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Short Communication

Optimization of the Ritz method to calculate axisymmetric natural vibration frequencies of cylinders

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Abstract

A straightforward accurate technique is presented for calculating natural frequencies for axisymmetric vibration of cylinders of any aspect ratio. The Ritz variational method is applied by using algebraic polynomials in the form of power series in the axial and radial coordinates. The adequate choice of the maximum exponents for each coordinate is important factor in order to achieve better accuracy, compatible with an admissible time of calculation. A computational model is developed that provides the optimum selection of exponents by following an automatic iterative process. This leads to a precise calculation of frequencies in a minimum time of calculation. The method is tested by comparing numerical and experimental results and a good agreement is obtained. The common hypothesis on the number of terms in the radial and axial coordinates is not sufficiently justified, as is proved.

1. Introduction

The study of the propagation of waves in cylinders, based on the classic theory of elasticity, is well known [1,2]. An analytical solution of the equations of motion that satisfies the boundary conditions is generally impossible to obtain for a finite cylinder. Thus, approximate solutions are sought. Appropriate solutions lead to stationary waves.

Several researchers have carried out three-dimensional analysis based on the equations of elasticity in order to find accurate natural frequencies for the vibrations of solid elastic cylinders. The work of Hutchinson [3,4] is particularly notable. He proposed a Bessel series solution of the general three-dimensional equations of linear elasticity; three of the six boundary conditions are exactly satisfied, the remaining three conditions are approximated. This method has been extended [5] to determine the vibration response of cylinders to arbitrary distribution of axisymmetric symmetric excitation on its surface.

The Ritz method is a widely used procedure for finding solutions for stationary vibrations. Rumermann and Raynor [6] expressed the components of displacement as series of the corresponding pure radial and axial modes of the infinite cylinder. Heyliger [7] used power series in the axial and radial coordinates as the

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approximation functions. The work of Leissa and So [8,9] in the study of finite cylinders with different boundary conditions deserves special mention; the displacement functions chosen are in the form of algebraic polynomials in the cylindrical coordinates. Liew et al. [10] studied free vibrations of elastic solid cylinders of polygonal cross-section, sets of one and two-dimensional orthogonal polynomials are constructed to represent the displacement components in the longitudinal and lateral surface directions.

When considering only axisymmetric modes of a vibrating cylinder, the azimuthal variable θ does not appear and, in addition, the displacements have only two non-null components; radial u and axial w. The torsional axisymmetric vibrations are therefore excluded. This study is focused on the free vibrations of a cylinder, i.e. ends and lateral surfaces are free and there are no bulk forces. By using non-dimensional coordinates: r, the radial coordinate divided by the radius of the cylinder; and z, the axial coordinate divided by its length, the supposed displacements in the cylinder become:

$$\begin{aligned} u(r,\theta,z,t) &= U(r,z)\sin(\omega t), \\ w(r,\theta,z,t) &= W(r,z)\sin(\omega t). \end{aligned} \tag{1}$$

In the Ritz method approximate solutions are assumed which satisfy the boundary conditions. The series used here for the radial and longitudinal displacements are algebraic polynomials of the form

$$U(r,z) = \sum_{i=1}^{I}\sum_{j=0}^{J} A_{ij} r^i z^j \text{ and } W(r,z) = \sum_{p=0}^{P}\sum_{q=0}^{Q} C_{pq} r^p z^q. \tag{2}$$

Convergence towards the "true" frequencies and mode shapes is obtained as the number of terms in the approximating expressions is increased.

For simplicity of formulation, the non-dimensional frequency

$$\Omega = \pi f\, D(\rho/G)^{1/2} \tag{3}$$

is used, where f is the ordinary frequency measured in Hertz, D the diameter, ρ the density, and G the shear modulus.

