons/m^2 crossed the London Millennium Footbridge [4]. Unexpected excessive lateral vibrations of the bridge occurred on the central span, when it was occupied by about 200 people, at frequencies just around 0.5 and 1 Hz, corresponding to the first symmetric ($n_1 = 0.48$ Hz, $M_{s1} = 130$ t) and the second skew-symmetric ($n_2 = 0.95$ Hz, $M_{s2} = 150$ t) lateral mode, respectively [5]. The oscillation amplitude of the central span has been visually estimated as 70 mm, and the maximum lateral acceleration experienced on the bridge was between 200 and 250 mg [4].

In this section, an attempt is made to explain the swaying motion of the London Millennium Bridge adopting the excitation mechanisms proposed in the literature (Section 2) and the parametric excitation mechanism proposed in this paper (Section 3); finally, the results obtained adopting the proposed model are compared with experimental observations.

5.1. Direct resonance

If pedestrians are assumed to walk with a lateral frequency around 1 Hz, the direct resonance can be achieved with reference to the second mode of vibration ($n_2 = 0.95$ Hz). However, since the modal shape of such a mode is skew-symmetric, the modal force is zero assuming a uniform pedestrian distribution. The only possibility to have a non-zero modal force is to assume that synchronized pedestrians are not uniformly distributed. In such a case, the response of the bridge would be harmonic with the sole component at $n_2 = 0.95$ Hz, whereas the frequency component corresponding to the first mode of vibration cannot be motivated in linear dynamics. It is almost impossible to assume that pedestrians walked so slowly as to directly excite the first mode of vibration ($n_1 = 0.48$ Hz). Thus, direct resonance is not able to reproduce the experimental evidence that both the first and the second natural frequencies occur in the Millennium Bridge oscillations.

Dallard et al. [5] made a remark that the mode at 0.48 Hz might have been excited due to slightly meandering path of people who were instinctively changing direction slightly every 3–4 footfalls.

5.2. Dynamic interaction

The model proposed by Dallard et al. [4] reproduces the limit stability condition since the parameter k has been defined by back analyses of the tests on the Millennium Footbridge itself. However, such a model is not able to reproduce the physical evidence that two harmonic components are experimentally identified in the response.

The mechanisms analyzed by Roberts [10] and Newland [9] could be appropriate to model the synchronization between pedestrians and the second skew-symmetric mode of vibration of the bridge, concerning Newland's model a not uniform distribution of synchronized pedestrians on the bridge span should be used. Results obtained with regard to the pedestrian limit number are consistent with the experimental crowd densities but the only physical explanation for oscillations on the first symmetric mode is to assume that pedestrians took two steps directed slightly to the right followed by two steps directed slightly to the left, giving rise to a force at the frequency of the first mode of vibration.

5.3. Internal resonance

The third vertical mode ($n_3 = 1.89$ Hz) and the second lateral mode ($n_2 = 0.95$ Hz) of the Millennium Bridge satisfy the necessary condition for internal resonance. However, the analysis of vertical forces and lateral oscillations of the bridge showed no correlation between such quantities [4]. Thus, internal resonance is not appropriate to explain lateral sway of the bridge.

5.4. Parametric excitation

In this section, the proposed parametric excitation mechanism is applied to the case study of the London Millennium Footbridge.

Fig. 6 shows the limit pedestrian number $N_{\mathrm{lim}} = m_p L/m_{\mathrm{ps}}$ in case of uniformly distributed pedestrians along the central span (Eq. (49) with m_p/m_s in place of M_{p_j}/M_{s_j}) as a function of the walking frequency $n = \Omega/2\pi$, setting $\omega_j = 2\pi n_1$ ($n_1 = 0.48$ Hz), $\lambda = 0.3$ (mean value of synchronized pedestrian percentage), $m_s = 2000$ kg/m, $m_{\mathrm{ps}} = 70$ kg, $\alpha_1 = 2\,\mathrm{m}^{-1}$, for three values of the damping ratio $\xi = 0.005$ (solid line), 0.007 (dashed line), 0.01 (dash–dotted line). The worst condition (minimum limit number of pedestrians) obviously occurs when pedestrians walk with a lateral frequency $n = 0.96$ Hz, corresponding to exactly twice the natural frequency n_1; in such a case, the number of uniformly distributed pedestrians which causes instability is