Hamilton's principle for harmonic motion leads to the condition of minimization for the difference between maximum kinetic $T_{\max}$ and maximum potential $V_{\max}$ energies, which are expressed and calculated in terms of U and W. The difference $T_{\max}-V_{\max}$ can be expressed [11] as a function of aspect ratio L/D, Poisson's ratio v, and Ω. Since the condition of minimum relates such quantities, axisymmetric non-dimensional natural frequencies depend on v and L/D, i.e., $\Omega = \Omega\ (v, L/D)$. The minimizing conditions imply that $\partial(T_{\max}-V_{\max})/\partial A_{ij} = 0$ and $\partial(T_{\max}-V_{\max})/\partial C_{pq} = 0$. The compatibility condition for this set of equations gives the admissible values for Ω^2. For each eigenvalue Ω^2, the set of linear equations supply the eigenvectors whose components are the unknown quantities A_{ij} and C_{pq}. A Maple program is developed to compute the eigenvalues and eigenvectors. The algebraic system of n equations is solved, and therefore n non-dimensional frequencies are obtained which correspond to the n eigenvalues of the system. The value of n equals the number of monomials in the displacement functions.

A mode of vibration is called symmetric mode if j takes even values (zero included) and q odd values. A mode is called antisymmetric if j takes odd values and q even values (zero included). The separation of the problem into symmetric and antisymmetric modes has at least three advantages: (1) the size of the matrices decreases, (2) the separation allows the problem of one type of symmetry to be solved with no need to solve the other, and (3) by studying both types of symmetry, the total number n of the natural frequencies of vibration of the cylinder are obtained.

A sequence of values of the non-dimensional parameter Ω is obtained from the application of the Ritz method. Different frequencies result from each combination of maximum exponents for a given value of number of term n. The upper limits I, J, P, and Q of the series are chosen for each value of L/D and v by following a trial with increasing values of the four indices so that the values of Ω_j obtained are as low as possible. The most accurate results achievable are only guaranteed when a substantial variety of sets of exponents are tested and is thereby a labour-intensive task for the researcher. Although satisfactory results are obtained with a feasible computational effort, a more effective method would be involving less human effort.

Therefore, it is highly recommendable to implement a program capable of determining the adequate maximum exponents with confidence; which is the objective of this work. A program is here developed that automatically determines the optimum maximum exponents in the series for the displacements. The procedure applied guarantees the best choice of exponents at any stage of the process. Such a procedure is described in the following section.

2. The Ritz method: optimization of exponents in the series

In order to shorten the calculation, the exponents must then be sufficiently low, but in order to adapt the precision to the requirements, the exponents must be sufficiently high. An evident challenge is to find the right balance.

The Ritz method is optimally applied here to find accurate values of the non-dimensional frequencies by following an automatic iterative process. The procedure begins with very low values for maximum exponents I, J, P, Q, by taking the parity of the studied symmetry into account. With these input data, non-dimensional frequencies Ω_1, $\Omega_2,\ldots$ are obtained. The next step is that first maximum exponent I is increased by the possible minimum amount, and hence the maximum exponents are I+1, J, P, Q. Then, the calculation of the frequencies gives new values Ω'_1, $\Omega'_2,\ldots$, which must be lower than the previous ones. The initial value of I is restored and the exponent J is increased to its next possible amount $J' = J+2$, and therefore yielding maximum exponents I, J', P, Q and frequencies Ω''_1, $\Omega''_2,\ldots$. This process is continued with I, J, $P' = P+1$, Q and I, J, P, $Q' = Q+2$. The values obtained for the first frequencies are compared Ω'_k, $\Omega''_k,\ldots$. That is, the frequencies on which more interest is focused are compared, and one chooses the lowest from among them. These are the best values from which the most effective increase in the exponents is deduced because it provides the highest decrement of these frequencies. The best set of exponents is fixed.

The process of sequential increase of each one of the four exponents is repeated and the new optimal exponent is settled. The maximum exponents continue to be increased up to the point of interest. This method allows several frequencies to be obtained with the required precision and of routine form.

Since in two referred papers [8,9], the first five frequencies are calculated, the criterion of minimizing the sum of the five lowest frequencies is chosen here and in the cases studied the program stops when n is about 90.

A sample of $L/D = 0.3$, $\nu = 0.286$ is firstly analysed. For symmetric modes the first loop starts from the lowest combination for the symmetric mode, $I = 1$, $J = 0$, $P = 0$, and $Q = 1$. The program proceeds with the loop of four calculations for the four combinations of indices obtained by increasing only one maximum exponent. These intermediate results are presented in Table 1 for the studied cylinder. All cases have $n = 3$ and they have 3 non-null frequencies. In order to continue with the next iteration, the proposed automatic decision is made. In this table, one can see that the most accurate or minimum sum of the lowest frequencies is obtained when $I = 1$, $J = 0$, $P = 1$, and $Q = 1$. Hence, this is the combination selected.

With this selected combination, the second loop starts, increasing now the maximum exponents from the previous selected values. Table 2 shows the results in the second loop for the case studied after fixing the above combination. Looking at Table 2, the minimum sum appears when $I = 2$, $J = 0$, $P = 1$, and $Q = 1$.

Table 3 shows the results for the first 25 loops. The maximum exponents are automatically selected by the minimum sum of the lowest five frequencies. Values of up to the fifth non-dimensional frequencies are given. Its last column shows the accumulated time in seconds with a Pentium4 processor.

Table 1
Results of the first loop for symmetric modes for a cylinder of $L/D = 0.3$, $\nu = 0.286$

I	J	P	Q	n	Ω_1	Ω_2	Ω_3
2	0	0	1	3	3.426837	9.959593	13.119605
1	2	0	1	3	3.715127	10.775174	13.614634
1	0	1	1	3	3.715127	10.775174	11.367276
1	0	0	3	3	3.712840	9.778217	39.717970

The maximum exponent P should be increased in the next loop.

Table 2
Results of the second loop for the symmetric modes

I	J	P	Q	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
2	0	1	1	4	3.377318	9.046206	10.976615	14.090759	
1	2	1	1	4	3.715127	9.925223	10.775174	14.699087	
1	0	2	1	4	3.715127	10.775174	11.291242	13.599183	
1	0	1	3	5	3.712840	9.778217	10.465254	39.717970	39.933730

The maximum exponent I should be increased in the next loop.

Table 3
Results automatically selected from the different sequential loops with the criterion minimal sum of 5 first frequencies

I	J	P	Q	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	t (s)
1	0	0	1	2	3.715127	10.775174				0.1
1	0	1	1	3	3.715127	10.775174	11.367276			0.3
2	0	1	1	4	3.377318	9.046206	10.976615	14.090759		0.6
2	2	1	1	6	3.367326	8.367333	10.479249	13.245898	15.063372	0.8
2	2	2	1	7	3.367247	8.304261	10.259217	11.078892	13.552165	1.2
3	2	2	1	9	3.361496	8.000400	8.779359	10.567395	12.040821	2.2
3	2	2	3	12	3.358660	7.596777	8.041083	9.518541	10.854255	3.5
3	4	2	3	15	3.358623	7.559812	7.762003	9.469062	10.678527	5.8
3	4	3	3	17	3.358609	7.550632	7.693942	9.461278	10.606202	10
4	4	3	3	20	3.358327	7.514999	7.659873	9.398330	10.295691	20
4	4	4	3	22	3.358327	7.486220	7.645685	9.299003	10.257547	34
4	4	5	3	24	3.358326	7.485053	7.643053	9.250771	9.882591	58
5	4	5	3	27	3.358325	7.470496	7.638924	9.204775	9.870014	89
6	4	5	3	30	3.358325	7.470222	7.638664	9.183654	9.791550	133
6	4	5	5	36	3.358324	7.468450	7.625433	9.178569	9.786672	207
6	4	6	5	39	3.358324	7.468152	7.625224	9.171362	9.786203	316
6	4	7	5	42	3.358324	7.468147	7.625118	9.168931	9.773145	499
6	6	7	5	48	3.358324	7.468127	7.622361	9.167615	9.772952	780
7	6	7	5	52	3.358324	7.468076	7.622333	9.166272	9.772735	1172
8	6	7	5	56	3.358324	7.468075	7.622312	9.165966	9.771140	1916
8	6	8	5	59	3.358324	7.468074	7.622285	9.165851	9.771121	3034
8	6	9	5	62	3.358324	7.468074	7.622279	9.165837	9.770973	4624
8	6	9	7	72	3.358324	7.468073	7.622220	9.165792	9.770954	6583
9	6	9	7	76	3.358324	7.468073	7.622213	9.165781	9.770952	94,66
9	8	9	7	85	3.358324	7.468073	7.622203	9.165774	9.770949	16,955