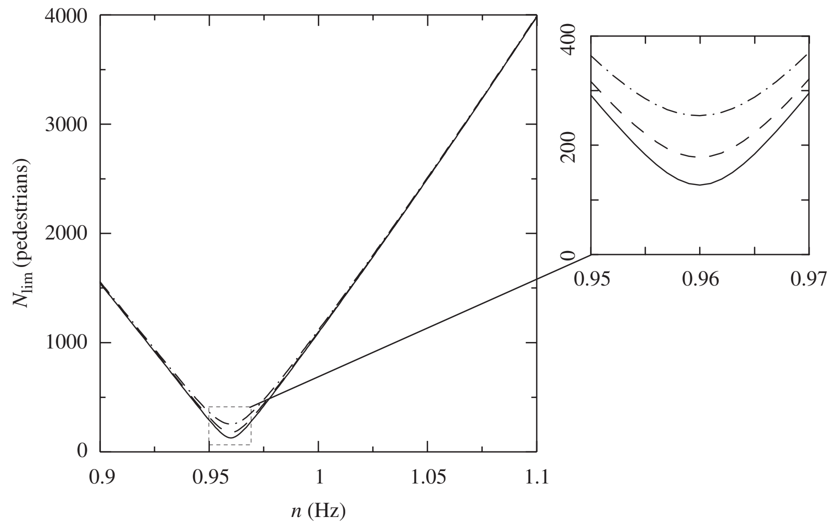

Fig. 6. Limit pedestrian number as a function of the walking frequency and of the damping ratio (— $\xi = 0.005$; – – $\xi = 0.007$; and –·– $\xi = 0.01$).

$N_{\lim} = 127$, 178, 254, respectively, for the three damping conditions. The limit value of pedestrians corresponding to the measured damping ratio ($\xi = 0.007$) is $N_{\lim} = 178$, consistent with experimental observations.

The second lateral mode of vibration of the central span of the bridge is assumed exactly skew-symmetric (i.e. $\varphi_2(L/2) = 0$), thus the bridge deck mid-span response is only associated with the first mode of vibration. The forced equation of motion of the first principal component (Eq. (30)) is numerically solved, assuming a sinusoidal shape for the first mode of vibration (i.e. $\varphi_1(x) = \sin(\pi x/L)$) and setting $\alpha_0 = 0.04$, $\alpha_1 = 2\,\mathrm{m}^{-1}$, $\lambda = 0.3$, $\omega_{\mathrm{j}} = \omega_1 = 2\pi n_1$, $m_s = 2000\,\mathrm{kg/m}$, $\xi = 0.007$ (measured value of the structural damping ratio). The number of pedestrians crossing the footbridge is considered variable, with a lateral walking frequency exactly twice the first natural frequency of the bridge n_1 (i.e. $\Omega = 2\omega_1$). The deck displacement at the central mid-span is finally evaluated taking into account the sole first mode of vibration ($\bar{N} = 1$ in the principal transformation (9)).

Fig. 7 shows the time histories of the bridge deck displacements at the central mid-span, together with their Fourier transforms, assuming $\xi = 0.007$, for a variable number of pedestrians crossing the bridge ($N_p = 50$, 180, 250 in Fig. 7a–c, respectively). When a small number of pedestrians crosses the bridge, the effect of parametric excitation is almost negligible and the system response is mainly at the excitation frequency (the Fourier transform of the response shows only one significant spike at the excitation frequency $n = 0.97\,\mathrm{Hz}$, Fig. 7a). When the number of pedestrians reaches the limit value corresponding to the selected damping coefficient, parametric excitation occurs and the response of the system shows two harmonic components (the Fourier transform of the response has two comparable spikes, respectively at $n = 0.97\,\mathrm{Hz}$ and $n_1 = 0.48\,\mathrm{Hz}$, the frequency of the first lateral mode of vibration, Fig. 7b). When the number of pedestrians