Fig. 1 shows the values of the frequencies in terms of the number of coefficients used in the calculation. It can be observed that the lowest frequencies converge quickly to an almost stationary value even for a small number of coefficients. However, the high frequencies converge slowly.

3. Results of numerical calculation

The method described in the previous section is applied to three cylinders whose slenderness ratios are 0.3, 0.853145, and 3.2, respectively. This selection has been done because the cylinder of ratio 0.3 can be assimilated to a disc and the cylinder of ratio 3.2 can be considered a long rod. Value 0.853145 corresponds to the so-called [11] universal slenderness ratio because a cylinder with this aspect ratio has its lowest symmetric natural frequency independent of Poisson's ratio and that slenderness can be seen as a boundary between the class of short cylinders and the class of long cylinders. The selected values of Poisson's ratio are 0, 0.286, and 0.49999 for each of the aforesaid ratios. These values of ν have been selected because the first and last values are extreme values in ordinary materials and the second value corresponds to the kind of steel used in the

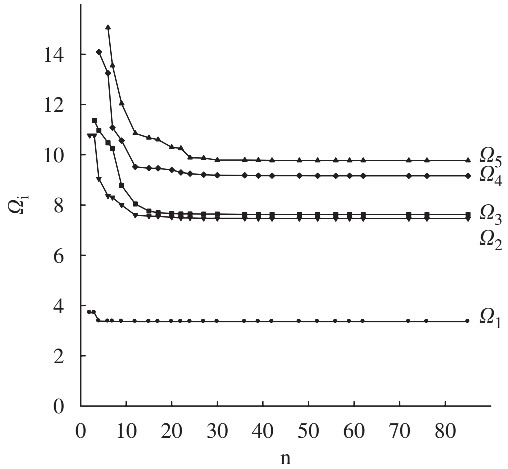

Fig. 1. Convergence of the first five lowest frequencies for the symmetric modes for a cylinder with aspect ratio $L/D = 0.3$ and Poisson's ratio $v = 0.286$.

Table 4
The five lowest frequencies of the symmetric modes of nine test cases

L/D	v	I	J	P	Q	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	t (s)
0.3	0	9	8	9	7	85	2.603827	6.501539	7.404805	7.428515	7.539799	11,592
0.3	0.286	9	8	9	7	85	3.358324	7.468073	7.622203	9.165774	9.770949	16,955
0.3	0.49999	5	18	3	19	90	3.975391	7.521591	8.739191	10.337127	12.609593	9060
0.853145	0	7	10	6	9	77	2.603826	2.603827	2.603827	4.664227	5.389498	10,763
0.853145	0.286	7	10	6	9	77	2.603827	3.213123	4.073039	5.228671	5.969739	57,774
0.853145	0.49999	9	8	8	7	81	2.603827	3.756643	4.982324	6.427426	7.671722	4216
3.2	0	5	14	6	15	96	0.694200	2.082601	2.365269	2.558300	2.603827	11,263
3.2	0.286	5	14	5	15	88	0.783195	2.209040	2.915038	3.071125	3.591413	29,915
3.2	0.49999	12	6	11	5	84	0.837488	2.263614	3.322413	4.106648	4.743365	1951

laboratory in this work. The theory and the criterion described above have been applied to these nine test samples.

Table 4 shows the resulting non-dimensional frequencies of the numerical calculation for symmetric modes of the nine samples. The calculation has been initiated with the lower maximum exponents, in order not to neglect any mode of vibration and to choose the best direction at any stage of the process. Note that the number of maximum terms reached, n, is of the same order of magnitude in all the cases due to the limitations of the calculation program. It can also be seen, by simple inspection of Table 4, that: (1) For aspect ratio 0.853145, corresponding to the first universal point, frequency Ω_1 is independent of Poisson's ratio. (2) For the universal slenderness and null Poisson's ratio the three lower frequencies are equal, which proves the multiplicity of the modes. (3) Frequency is an increasing function of v in the other cases. (4) The difference in the values of the maximum exponents is significant. In particular, maximum exponents I and J corresponding to the series U, are different to each other as also happens to exponents P and Q.

The method of optimization for the nine test cases has been repeated for the antisymmetric modes. Table 5 gives the values of the five lowest frequencies of the nine test samples.

4. Experimental results

The procedure for generating and detecting the vibration of a sample is described in a previous paper [12]. A simple pendulum is used to apply an axial impact to the centre of one of the ends of the cylinder.

Table 5
The five lowest frequencies of the anti-symmetric modes

L/D	v	I	J	P	Q	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	t (s)
0.3	0	9	7	9	6	76	1.503205	4.324353	6.489739	8.128924	9.589480	8389
0.3	0.286	9	7	9	6	76	1.875431	4.859977	6.829216	8.724917	10.155172	9863
0.3	0.49999	10	5	10	4	63	2.236864	5.340333	7.119669	9.194226	10.608075	7268
0.853145	0	7	9	8	10	89	2.241321	3.541322	4.530877	5.207653	5.860811	23,812
0.853145	0.286	7	9	7	10	83	2.768114	3.828274	5.043911	6.049151	6.520905	18,431
0.853145	0.49999	10	5	9	6	70	3.141307	3.957009	5.589955	6.222026	6.862911	14,798
3.2	0	5	15	5	14	88	1.388401	2.363052	2.579648	2.632320	2.776802	19,529
3.2	0.286	4	15	5	14	86	1.537746	2.709765	2.921827	3.368714	3.608065	11,115
3.2	0.49999	9	7	8	8	81	1.600620	2.717486	3.326428	3.871972	3.960011	6579

Table 6
Calculated (f_c) and experimental (f_e) frequencies and their differences (f_c-f_e) for the steel cylinders

L/D	f_c (Hz)	f_e (Hz)	f_c–f_e (%)
0.30	37,477	37,500	−0.23
	67,110	67,025	0.85
	97,118	97,150	−0.32
0.85	52,033	51,725	3.08
	55,316	55,325	−0.09
	64,209	64,150	0.59
3.20	15,651	15,550	1.01
	30,729	30,510	2.16
	44,144	43,780	3.64

The cylinder is then left to vibrate almost freely. A heterodyne optical interferometer [13] is used to detect the resulting vibration at the centre of the opposite end. The normal component of the displacement is here detected. The Fourier transform of the detected signal gives the natural frequencies.

For a specific cylinder, the natural frequencies in the spectrum are denoted in ascending order by f_1, f_2, f_3,…. Proportionality of Ω and f is the result of the definition of non-dimensional frequency, Eq. (3).

The three cylinder tested are of stainless steel, with a diameter $D = 49.90$ mm, and density $\rho = 7889\,\text{kg/m}^3$. Its elastic constants are $v = 0.286$ and $G = 77.42$ GPa, both calculated from the measurement of the P- and S-wave velocities.

Table 6 shows L/D of each cylinder and the respective lowest experimental frequencies f_e. The resolution is 25 Hz for the two first samples and 10 Hz for the third one.

5. Comparison of results

5.1. Comparison of computed results versus experimental ones

From Tables 4 and 5, it is deduced that for the steel test sample of aspect ratio 0.300, the three lowest non-dimensional frequencies are: Ω_1 (antisymmetric) = 1.875431, Ω_1 (symmetric) = 3.358324, and Ω_2 (antisymmetric) = 4.859977. Considering the diameter, the density, and the shear modulus, Eq. (2) gives the respective frequencies: $f_1 = 37\,477$ Hz, $f_2 = 67\,110$ Hz, and $f_3 = 97\,118$ Hz.