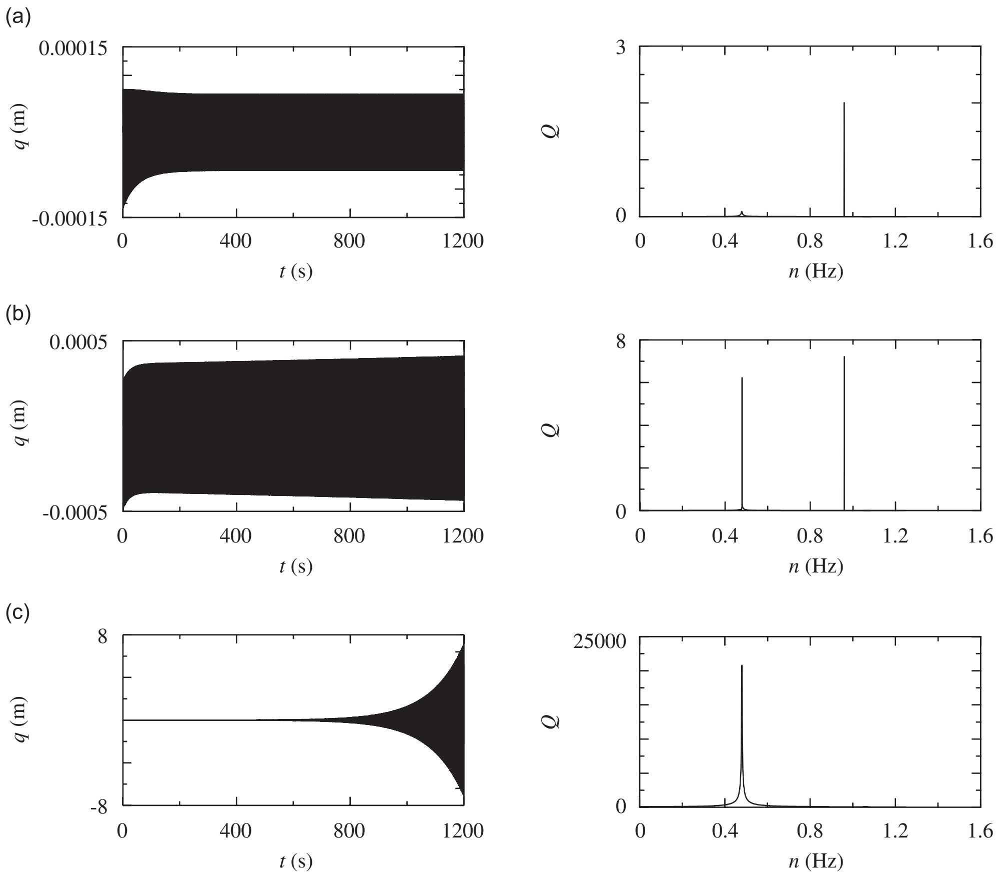

Fig. 7. Bridge response (left) and harmonic content Q (right) for three values of the number of pedestrians: (a) $N_p = 50$, (b) $N_p = 180$, and (c) $N_p = 250$.

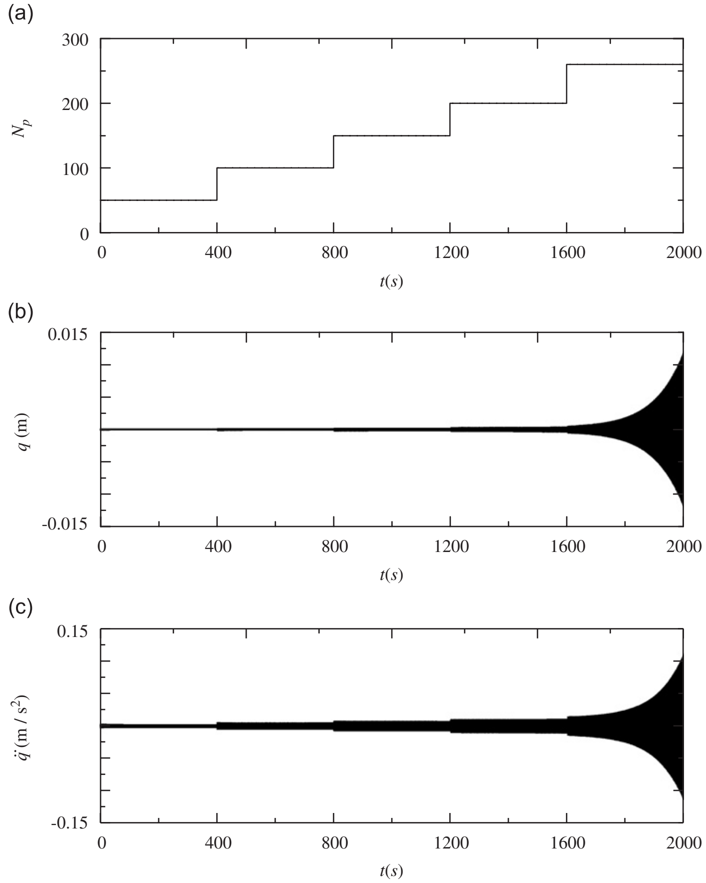

Fig. 8. Bridge response (b) and acceleration (c) on increasing the number of pedestrians (a).