These numerically calculated values appear in the second column of Table 6, labelled f_c. In the fourth column the differences between the numerical and the experimental values are shown, expressed as a percentage.

The calculated and experimental values of frequencies for the samples of $L/D = 0.85$ and 3.2 are also shown in Table 6.

From Table 6 it may be deduced that:

(a) The values of the calculated frequencies are close to those of the experimental ones; the differences ranging from −0.09% to 3.64%, which indicates a close agreement between the theory and experiments. Therefore the two methodologies are adequate.
(b) Some differences of frequencies are negative. This fact seems to be paradoxical. In effect, the Ritz method always gives frequencies higher than the exact values. Some simple explanations of that result can be given: (1) The exact values of frequencies can never be known, therefore it is not possible to confirm if the computed values are higher. (2) The elastic constants are measured with certain uncertainties, therefore, Poisson's ratio taken as datum to calculate the non-dimensional frequencies is affected by error. Moreover, the value of the shear modulus introduced as datum to calculate the frequencies is not precise. (3) All the experimental set-up is affected by systematic and random uncertainties.

5.2. Comparison with other computed results

In order to compare the results attained by application of the method of optimization proposed with those obtained by Leissa and So [8], the non-dimensional frequencies for an $L/D = 1$ and $\nu = 0.3$ cylinder have been calculated. The results appear in Table 7. From the comparison of the two tables is deduced that:

(a) Table 7 includes for n the interval (3–97) whereas it is (8–96) in Table 1 of Leissa and So. The final results from both tables are equal for six significant figures except for the fifth frequency, which is 5.59265 in Table 7 whereas such a frequency is 5.59266 in the table of Leissa and So. In the aforesaid tables, the first frequency converges to six significant figures with the smallest determinant size (order 40 in both cases). However, in Table 7, determinants of order 55 and 71 are required to obtain six-figure convergence for the

Table 7
Calculated non-dimensional symmetric frequencies for a free cylinder of $L/D = 1$ and $\nu = 0.3$

I	J	P	Q	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
1	1	1	0	3	2.54950976	4.89897949	5.33853913		
1	3	1	0	4	2.54950976	3.99009196	4.89897949	6.89776530	
1	3	2	0	5	2.54950976	3.94526099	4.89897949	6.16068010	9.61670091
1	5	2	0	6	2.54950976	3.91563044	4.89897949	5.79684754	8.25547191
2	5	2	0	9	2.51896754	3.44499774	4.43991436	5.56087175	7.95091844
2	5	2	2	12	2.32866414	3.10260824	4.05716552	5.41928467	7.37403035
2	5	2	4	15	2.32797139	3.10105004	4.02991165	5.22118334	6.94998267
2	7	2	4	17	2.32796997	3.10103481	4.02310791	5.05490901	6.30318310
3	7	2	4	21	2.32679925	3.10019831	4.01231923	4.99339282	5.92335230
3	7	3	4	24	2.32633594	3.06900845	3.99387981	4.94528886	5.89241946
3	7	4	4	27	2.32633004	3.06893946	3.99305815	4.93528601	5.69795413
3	7	4	6	32	2.32632976	3.06892531	3.99181182	4.90959263	5.64378820
3	9	4	6	35	2.32632974	3.06891208	3.99167163	4.90171631	5.61655615
4	9	4	6	40	2.32630255	3.06720301	3.98951405	4.89815222	5.60823529
5	9	4	6	45	2.32630237	3.06719723	3.98944400	4.89721246	5.59955553
5	9	4	8	50	2.32630236	3.06719607	3.98943271	4.89658116	5.59633658
5	9	5	8	55	2.32630223	3.06714254	3.98929939	4.8962713	5.59578791
5	9	6	8	60	2.32630222	3.06714183	3.98929598	4.89617629	5.59367125
5	11	6	8	65	2.32630222	3.06714165	3.98929569	4.89607802	5.59279194
6	11	6	8	71	2.32630222	3.06714035	3.98928820	4.89604537	5.59276086
6	11	6	10	78	2.32630222	3.06714023	3.98928801	4.89603857	5.59271366
7	11	6	10	84	2.32630222	3.06714009	3.98928774	4.89603703	5.59266445
7	13	6	10	91	2.32630222	3.06714008	3.98928768	4.89603551	5.59265638
7	13	7	10	97	2.32630222	3.06713998	3.98928734	4.89603087	5.59265247