exceeds the stability limit, the response is no longer periodic but tends to diverge in an oscillatory way and it is totally governed by parametric excitation: the Fourier transform has a dominant spike at $n_1 = 0.48\,\text{Hz}$ (Fig. 7c). Actually, when deck vibrations become excessive, pedestrians stop walking and the phenomenon is naturally self-limiting.

Fig. 8 is an attempt to reproduce the experimental results obtained during the pedestrian loading tests on the Millennium Bridge [5]: it shows the time history of the displacement (Fig. 8b) and of the acceleration (Fig. 8c) of the bridge on increasing the number of pedestrians crossing the deck, from 50 up to 260 people (Fig. 8a). It can be observed that, when the number of pedestrians exceeds 200 units, the deck oscillation begins to increase and both displacements and accelerations reach values of the same order of magnitude as those experimentally observed.

Therefore, the parametric excitation of the first lateral mode of vibration ($n_1 = 0.48\,\text{Hz}$) can explain the excessive lateral sway of the London Millennium Footbridge at the deck mid-span. Since the Millennium Bridge is characterized by a second skew-symmetric lateral mode of vibration with a natural frequency around 1 Hz, also direct resonance could be activated and could probably be responsible for bridge vibrations in other positions of the deck, where the contribution of the second mode of vibration is significant. However, only a mechanism of parametric excitation is able to motivate the significant excitation of the first lateral mode of vibration.

6. Conclusions and prospects

In this paper, a new excitation mechanism has been proposed to motivate the excessive lateral sway motion caused by crowds walking across footbridges, based on a forcing model considering an amplitude-dependent dynamic loading factor. The footbridge is thus analyzed as a parametrically excited system and a stability limit is identified, depending on the ratio between the structural and excitation frequencies, on the ratio of the structural and pedestrian mass, and on the structural damping. Differently from the direct resonance and internal resonance mechanisms, the parametric excitation mechanism produces an instability phenomenon, whose critical conditions can be achieved for light, lowly damped footbridges, with a lateral natural frequency around 0.5 Hz, corresponding to a half of the lateral walking frequency. The application of the proposed formulation to the case of the London Millennium Footbridge provides results in accordance with experimental observations: when the pedestrian mass is greater than the deduced limit value, the amplitude of motion tends to dramatically increase, in a way very similar to experimental measurements.

The domain of occurrence of the different crowd-induced mechanisms analyzed in this paper and, therefore, the bounds of applicability of the proposed forcing model is worth discussing. When a footbridge is sufficiently stiff in the lateral direction, that is its first lateral frequency is around 1 Hz (e.g. the T-Bridge [3]) and certainly not lower than 0.7 Hz (e.g. the Solferino Footbridge [6]), the dynamic interaction mechanisms in resonant conditions appear predominant and the criterion proposed in this paper is not applicable. When a footbridge is very flexible in the lateral direction, that is its first lateral frequency is around 0.5 Hz (e.g. the Millennium Bridge [4,5] and the Maple Valley Great Suspension Bridge, recently studied by Nakamura and Kawasaki [22]), the proposed parametric excitation mechanism seems the only one able to lead to a predictive criterion, producing both a reliable value of the pedestrian limit number and an interpretation of the response frequency spectrum, without the need for back analyses on the real structure. In these cases, however, it is probable that the footbridge exhibits lateral frequencies around 1 Hz too, leading to a mixed resonant-parametric excitation mechanism, as probably happened for the London Millennium Footbridge.

Even though the proposed mechanism is supported by the previously cited experimental evidence on a real structure, the main limit of the current formulation is the effective reliability of the forcing model adopted here, because of the incompleteness of the actual experimental measures concerning the interaction between the human walking and the generic movement of the platform where the pedestrian motion occurs. Therefore, the authors highlight the effective necessity of carrying out free field tests on flexible footbridges, recording simultaneously the bridge response and the pedestrian motion, in order to provide reliable models of forces induced by pedestrians on a moving footbridge and, thus, to definitely confirm the applicability of the proposed forcing model.