Table 8
Sum of the five calculated non-dimensional symmetric frequencies

n	I	J	P	Q	TR	TZ	Σ_{LS}	I	J	P	Q	Σ_{Op}
12	3	2	2	3	3	2	23.40333	2	4	2	3	22.28175
24	3	6	2	7	3	4	20.25094	3	6	3	5	20.22693
32	4	6	3	7	4	4	20.10651	3	6	4	7	19.94045
40	5	6	4	7	5	4	19.91512	4	8	4	7	19.88941
60	5	10	4	11	5	6	19.87479	5	8	6	9	19.87259
84	7	10	6	11	7	6	19.87143	7	10	6	11	19.87143

second and third frequencies, respectively, while in Table 1 [8] such a convergence is achieved with determinant orders of 72. The value of the fourth frequency is equal in both works.

(b) With the aim of comparing the first five frequencies simultaneously, Table 8 has been constructed, in which the lowest sums of the first five frequencies of Leissa and So, Σ_{LS}, are shown, together with the sums Σ_{Op} obtained from Table 7, for those calculations with same number of terms ($n = 12, 24, 32, 40, 60, 84$). It can be seen that all the sums of the optimization method are less than those in Ref. [8], except for the last sum, which is equal. Therefore, the optimization method proposed requires either smaller or equal determinant sizes to obtain the same degree of convergence. The equality of sum in the last row does not mean equality in the effectiveness of the methods, in effect, since the four maximum exponents are equal, then the results must necessarily be equal. In Ref. [8], different combinations of exponents are tested, the convergence scheme is repeated with increasing upper limits of the series, the results of the trial are shown in a table; an analysis of the table permits the determination of the lowest frequency obtained for a mode with the smallest determinant size. On the other hand, the optimum procedure proposed makes use of a computer to automatically calculate the best choice of exponents. As a result of the calculations, for a given determinant size, the computer program gives the best combination of the upper limits I, J, P, Q.

(c) In the sixth and seventh columns of Table 8, the amounts TR and TZ appear. TR indicates the number of terms of the series U where r appears, which agrees with the number of terms in r of the series of W. TZ indicates the number of terms in z that appear in each series U and W. The supposition that the number of terms in r is equal for U and W and that the number of terms in z for U and W is equal, is not sufficiently justified, although it has been used with some frequency [8,9]. Better options appear in Table 8.

(d) The hypothesis [8] that the best choice of number of terms in the axial direction is equal to the number of terms in the radial direction multiplied by the length-to-diameter ratio is not consistent with Table 8. When continuing Table 7 up to $n = 112$, the best values of frequencies are for the maximum exponents $\{7, 12, 9, 13\}$, which are in disagreement with the aforementioned hypothesis.

6. Conclusions

An intuitive procedure to select the number of terms in the series solution to calculate axisymmetric natural vibration frequencies of a cylinder is presented. The procedure can be used in any problem where it is necessary to determine the optimum number of terms in a series of two independent functions that can be expressed in terms of two independent variables.

The Ritz method is optimized by automatically determining the upper limits of the exponents of the series used as displacements functions. The proposed procedure yields the best combination of exponents for a determinant size. One advantage of the proposed procedure is that no tables are needed to compare the results in order to seek the best solution. The computer program developed chooses the best combination of exponents at any stage.

For large determinant sizes, the proposed technique yields similar results to those obtained by other authors for six-figure convergence; because when many polynomials terms are taken convergence is guaranteed. However, finding the upper limits in the series is a time-consuming task for the researcher.

The optimization procedure is proved to yield either more or equal accurate frequencies for equal number of terms in the series.

The proposed method is an improvement on the methods to calculate the natural frequencies in a more straightforward way. The calculated frequencies agree with the experimental ones better than others calculated values.

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