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References

[1] S. Zivanovic, A. Pavic, P. Reynolds, Vibration serviceability of footbridges under human-induced excitation: a literature review, *Journal of Sound and Vibration* 279 (2005) 1–74.
[2] H. Grundmann, H. Kreuzinger, M. Schneider, Dynamic calculation of footbridges, *Bauingenieur* 68 (1993) 215–225.
[3] Y. Fujino, B.M. Pacheco, S. Nakamura, P. Warnitchai, Synchronization of human walking observed during lateral vibration of a congested pedestrian bridge, *Earthquake Engineering and Structural Dynamics* 22 (1993) 741–758.
[4] P. Dallard, A.J. Fitzpatrick, A. Flint, S. Le Bourva, A. Low, R.M. Ridsdill Smith, M. Willford, The London Millennium Footbridge, *The Structural Engineer* 79 (22) (2001) 17–33.
[5] P. Dallard, A.J. Fitzpatrick, A. Flint, A. Low, R. Ridsdill Smith, M. Willford, M. Roche, London millennium bridge: pedestrian-induced lateral vibration, *Journal of Bridge Engineering ASCE* 6 (6) (2001) 412–417.
[6] F. Danbon, G. Grillaud, Dynamic behaviour of a steel footbridge. Characterization and modelling of the dynamic loading induced by a moving crowd on the Solferino Footbridge in Paris, *Proceedings Footbridge 2005*, 2005.
[7] A.N. Blekherman, Swaying of pedestrian bridges, *Journal of Bridge Engineering* 10 (2) (2005) 142–150.
[8] S. Nakamura, Model for lateral excitation of footbridges by synchronous walking, *Journal of Structural Engineering ASCE* 130 (19) (2004) 32–37.
[9] D.E. Newland, Pedestrian excitation of bridges, *Proceedings of the Institution of Mechanical Engineers 128c, Journal of Mechanical Engineering Science* (2004) 477–492.

[10] T.M. Roberts, Lateral pedestrian excitation of footbridges, *Journal of Bridge Engineering ASCE* 10 (1) (2005) 107–112.
[11] T.M. Roberts, Synchronised pedestrian lateral excitation of footbridges, *Proceedings Eurodyn 2005*, 2005, pp. 1089–1094.
[12] T.M. Roberts, Probabilistic pedestrian lateral excitation of bridges, *Proceedings of ICE, Bridge Engineering* 158 (BE2) (2005) 53–61.
[13] Y. Fujino, P. Warnitchai, B.M. Pacheco, An experimental and analytical study of autoparametric resonance in a 3 DOF model of a cable-stayed beam, *Nonlinear Dynamics* 4 (1993) 111–138.
[14] A.H. Nayfeh, *Introduction to Perturbation Techniques*, Wiley, New York, 1993.
[15] A.H. Nayfeh, D.T. Mook, *Nonlinear Oscillations*, Wiley, New York, 1979.
[16] Y. Matsumoto, T. Nishioka, H. Shiojiri, K. Matsuzaki, Dynamic design of footbridges, *IABSE Proceedings*, No. P-17/78, 1978, pp. 1–15.
[17] H. Bachmann, A.J. Pretlove, H. Rainer, Dynamic forces from rhythmical human body motions, in: *Vibration Problems in Structures: Practical Guidelines*, Birkhauser, Basel, 1995.
[18] S.A. Gard, S.C. Miff, A.D. Kuo, Comparison of kinematic and kinetic methods for computing the vertical motion of the body center of mass during walking, *Human Movement Science* 22 (2004) 597–610.
[19] V.T. Inman, H.J. Ralston, F. Todd, *Human Walking*, Williams & Wilkins, Baltimore USA, 1981.
[20] A.D. Pizzimenti, F. Ricciardelli, Experimental evaluation of the dynamic lateral loading of footbridges by walking pedestrians, in: C. Soize, G.I. Schueller (Eds.), *Proceedings of the Eurodyn 2005*, Millpress, Rotterdam, 2005, pp. 435–440.
[21] A.P. Seyranian, A.A. Mailybaev, *Multiparameter Stability Theory with Mechanical Applications*, World Scientific, Singapore, 2003.
[22] S. Nakamura, T. Kawasaki, Lateral vibration of footbridges by synchronous walking, *Journal of Constructional Steel Research* 62 (2006) 1148–1160.